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Scale recurrence lemma and dimension formula for Cantor sets in the complex plane

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Abstract. We prove a multidimensional conformal version of the scale recurrence lemma of Moreira and Yoccoz [Stable intersections of regular Cantor sets with large Hausdorff dimensions. *Ann. of Math. (2)* **154**(1) (2001), 45–96] for Cantor sets in the complex plane. We then use this new recurrence lemma, together with Moreira's ideas in [Geometric properties of images of Cartesian products of regular Cantor sets by differentiable real maps. *Math. Z.* **303** (2023), 3], to prove that under the right hypothesis for the Cantor sets $K_1, \ldots, K_n$ and the function $h : \mathbb{C}^n \to \mathbb{R}^l$, the following formula holds:

$$HD(h(K_1 \times K_2 \times \cdots \times K_n)) = \min\{l, HD(K_1) + \cdots + HD(K_n)\}.$$

Key words: dynamically defined Cantor set, Hausdorff dimension, Marstrand projection theorem
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1. *Introduction*

In this paper, we prove a version of the *scale recurrence lemma* of Moreira and Yoccoz (see [7, §3.2]) in the context of Cantor sets in the complex plane. We will use this new version, together with other results, to prove a dimension formula for projections of products of complex Cantor sets. More precisely, given conformal regular Cantor sets $K_1, \ldots, K_n$ in $\mathbb{C}$, and a C^1 function $h : \mathbb{C}^n \to \mathbb{R}^l$, we prove that, under natural hypothesis, one has

$$HD(h(K_1 \times K_2 \times \cdots \times K_n)) = \min\{l, HD(K_1) + \cdots + HD(K_n)\}. \tag{1}$$

Our results will be proved for conformal regular Cantor sets. Those are Cantor sets that are maximal invariant sets for an expanding map, whose derivative is conformal at the points in the Cantor set. Rigorous definitions will be given in §2. The investigation of such Cantor sets is important because they appear in the study of homoclinic bifurcations of automorphisms of $\mathbb{C}^2$, as shown by Araújo and Moreira in [1]. We expect that conformal regular Cantor sets in $\mathbb{C}$ will play a role in the study of homoclinic bifurcations of automorphisms of $\mathbb{C}^2$, similar to regular Cantor sets in $\mathbb{R}$ in the study of homoclinic bifurcations of surface diffeomorphisms.

The study of homoclinic bifurcations has proved to be fruitful in the understanding of dynamics for surface diffeomorphisms. Complicated dynamical phenomena arise from them. For example, arbitrarily close to any diffeomorphism exhibiting a generic homoclinic tangency, there are open regions in which any diffeomorphism belonging to a residual set has an infinite number of sinks—this is the so-called Newhouse phenomenon. Looking for analogous results and using similar ideas to those of Newhouse, Buzzard [2] proved the existence of an open set of automorphisms of $\mathbb{C}^2$ with stable homoclinic tangencies.

The scale recurrence lemma was an important step in the solution to the Palis conjecture, about the arithmetic difference of Cantor sets, by Moreira and Yoccoz. They proved that there is an open and dense subset, inside the set of pairs of regular Cantor sets with sum of Hausdorff dimensions bigger than one, such that any pair (K_1, K_2) in this subset verifies $int(K_1 - K_2) \neq \emptyset$. The theorem of Moreira and Yoccoz is for regular Cantor sets in the real line. Together with Araujo, we are close to proving an analogous result for conformal regular Cantor sets in the complex plane. The scale recurrence lemma in these papers is a fundamental tool for this work in progress; from our point of view, this is the main motivation for proving the conformal version of the lemma.

