Canad. Math. Bull. Vol. **54** (3), 2011 pp. 456–463
doi:10.4153/CMB-2011-074-3

On Operator Sum and Product Adjoints and Closures

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Abstract. We comment on domain conditions that regulate when the adjoint of the sum or product of two unbounded operators is the sum or product of their adjoints, and related closure issues. The quantum mechanical problem PHP essentially selfadjoint for unbounded Hamiltonians is addressed, with new results.

1 Introduction

A preprint of a paper by M. Mortad [9], dealing with when $(A + B)^* = A^* + B^*$, caused me to go back to some old notes of mine [3] in which the same question was investigated but with a more specific goal: when is $A + B$ essentially selfadjoint. In [3] $A + B$ is a quantum mechanical Hamiltonian, *e.g.*, $A = -\Delta$ on a core domain $D(A)$, and B is a quantum mechanical potential. The notes [3] contained a number of sufficient conditions, but they were not what I wanted, so were never published. Here I would like to touch on some main points from [3] to augment [9]. Recall that $A + B$ essentially selfadjoint means that, for A selfadjoint and B a regular symmetric perturbation, *i.e.*, $D(B) \supset D(A)$, one has $A + B \subset \overline{A + B} = (A + B)^*$. Here $\overline{T}$ denotes the closure of a closeable operator T.

Also, in [6] I treated the product adjoint problem $(AB)^* = B^*A^*$ using only domain/range conditions. As pointed out in [9], Mortad's intent was to follow the philosophy of [6] for the $(A + B)^* = A^* + B^*$ problem. However, there are intimate connections between sum and product adjoints and sum and product closures, as shown in [3] and related later papers [4,7,8] and in an earlier paper [2]. These could be of interest to the various authors cited in [9] who are now following a similar path, hence this paper.

However, there was an error in [6, Lemma 1], which carried forward into [6, Lemmas 2, 3, and 4]. Therefore in Section 2 we identify and discuss this oversight, in its way a subtle one, and then present a correct version of [6, Lemma 1]. The point of [6] was to simplify and generalize an interesting lemma of Albevario, Hoegh, and Streit [1]. In looking more closely now at the proof given in [1], I find discrepancies and an error there too. In sum: since [9] followed [6] which followed [1], we first need to identify and correct the errors in [1] and [6]. Then we may proceed to other things.

Section 2 corrects an error in an earlier draft of this paper. Section 3 discusses related issues from [3, 4, 7, 8]. In particular, we look into an important issue treated in

Received by the editors August 6, 2008; revised August 16, 2010.
Published electronically April 25, 2011.
AMS subject classification: **47A05**.
Keywords: unbounded operators, adjoints of sums and products, quantum mechanics.

[7, 8] to improve our understanding of the PHP measurement problem from quantum mechanics and obtain some new results.

2 Correction of [6] and [1]

The paper [6] was motivated by the following interesting lemma [1, Lemma 2].

Lemma 2.1 *Let A and B be selfadjoint operators in a Hilbert space $\mathcal{H}$ and suppose that $D(AB)$ is dense, that AB is closable, that $D(B)$ contains the range of A, that B^{-1} exists, and that one has $D(B^{-1}(AB)^*) \supset D((AB^*)$. Then $(AB)^* = B^*A^*$.*

My proposed simplification and generalization was the following [6, Lemma 1].

Lemma 2.2 *Let A and B be selfadjoint operators in a Hilbert space $\mathcal{H}$ and suppose that $D(AB)$ is dense, that $R(B) \supset D(A)$, and $D(B) \supset R(A)$. Then $(AB)^* = B^*A^*$.*

My proof was the following. Since, when all adjoints are defined, we always have $(AB)^* \supset B^*A^*$, one needs only to go the other way. We write $A = (AB)B^{-1}$, then $A \equiv A^* \supset B^{-1}(AB)^*$, and applying B from the left, $BA \supset (AB)^*$.

