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LOCAL SYSTEMS OF LOCALLY SUPERSOLUBLE FINITARY GROUPS

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Abstract. We show that locally supersoluble finitary groups over certain division rings (e.g. fields) have local systems of hypercyclic normal subgroups.

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In 1960, M. S. Garaščuk [**1**] proved that a locally nilpotent linear group is hypercentral and in 1971, B. A. F. Wehrfritz extended this result by proving that a locally supersoluble linear group is hypercyclic ([**7**] Theorem B). A. E. Zalesskii investigated matrix groups over division rings in his 1969 paper [**12**]. One of the results of this paper was the following.

THEOREM 1 ([**12**]). *Let D be a locally finite-dimensional division algebra and let G be a locally nilpotent matrix group over D. If G is completely reducible or D has characteristic zero, then G is hypercentral.*

Zalesskii leaves the general case (i.e. in positive characteristic when the group is not necessarily completely reducible) as a conjecture. Despite much effort, this conjecture has not been settled.

I. A. Stewart, in his Ph.D. thesis [**5**], generalized Zalesskii's result to locally supersoluble groups; that is, he showed that Theorem 1 holds with "nilpotent" replaced by "supersoluble" and "hypercentral" replaced by "hypercyclic".

A *finitary skew linear group* G on the (left) vector space V over the division ring D, is a subgroup of

$$\mathrm{FGL}(V) = \{g \in \mathrm{GL}(V) : \dim_D V(g-1) < \infty\}.$$

We drop the word "skew" in the above if D is a field. For an introduction to finitary linear groups, see [**3**].

We shall be interested in these groups only when D is a locally finite-dimensional division algebra over a perfect field. We assume that D satisfies this condition unless otherwise stated.

Now there exist locally nilpotent finitary linear groups which are as far away from being hypercentral as possible. For example, the McLain group $\mathrm{M}(\mathbb{Q}, \mathbb{C})$ (see [**10**] page 421) is locally nilpotent, finitary linear and has trivial hypercentre.

Wehrfritz has studied nilpotence in the finitary skew linear groups that we are interested in (see [**10**] and [**9**]). One of his results yields the following proposition, which

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says that locally nilpotent finitary groups, despite not necessarily being hypercentral, have a rich local structure of hypercentral normal subgroups.

PROPOSITION 2. *Let G be a locally nilpotent finitary skew linear group over D. Then G has a local system of hypercentral normal subgroups. That is, if X is a finite subset of G, then $\langle X^G \rangle$ is hypercentral.*

Proof. For by **[10]** Corollary 1.3, G is generated by normal subgroups which are hypercentral of central height at most $\omega 2$.

In this note, we shall prove a generalization of Proposition 2:

THEOREM 3. *Let G be a locally-nilpotent by abelian subgroup of* $\mathrm{FGL}(V)$*. Then G has a local system of hypercentral by finitely-generated-abelian normal subgroups.*

Let G be a locally supersoluble subgroup of $\mathrm{FGL}(V)$. Then any finite subset X of G lies in a hypercentral by finitely-generated abelian normal subgroup $H \lhd G$. Since H is locally supersoluble, H is hypercyclic by **[8]** Lemma 11.19. Consequently:

COROLLARY 4. *Let G be a locally supersoluble subgroup of* $\mathrm{FGL}(V)$*. Then G has a local system of hypercyclic normal subgroups.*

We need some auxilary results. Note that with our assumptions on D, our results will include the general field case; for when D is a field, we may suppose that it is algebraically closed. Also, with our hypotheses in Theorem 3, any subgroup of G of $\mathrm{FGL}(V)$ has a maximal unipotent normal subgroup $\mathcal{U}(G)$ and any locally nilpotent subgroup N of $\mathrm{FGL}(V)$ has a Jordan decomposition in $\mathrm{FGL}(V)$ into a unipotent part N_u and a d-part N_d (see **[9]** Lemma 2.8). The following lemma is a restatement of **[9]** 4.2(d).

LEMMA 5 (Wehrfritz). *Let $G \leq \mathrm{FGL}(V)$ and let N be a locally nilpotent normal subgroup of G with $\mathcal{U}(N) = 1$. For every finite subset X of G, there is a normal subgroup K of G with $X \subseteq K$ and $N \cap K$ is hypercentral (of central height $\leq \omega 2$).*

The Hirsch-Plotkin radical of the group G is denoted by $\eta(G)$.