Furthermore, Moreira and Yoccoz were able to use their solution to the Palis conjecture in the study of homoclinic bifurcations for surface diffeomorphisms (see [9]). They proved that given a surface diffeomorphism F with a homoclinic quadratic tangency associated to a horseshoe with dimension larger than one, the set of diffeomorphisms close to F presenting a stable tangency has positive density at F. One of the main reasons to study conformal regular Cantor sets is to apply the ideas in [9] to the context of homoclinic bifurcations of automorphisms of $\mathbb{C}^2$. The work [1] already proves that this approach is rewarding; it shows that Buzzard's example [2] can be recast in terms of the theory of conformal Cantor sets (where the concepts of limit geometries and recurrent compact set are applied).

Another development in the subject was given by Lopez [4]. He generalized the work [7] for a product of several Cantor sets in the real line. In this paper, we will consider a scale recurrence lemma for a product of several Cantor sets in the complex plane.

In brief, we believe that the study of conformal regular Cantor sets plays an important role in the theory of dynamics of automorphisms of $\mathbb{C}^2$. We think that such Cantor sets should be further investigated, as we are convinced that the scale recurrence lemma is a key step towards a complex version of Palis's conjecture, and this will have implications in the bifurcation theory of automorphisms of $\mathbb{C}^2$.

At the same time, we have the dimension formula as an application of the scale recurrence lemma. The study of this type of dimension formulas is motivated by a classical theorem of Marstrand, generalized by Mattila and others. Denote by $G(m, l)$ the set of l-dimensional linear subspaces of $\mathbb{R}^m$ and for $s \in G(m, l)$, denote by π_s the orthogonal projection on s. The Marstrand theorem states that given $F \subset \mathbb{R}^m$, a Borel subset, we have

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$$HD(\pi_s(F)) = \min\{l, HD(F)\}$$

for almost all $s \in G(m, l)$, with respect to a volume measure on $G(m, l)$. In the particular case when $F = K_1 \times \cdots \times K_n$ is a product of regular Cantor sets, one has $HD(K_1 \times \cdots \times K_n) = HD(K_1) + \cdots + HD(K_n)$. Thus, our formula, equation (1), corresponds to the Marstrand formula replacing π_s by h. The difference between our result and the classical Marstrand theorem is that our theorem is not an 'almost all' result; it holds under explicit generic conditions on the map h and the Cantor sets. The formal statement of the dimension formula proven in this paper is the following main theorem.

THEOREM B. (Dimension formula) *Let $K_1, \ldots, K_n$ be C^m, $m \geq 2$, conformal regular Cantor sets generated by expanding maps $g_1, \ldots, g_n$, respectively. Suppose all of them are not essentially affine. Assume that there exist periodic points $p_j \in K_j$, with period n_j, for $1 \leq j \leq n$, such that if we write $Dg_j^{n_j}(p_j) = (1/r_j)R_{-v_j}$, where R_v is the rotation matrix by an angle $v \in \mathbb{T}$, then*

$$\begin{gathered}(\log r_1, 0, \ldots, 0; v_1, 0, \ldots, 0),\\ \vdots\\ (0, \ldots, \log r_{n-1}; 0, \ldots, v_{n-1}, 0),\\ (-\log r_n, \ldots, -\log r_n; 0, \ldots, 0, v_n),\end{gathered}$$

generate a dense subgroup of $\mathbb{R}^{n-1} \times \mathbb{T}^n$. Let h be any C^1 function defined on a neighborhood of $K_1 \times \cdots \times K_n$ into $\mathbb{R}^l$ such that there exists $x_0 \in K_1 \times \cdots \times K_n$ where $Dh(x_0)$ verifies the transversality hypotheses. Then

$$HD(h(K_1 \times \cdots \times K_n)) = \min\{l, HD(K_1) + \cdots + HD(K_n)\}.$$

The transversality hypotheses mean that for any subset $A \subset \{1, \ldots, n\}$, the linear map $Dh(x_0) : \mathbb{C}^n \to \mathbb{R}^l$ satisfies

$$\dim(\mathrm{Im}(Dh(x_0)|_{\{z_j=0: j \notin A\}})) = \min\{l, 2 \cdot \#A\}.$$

This is the minimum assumption one needs to have the dimension formula for linear maps. Proper definitions of all other objects are given in the next section.