Here is a counterexample (see Acknowledgments). Take A to be an unbounded selfadjoint operator with domain $D(A)$ and range $R(A)$ not equal $\mathcal{H}$ and trivial kernel $N(A) = 0$. Let $B = A^{-1}$. Then AB is the Identity operator restricted to the properly dense $R(A)$, $(AB)^*$ is the Identity operator on the whole space $\mathcal{H}$, and that operator strictly contains B^*A^*, which is the Identity operator restricted to $D(A)$.

Somehow in my haste in writing [6], an invited paper in honor of Sergio Albeverio, I had overlooked the fact that the left side of the sought $(AB)^* = B^*A^*$ is always a closed operator. So a necessary condition on the right side is that it also be a closed operator. The above counterexample to the lemma is easily accommodated by just asserting a conclusion that $(AB)^* = \overline{B^*A^*}$, noting that the domination $(AB)^* \supset B^*A^*$ guarantees that the right side is always closeable. In other words, we have the little result that whenever the right side B^*A^* is a densely defined bounded operator, as it is in the counterexample, then $(AB)^* = \overline{B^*A^*}$, because the closure of an operator is always the smallest closed extension, and when B^*A^* is bounded the domain of its closure is the whole space $\mathcal{H}$.

Could just taking $\overline{B^*A^*}$ fix the situation in the unbounded case? One quickly finds that the issue is more subtle than that. Thus, to get some feeling for what is needed, let us look more carefully at the proof from [6] given above.

We may write $A = (AB)B^{-1}$ if and only if $D(B^{-1}) \equiv R(B) \supset D(A)$. But unless $R(B)$ is exactly $D(A)$, we must write more carefully $A = (AB)B_A^{-1}$, where B_A^{-1} denotes B^{-1} restricted to $D(A)$. Then by adjoints we have $A = A^* = ((AB)B_A^{-1})^* \supseteq (B_A^{-1})^*(AB)^*$. We may write this because $D(A)$ is dense and B_A^{-1} is symmetric on it, so $(B_A^{-1})^*$ exists, although it could be bigger than $(B^{-1})^* \equiv B^{-1}$ on $R(B)$. Thus $(B_A^{-1})^*(AB)^* \supset B_{R(B)}^{-1}(AB)^*$, where we perhaps redundantly put the subscript on B^{-1} to remind ourselves of its domain. So we have $A \supset B_{R(B)}^{-1}(AB)^*$. We would like to "apply B from the left" to get $BA \supset (AB)^*$ and be done.

A sufficient condition for that is the assumption in [1] that $D(B^{-1}(AB)^*) \supset D((AB)^*)$. We will look a little more closely at this condition below. But notice at this point that the proof of the lemma in [1] also contained an error. In their first step (see [1], or [6] where I reproduce their proof verbatim), they say "By the assumption that B^{-1} exists on a domain containing $D(A)$ we have $(AB)^{**}B^{-1} \supset A$." As the reasoning behind this step is not elaborated in [1], consider the following. If $R(B) \supset D(A)$, then one can write $(AB)^{**} = \overline{AB} \supset AB$. Since $R(B)$ covers $D(A)$, we know that the $D(AB)$ is generally some proper subset of $D(B)$, equal to $D(B)$ if and only if $R(B) = D(A)$ exactly. Recall that $D(AB)$ was assumed to be at least dense, so that $(AB)^*$ exists. Let x be in this subset $D(AB)$ of $D(B)$ and let $x = B^{-1}y$. Then we may write $(AB)^{**}x = (AB)^{**}B^{-1}y = (AB)B^{-1}y = Ay$ and thus justify the first assertion in [1] that $(AB)^{**}B^{-1} \supset A$. But that should be more precisely written as $(AB)^{**}B_R^{-1} \supset A$, where B_R^{-1} denotes B^{-1} restricted to the range of B on $D(AB)$. Next, the proof in [1] takes adjoints, whereby $(B_R^{-1})^*(AB)^{***} \equiv (B_R^{-1})^*(AB)^* \subset A^* = A$. Hence for all x in $D(B_R^{-1})^*(AB)^*$ one may write $(B_R^{-1})^*(AB)^*x = Ax$. The authors of [1] did not deal with $(B_R^{-1})^*$ and simply wrote $B^{-1}(AB)^*x = Ax$. Nonetheless, at this point one would like to conclude their proof by now "applying" B to both sides to get $(AB)^*x = BAx$, and then by the assumption $D(B^{-1}(AB)^*) \supset D((AB)^*)$, arrive at the desired $BA \supset (AB)^*$. It is in this last step that the assumption that $D(B) \supset R(A)$ is used. There seems no essential problem caused by our better precision $(B_R^{-1})^*$, because $(B_R^{-1})^*$ contains $(B^{-1})^* = B^{-1}$ provided that B_R^{-1} is densely defined.