LEMMA 6. *Let $G \leq \mathrm{FGL}(V)$ and let X be any subset of G for which $n = \dim_D[V, X]$ is finite. Put $N = \eta(\langle X^G \rangle)$. Then N_u is nilpotent of class $\leq 2n$.*

Proof. Let $\overline{G} = G(N_d \times N_u)$. Pick any D-$\overline{G}$ composition series of V, say $(V_\alpha, \Lambda_\alpha)_{\alpha \in I}$. Intersecting this series with $[V, X]$ and removing repititions, we obtain a finite series

$$\begin{aligned} 0 = [V, X] \cap V_{\alpha_1} &\leq [V, X] \cap \Lambda_{\alpha_1} = [V, X] \cap V_{\alpha_2} \leq \dots \\ &\leq [V, X] \cap \Lambda_{\alpha_{i-1}} = [V, X] \cap V_{\alpha_i} \leq \dots \\ &\leq [V, X] \cap \Lambda_{\alpha_n} = [V, X] \end{aligned}$$

where $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ are elements of I.

Consider the series

$$0 \leq V_{\alpha_1} \leq \Lambda_{\alpha_1} \leq V_{\alpha_2} \leq \dots \leq V_{\alpha_n} \leq \Lambda_{\alpha_n} \leq V. \tag{1}$$

Now $[V, X] = [V, X] \cap \Lambda_{\alpha_n}$ and $[V_{\alpha_1}, X] \leq [V, X] \cap V_{\alpha_1} = 0$. Also if $1 < i \leq n$ then

$$\left[V_{\alpha_i}, X\right] \leq [V, X] \cap V_{\alpha_i} = [V, X] \cap \Lambda_{\alpha_{i-1}} \leq \Lambda_{\alpha_{i-1}}.$$

Furthermore, the series 1 is $\overline{G}$-invariant and since $N \leq \langle X^{\overline{G}} \rangle$, we have $[V, N] \leq \Lambda_{\alpha_n}, [V_{\alpha_1}, N] = 0$ and $[V_{\alpha_i}, N] \leq \Lambda_{\alpha_{i-1}}$ for $1 < i \leq n$.

Let $B \leq A$ be D-$\overline{G}$ modules with $[A, N] \leq B$ and choose $n \in N$. On the factor A/B, we have $1 \equiv n \equiv n_u n_d$ as a Jordan decomposition for n. By the uniqueness of Jordan decomposition, $n_u \equiv 1$ on the factor A/B. In other words, $[A, N_u] \leq B$. Thus $[V, N_u] \leq \Lambda_{\alpha_n}, [V_{\alpha_1}, N_u] = 0$ and $[V_{\alpha_i}, N_u] \leq \Lambda_{\alpha_{i-1}}$ for $1 < i \leq n$.

Put $C_\alpha = C_{\overline{G}}(\Lambda_\alpha / V_\alpha)$. Then $N_u C_\alpha / C_\alpha$ is a unipotent normal subgroup of the irreducible group $\overline{G}/C_\alpha$ for every $\alpha \in I$. By [**11**] 2.2, $N_u \leq C_\alpha$ and so $[\Lambda_\alpha, N_u] \leq V_\alpha$ for every $\alpha \in I$.

Consequently, N_u stabilizes the series 1 and thus N_u is nilpotent of class $\leq 2n$ (for example, by [**2**] Theorem 1.C.1).

Proof of 3

Let X be a finite subset of G and put $H = \langle X^G \rangle$. Set $N = \eta(H) = H \cap \eta(G)$. Since $G/\eta(G)$ is abelian,

$$H/N \cong H\eta(G)/\eta(G) = \langle \eta(G)x : x \in X \rangle.$$

Thus H/N is a finitely generated abelian group. Now it is sufficient to prove that N is hypercentral.

There is an epimorphism $N \longrightarrow N_d$ with kernel $U = \mathcal{U}(N)$ (see [**9**] Lemma 2.8). Also $U \lhd G$; for U^g is a unipotent normal subgroup of N for every $g \in G$. Let $\overline{X}$ be the set $\{Ux : x \in X\}$. Now N/U is a locally nilpotent normal subgroup of H/U with $\mathcal{U}(N/U) = 1$ on its action on the sum of the composition factors of V as a D-G bimodule. Thus by 5, there is $K \lhd G/U$ with $\overline{X} \subseteq K$ and $(N/U) \cap K$ hypercentral. Also $\langle \overline{X}^G \rangle = H/U$, so $N_d \cong N/U$ is hypercentral.

By 6, N_u is nilpotent and thus $N_u N_d$ is hypercentral (of height $\leq \omega 2$). Since $N \leq N_u N_d$, we have that N is hypercentral, as required.

To finish, we note the following result. It is a small extension of [**6**] Theorem 2.3.

PROPOSITION 7. *Let G be a locally supersoluble finitary skew linear group over the division ring D, which is locally finite-dimensional over some (not necessarily perfect) subfield F. Then G is locally-nilpotent by periodic-abelian.*

Proof. Any finitely generated subgroup of G is supersoluble and thus nilpotent by finite-abelian. The result follows using the local Zariski topology (for example, see [**4**] 2.2 Part 1).

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