This type of problem has already been investigated by other authors, we mention some of them. Peres and Shmerkin [11] proved that for $K_1, K_2 \subset \mathbb{R}$ attractors for self-affine iterated functions system (i.f.s.) given by maps $\{r_i x + t_i\}_{i=1}^{n}$, $\{r_i' x + t_i'\}_{i=1}^{n'}$, if there are j, k such that $\log(r_j)/\log(r_k')$ is irrational, then

$$HD(K_1 + \lambda \cdot K_2) = \min\{1, HD(K_1) + HD(K_2)\}$$

for all $\lambda \neq 0$.

However, Moreira [8] studied the same formula for $K_1, K_2 \subset \mathbb{R}$ regular Cantor sets. He proved that the formula holds provided one of the Cantor sets is not essentially affine. Moreira's proof uses the scale recurrence lemma of [7].

In another work, Hochman and Shmerkin [3] proved a dimension formula without assuming any type of affinity or non-affinity in the attractors or Cantor sets. They proved

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(in fact, this is a corollary of their main theorem) that for $K_1, \ldots, K_n$ attractors for i.f.s. on $\mathbb{R}$, one has

$$HD(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \min\{1, HD(K_1) + \cdots + HD(K_n)\}$$

for all $\lambda_i \neq 0$, $i = 1, \ldots, n$, provided that a certain set is dense in the group $(\mathbb{R}^n, +)/\Delta$, where Δ is the diagonal. This set depends on the derivative of the contractions of the i.f.s. on periodic points. The technique used by Hochman and Shmerkin is different from the approach of Moreira.

Apart from the motivations given by the Marstrand theorem and dynamical systems, there are other reasons to study sets of the form $K_1 + K_2$, where K_1, K_2 are dynamically defined Cantor sets. There are applications in number theory as well. In [6], Moreira used his dimension formula to prove that fractal dimensions of the Lagrange spectrum grow continuously. More precisely, he proved that the function

$$d(t) = HD(L \cap (-\infty, t)),$$

where $L \subset \mathbb{R}$ is the Lagrange spectrum, is continuous.

In this paper, we will adapt the methods used by Moreira and Yoccoz to the context of Cantor sets in the complex plane. We will consider an arbitrary finite number of Cantor sets, not just two. This will leave us facing different difficulties. First, we need to find the right definition for the renormalization operators and the right statement for the scale recurrence lemma. Those are mainly influenced by Lemma 3.1 and equation (2). However, the high dimensionality of the context requires a more detailed analysis. This can be seen in the proof of Proposition 3.1 and the use of Lemma 3.5. The proof of the conformal scale recurrence lemma requires a new type of hypothesis, the fact that each of the Cantor sets we are working with is not contained in a C^1 embedded curve. We call this property not essentially real, it is introduced in §2.4. Moreover, the proof of the scale recurrence lemma in [7] has a minor flaw, which we need to deal with; in §4, we comment on how to solve it.

The paper is organized as follows. Section 2 contains basic definitions and results. In this section, we state, without proof, the scale recurrence lemma. Section 3 is dedicated to the proof of the dimension formula. Finally, in §4, we prove the scale recurrence lemma.

2. *Basic definitions*

In this section, we define the objects and present the principal tools that will play a role in the paper. Most of the proofs of the facts stated in this section follow from standard techniques, so we leave them without proof. For proofs, we refer the reader to [13, Ch. 1].

2.1. *Conformal regular Cantor set.*

A C^m *regular Cantor set* (or *dynamically defined Cantor set*) on the complex plane is given by the following data: a finite set $\mathbb{A}$ of *letters* and a set $B \subset \mathbb{A} \times \mathbb{A}$ of *admissible* pairs, for each $a \in \mathbb{A}$ a compact connected set $G(a) \subset \mathbb{C}$, and a C^m function $g : V \to \mathbb{C}$ defined in an open neighborhood V of $\bigsqcup_{a \in \mathbb{A}} G(a)$. This data must verify the following assumptions.