But their first step required, as I have explained above, that $R(B) \supset D(A)$, which was not in their lemma's hypothesis.

So there were errors in both [1] and [6].

From my analysis above we have proved the following corrected version of the sought lemma.

Lemma 2.3 *Let A and B be selfadjoint operators in a Hilbert Space $\mathcal{H}$ and suppose that $D(AB)$ is dense, that $R(B) \supset D(A)$, and that $D(B^{-1}(AB)^*) \supset D((AB)^*)$. Then $(AB)^* = B^*A^*$.*

We do not need to explicitly assume that B is 1–1, because that is automatically the case when B is selfadjoint with dense range; see, for example, [5], dealing with operator state diagrams. We do not need, nor want, the condition $D(B) \supset R(A)$.

However, we may now turn that hypothesis to advantage to explain the meaning of the $D(B^{-1}(AB)^*) \supset D((AB)^*)$ condition that we had to accept from [1] in our corrected Lemma 2.3. Recall that one always has $D(BA) \subset D(A)$.

Lemma 2.4 *We have $D(B) \supset R(A)$ if and only if $D(BA) = D(A)$ if and only if $D(BA) \supset D(A)$. Therefore, in particular, the following are all equivalent:*

- $D(B^{-1}(AB)^*) \supset D((AB)^*)$
- $D(B^{-1}(AB)^*) = D((AB)^*)$
- $D(B^{-1}) \supset R((AB)^*)$
- $R(B) \supset R((AB)^*)$, *which* $\supset R(BA)$.

The assumption $D(B) \supset R(A)$ of [1] is strong enough to cause $D(B^*A^*) \equiv D(BA)$ to already be equal to $D(A)$ on the right side of the sought $(AB)^* = BA$. It is also relevant to remember that for bounded selfadjoint A and B, one has AB selfadjoint if and only if A and B commute. Thus the innocent looking assumption $D(B) \supset R(A)$

already says, in the unbounded case we are considering, that $(AB)^*$, whatever it is, will have the domain $D(BA)$ that it would have should the two operators commute. That to me is always a special easier case and I did not want it built into Lemma 2.3.

About the other condition $D(B^{-1}(AB)^*) \supset D((AB)^*)$ that I did accept here to correct the proof I gave in [6], I have identified in Lemma 2.4 above the equivalent meanings it gives to ranges. I am not satisfied with having to accept this assumption, but the lemma is now corrected. One can write down the corrected versions of [6, Lemmas 2, 3, and 4] making similar assumptions. The operator state diagrams of [5] can guide this process, but to do that carefully, especially trying at the same time to get rid of the onerous assumption discussed above, is better left to another paper.

Having corrected [6, Lemma 1] for the adjoint operator $(AB)^*$, let us now turn to $(A + B)^*$ sum adjoint results in [9].

These are typified by the following result.

Proposition 2.5 *Let A and B be densely defined operators in a Hilbert space $\mathcal{H}$ and suppose $D(A) \subset D(B)$ and $D((A + B)^*) \subset D(B^*)$. Then $(A + B)^* = A^* + B^*$.*

Proof See [9, Theorem 2.1]. ■

Corollary 2.6 *Let A be an unbounded densely defined closed operator satisfying $D(A) \subset D(A^*)$ and $D[(A + A^*)^*] \subset D(A)$. Then $(A + A^*)^* = A + A^*$,* i.e., *$A + A^*$ is selfadjoint.*