- The sets $G(a)$, $a \in \mathbb{A}$, are pairwise disjoint.
- $(a, b) \in B$ implies $G(b) \subset g(G(a))$, otherwise $G(b) \cap g(G(a)) = \emptyset$.

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- For each $a \in \mathbb{A}$, the restriction $g|_{G(a)}$ can be extended to a C^m diffeomorphism from an open neighborhood of $G(a)$ onto its image such that $m(Dg) > 1$ (where $m(A) = \inf_{v \neq 0} |Av|/|v|$ is the minimum norm of the linear map A).
- The subshift (Σ^+, σ) induced by B

$$\Sigma^+ = \{\underline{a} = (a_0, a_1, \ldots) \in \mathbb{A}^{\mathbb{N}} : (a_i, a_{i+1}) \in B \text{ for all } i \geq 0\},$$

$\sigma(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$ is topologically mixing.

Once we have such data, we can define a Cantor set (that is, totally disconnected, perfect compact set) on the complex plane $K = \bigcap_{n \geq 0} g^{-n}(\bigsqcup_{a \in \mathbb{A}} G(a))$.

We will say that the regular Cantor set is conformal if for all $x \in K$, the linear map $Dg(x) : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal, that is, $m(Dg(x)) = \|Dg(x)\|$. The assumption that Dg is not necessarily conformal outside of the Cantor set was introduced by Araujo in his PhD thesis, where he studied Cantor sets associated to complex horseshoes for automorphisms of $\mathbb{C}^2$. It plays an important role in the investigation of a complex version of Palis's conjecture. We will write only K to represent all the data that are required to define a conformal regular Cantor set. All Cantor sets in this paper will be conformal regular Cantor sets; we will usually refer to them just as Cantor sets.

The degree of differentiability, m, can be any real number bigger than one. If m is not an integer, then g being C^m means that it is $C^{[m]}$, where $[m]$ is the integer part of m, and $D^{[m]}g$ is Holder with exponent $m - [m]$. To prove our results, we will assume that $m \geq 2$.

We can actually suppose that the sets $G(a)$ verify $G(a) = \overline{\text{int}(G(a))}$, this is a consequence of the next lemma.

LEMMA 2.1. *Let K be a C^m conformal Cantor set, then there exist a family of open and connected sets $G^*(a) \subset \mathbb{C}$ for $a \in \mathbb{A}$, such that we have the following properties.*

(i) *$G(a) \subset G^*(a)$, and $g|_{G(a)}$ can be extended to an open neighborhood of $\overline{G^*(a)}$ such that it is a C^m diffeomorphism from this neighborhood onto its image and $m(Dg) > 1$.*

(ii) *The sets $\overline{G^*(a)}$, $a \in \mathbb{A}$, are pairwise disjoint.*

(iii) *$(a, b) \in B$ implies $\overline{G^*(b)} \subset g(G^*(a))$, and $(a, b) \notin B$ implies $\overline{G^*(b)} \cap \overline{g(G^*(a))} = \emptyset$.*

2.2. *Limit geometry.* Associated to K, we define the sets $\Sigma^{fin} = \{(a_0, \ldots, a_n) : (a_i, a_{i+1}) \in B\}$ and $\Sigma^- = \{(\ldots, a_{-n}, \ldots, a_{-1}, a_0) : (a_i, a_{i+1}) \in B\}$.

Given $\underline{a} = (a_0, \ldots, a_n)$, $\underline{b} = (b_0, \ldots, b_m)$, $\underline{\theta}^1 = (\ldots, \theta^1_{-1}, \theta^1_0)$, $\underline{\theta}^2 = (\ldots, \theta^2_{-1}, \theta^2_0)$, we will use the following notation.