Proof 1. See [9, Proposition 2.1]. ■

Proof 2. I would like to give a different proof here, in hopes of revealing some further content of the domain assumptions. By the assumption $D(A) \subset D(A^*)$ we may form the real part of A, also sometimes called the symmetric part of A, namely $T = (A+A^*)/2$. Thus $A+A^*$ is a symmetric operator on $D(A)$ and thus $A+A^* \subset (A+A^*)^*$. But the second domain assumption makes that operator relation an equality, *i.e.,* $A + A^*$ is selfadjoint. ■

Corollary 2.7 *Let A and B be selfadjoint operators such that $D(B)$ contains both $D(A)$ and $D[(A + B)^*]$. Then $A + B$ is selfadjoint.*

Proof (See [9, Corollary 2.4]). I will give another, "positivity" version in the next section. ■

3 Some Issues and Results from [3, 4, 7, 8]

[3] was motivated by quantum mechanical scattering theory in which a major useful theorem is the Kato-Rellich Theorem, which historically established the essential selfadjointness of the quantum mechanics Schrödinger Hamiltonians.

Theorem 3.1 *Let A be essentially selfadjoint and B symmetric with $D(B) \supset D(A)$ and such that B is relatively bounded with $b < 1$,*

$$\|Bx\| \leqq a\|x\| + b\|Ax\|, \quad x \in D(A).$$

Then $A + B$ is essentially selfadjoint.

https://doi.org/10.4153/CMB-2011-074-3 Published online by Cambridge University Press

Proof See [4], where it is also proved in real Hilbert space. ■

What I was after in [3] was a "positivity" alternative, which would be useful in a number of situations in quantum mechanics and elsewhere. Most Schrödinger PDE operators are bounded below. My conjecture was: Let A be essentially selfadjoint and B a regular symmetric perturbation, $D(B) \supset D(A)$, and both A and B bounded below, then $A + B$ is essentially selfadjoint. With no loss of generality one can posit that A and B are both strongly positive. But the conjecture is not true. I gave a counterexample in [4], where I also gave the following true version, working from my notes in [3].

Theorem 3.2 *Let A and B be strongly positive selfadjoint operators with $D(B) \supset D(A)$. Let $C = I + A^{-1/2}B^{1/2} \cdot B^{1/2}A^{-1/2}$. Then $A + B$ is selfadjoint if and only if C maps $D(A^{1/2})$ onto itself if and only if $D((A + B)^*) \cap D(A^{1/2}) \subset D(B)$. A is essentially selfadjoint if and only if $\overline{A^{1/2}C} = A^{1/2}\overline{C}$ if and only if $(A + B)^2$ is densely defined.*

Proof See [4, p. 208]. ■

Notice that this result implies a "positivity" version of Corollary 2.7 [9, Corollary 2.4], under weaker assumptions on $[D(A+B)^*]$. That is, we only need $D[(A+B)^*] \cap D(A^{1/2})$ to be contained in $D(B = B^*)$.

The point to emphasize here is that the $D[(A+B)^*] \subset D(B^*)$ sufficiency assumption used in [9] is a very strong one, close to a necessary condition in many cases. Although perhaps less obvious, the same can be said of the $D(A + B)^*$ condition in Theorem 3.2. A second point, also evident in Theorem 3.2, is the close and mixed relationships between sum and product adjoint issues and product and sum closure questions, even when some of the operators at issue are bounded and densely defined. For example, writing C above as $C = I + TT^*$, we know that if $T = A^{-1/2}B^{1/2}$ is closed, then C is a bounded everywhere defined strongly positive selfadjoint operator. So product closure becomes important to sum adjoints.

I would now like to investigate another important instance of these issues, for which the reader can find more background in [7], see also [8]. In the measurement theory of quantum mechanics, one is led to consider projected operators PAP. Briefly, P is an orthogonal projector representing the measurement; A is a selfadjoint observable. There are issues about how well (von Neumann Postulate) you can represent a measurement with P, but let us accept that postulate here. Then, in a great many treatments in textbooks and papers in the quantum mechanics literature, the approach is to go to the so-called Heisenberg, or density operator, formulation, in which A is a bounded operator, or even of trace class or Hilbert-Schmidt class. This avoids all domain considerations. Also immediately $(PAP)^* = PAP$ is selfadjoint.

I prefer the more physical Schödinger formulation, in which one needs to consider unbounded selfadjoint Hamiltonians A. Long ago ([2]) I gave the following general sufficient condition for BAB to be selfadjoint.

Corollary 3.3 *If A and B are selfadjoint and B is bounded with closed range $R(B)$ and finite dimensional null space $N(B)$, then BAB is selfadjoint.*

Proof See [2, Corollary]. This result was established by what I call the conventional, *e.g.,* Fredholm theory, viewpoint. However, in my later experience in physical quantum mechanical measurement theory, see [7, 8] and the citations therein, when considering projected Hamiltonians PHP, where H is a physical unbounded Hamiltonian and P is an orthogonal measurement projector, unless one is just projecting onto finite dimensional bound states, there is no justification for making any Fredholm assumptions on P. I also do not accept the often-made assumption (called compatibility) that H commutes with P, although I will discuss that situation below. Therefore I established the following result in [3], see [7, 8]. ∎

Theorem 3.4 *Let H be an unbounded selfadjoint Hamiltonian that does not commute with the orthogonal projection P. Consider the operator PHP. Then PHP is a symmetric operator if and only if $D(HP)$ is dense. Then PHP is selfadjoint if PH is closed.*

Proof See [7] or [8]. I would like to go further in this paper and consider some important issues related to Theorem 3.4. Let me start by proving Theorem 3.4. Remember that $(AB)^* = B^*A^*$ whenever $A \in B(H)$. Recall that $D(PHP) = D(HP)$ is dense if and only if PH is closeable, and that we may not even speak of $(PHP)^*$ if that condition is not met. Then $(PHP)^* = (HP)^*P \supseteq PHP$ and PHP is symmetric. Next we note that

$$(HP)^* \supseteqq \overline{PH} = (PH)^{**} = ((PH)^*)^* = (HP)^*$$

and thus: $\overline{PH} = (HP)^*$. Here we used the facts that $\overline{PH}$ is to be the smallest closed extension of PH, and that $(HP)^*$ is closed. It is an easy exercise to check that HP itself is closed, a fact we will use below. To complete the proof of the theorem, assume PH is closed. Then $(PHP)^* = (HP)^*P = PHP$ is selfadjoint. ∎

Thus, what emerges from this is that if you still want PHP to be a generator of a unitary evolution, *i.e.,* to be selfadjoint, then in view of Theorem 3.4, the principal issues are (a) is $D(HP)$ dense? and (b) is PH closed? For this "finer" quantum mechanical measurement theory, I interpret condition (a) as requiring at least a modicum of consistency between H and P in order to be able to quantum mechanically measure. As I will show below, the usual (much stronger) assumption that H commutes with P certainly guarantees that, but essentially forces the measurement process to be already within the spectral calculus of the observable H. As for condition (b), one might offer an interpretation that it reflects a "completion" or closure of the measurement operation.

First let us generalize Theorem 3.4. Then we will return to look more closely at the principal issues (a) and (b) just mentioned in the (unbounded) commutative case.

Theorem 3.5 *Let A be a generally unbounded selfadjoint operator and B a bounded selfadjoint operator which does not commute with A. Then BAB is a symmetric operator if and only if $D(AB)$ is dense. Then BAB is selfadjoint if BA is closed. The polar factors satisfy $|\overline{BA}|^2 = AB\overline{BA} \supset AB^2A$ and $|AB|^2 = \overline{BA}AB \supset BA^2B$.*

Proof We used no projector properties of P and no special Hamiltonian properties of H in establishing Theorem 3.4. I add the polar factor characterizations in Theorem 3.5 because they can be useful, they highlight the importance of the properties (a) and

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(b) discussed above, and because I was a bit careless in how I stated them in [7, 8] for P and H where the physics papers I was referring to there sometimes assumed H strongly positive, sometimes assumed H commuting with P, etc. The correct versions and proofs with all details are contained in the following:

$$(HP^2H)^* \supset |\overline{PH}|^2 = (\overline{PH})^*\overline{PH} = HP\overline{PH} \supset HP^2H$$

and

$$(PH^2P)^* \supset |HP|^2 = (HP)^*HP = \overline{PH}HP \supset PH^2P.$$

I assumed $D(HP^2H) = D(HPH) = D((PH)^2)$ and $D(PH^2P) = D(H^2P)$ dense when writing those operator's adjoints to the left in the relations above. Note that on those domains both HP^2H and PH^2P are then symmetric operators with real nonnegative sesquilinear forms $\langle HP^2Hx, x\rangle = \langle P(Hx), (Hx)\rangle$ and $\langle PH^2Px, x\rangle = \|HPx\|^2$. Thus they both have selfadjoint Friedrichs extensions. One could go further into such form operators and their square roots but we won't do so here. The A and B versions follow in exactly the same way. ∎

One could generalize these considerations to Banach space versions should that be warranted, even beyond. To conclude this section, I return to the PHP situation when H commutes with P. Recall that situation $H \smile P$ means the following:

$$PH \subset HP \quad \text{and} \quad HP = PHP = PH \quad \text{on} \quad D(H).$$

Also H maps the range $R(P)$ into $R(P)$ and the null space $N(P)$ into $N(P)$, and P maps $D(H)$ into $D(H)$. Thus already we know that $D(HP)$ is dense and condition (a) is met. Generally $D(HP) = D(PHP)$ will be larger than $D(PH) = D(H)$.

Corollary 3.6 *Let H be an unbounded selfadjoint operator which does commute with an orthogonal projection P. Then PHP, PH, and $\overline{PH}$ are symmetric, and PH is essentially selfadjoint if and only if HP is selfadjoint if and only if HP is symmetric. In that case, PHP is essentially selfadjoint, and PHP is selfadjoint if and only if PHP is closed.*

Proof We know by Theorem 3.4 that PHP is symmetric since $D(HP)$ is dense. From $PH \subset HP$ and the above proved $\overline{PH} = (HP)^*$ we have $PH \subset \overline{PH} = (HP)^* \subset HP = (PH)^* = (\overline{PH})^*$, using the minimality of operator closure. Symmetric PH is essentially selfadjoint if and only if equality holds in the last inclusion, and that occurs if and only if HP is selfadjoint if and only if HP is symmetric. In that case, we have $(PHP)^* = (HP)^*P = HPP = HP$ selfadjoint, so $(PHP)^{**} = (PHP)^*$, and PHP is essentially selfadjoint. ∎

I would like to make three final comments. First, in the frequently (and very convenient) assumed case of H commuting with P when both are bounded, one simply has $HP = PHP = PH$ bounded selfadjoint. Trivial as that may seem, you can see that the PH and HP symmetry properties established in Corollary 3.6 are natural. Second, one could generalize Corollary 3.6 to A and B as was done in Theorem 3.5 if you assume A decomposes the Hilbert space in the same way P did. Also you could allow P to be an oblique projection, but then you would need to allow P^* to come into the expressions. Third, I am not sure that my results here are the tightest possible, since I have not looked in full detail at condition (b), namely, at the issue of PH or PHP closed. One can of course write down either Fredholm or domain/range sufficient conditions for condition (b), or go to the operator theory for known general conditions, *e.g.,* B^{-1} continuous, or H^{-1} respectively A^{-1} continuous. But for the general, noncommutative, unbounded, quantum mechanical measurement theory I am espousing here, P will generally have infinite dimensional null space and range, and H will be noninvertible, *e.g.,* not necessarily positive, so as to enable treatment of general momentum operators, be they of first or second order partial derivatives, be they of discrete or continuous spectrum, be that semibounded or not.

4 Conclusion

We have corrected, augmented, and extended the discussions of [1–4, 6–9] for unbounded operator sum and product adjoints. A quantum mechanical measurement issue involving operators PHP has been addressed. To my knowledge, Theorem 3.5 and Corollary 3.6 are new.

Acknowledgment I thank H. Mortad for passing on to me the counterexample to [6, Lemma], which is due to a referee on a related paper, whom I must therefore also thank.

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