- If $a_n = b_0$, $\underline{ab} = (a_0, \ldots, a_n, b_1 \ldots, b_m)$.
- If $\theta^1_0 = a_0$, $\underline{\theta}^1\underline{a} = (\ldots, \theta^1_{-1}, \theta^1_0, a_1, \ldots, a_n)$.
- If $\theta^1_0 = \theta^2_0$, $\underline{\theta}^1 \wedge \underline{\theta}^2 = (\theta^1_{-j}, \ldots, \theta^1_0)$, where j is such that $\theta^1_{-i} = \theta^2_{-i}$ for all $0 \leq i \leq j$, and $\theta^1_{-j-1} \neq \theta^2_{-j-1}$.
- If $\theta^1_0 = a_n$, $\underline{\theta}^1 \wedge \underline{a} = (\theta^1_{-j}, \ldots, \theta^1_0)$, where j is such that $\theta^1_{-i} = a_{n-i}$ for all $0 \leq i \leq j$, and $\theta^1_{-j-1} \neq a_{n-j-1}$.

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For $\underline{a} = (a_0, \ldots, a_n) \in \Sigma^{fin}$, define $G(\underline{a}) = \{x \in \bigsqcup_{a \in \mathbb{A}} G(a) : g^j(x) \in G(a_j),$ $j = 0, 1, \ldots, n\}$, and the function $f_{\underline{a}} : G(a_n) \to G(\underline{a})$ given by

$$f_{\underline{a}} = g|_{G(a_0)}^{-1} \circ \cdots \circ g|_{G(a_{n-1})}^{-1}.$$

Denote by $K(\underline{a})$ the set $K \cap G(\underline{a})$. For each $a \in \mathbb{A}$, we choose an arbitrary point $c_a \in K(a)$. Using this, define $c_{\underline{a}} \in G(\underline{a})$ by $c_{\underline{a}} = f_{\underline{a}}(c_{a_n})$.

Notice that $Df_{\underline{a}}(c_{a_n})$ is a conformal matrix in $\mathbb{R}^2$, then it is equal to a positive real number times a rotation matrix, and denote the angle of rotation by $v_{\underline{a}} \in \mathbb{R}/(2\pi\mathbb{Z})$. In this way, we have a preferred point and direction for each $G(\underline{a})$. We also define $r_{\underline{a}} = \operatorname{diam}(G(\underline{a}))$, where diam means diameter.

Given $\underline{\theta} = (\ldots, \theta_{-n}, \ldots, \theta_{-1}, \theta_0) \in \Sigma^-$, let $\underline{\theta}^n = (\theta_{-n}, \ldots, \theta_0)$ and define $k_n^{\underline{\theta}} : G(\theta_0) \to \mathbb{C}$ by

$$k_n^{\underline{\theta}} = \phi_{\underline{\theta}^n} \circ f_{\underline{\theta}^n},$$

where $\phi_{\underline{\theta}^n}$ is the unique map in $Aff(\mathbb{C}) = \{A(z) = az + b : a, b \in \mathbb{C},\ \ a \neq 0\}$ such that $\phi_{\underline{\theta}^n}(c_{\underline{\theta}^n}) = 0$, $D\phi_{\underline{\theta}^n}(c_{\underline{\theta}^n})e^{iv_{\underline{\theta}^n}} \in \mathbb{R}^+$, $\operatorname{diam}(\phi_{\underline{\theta}^n}(G(\underline{\theta}^n))) = 1$. For the next theorem, we consider $k_n^{\underline{\theta}}$ extended to a small open neighborhood $G^*(\theta_0)$ of $G(\theta_0)$ (as in Lemma 2.1).

THEOREM 2.1. *Let K be a C^m conformal Cantor set. For any $\underline{\theta} \in \Sigma^-$, the family of functions $k_n^{\underline{\theta}} : G^*(\theta_0) \to \mathbb{C}$ converges in the $C^{[m]}$ topology, with an exponential rate of convergence independent of $\underline{\theta}$, to a C^m function $k^{\underline{\theta}} : G^*(\theta_0) \to \mathbb{C}$. The function $k^{\underline{\theta}}$ is a diffeomorphism onto its image and the derivative $Dk^{\underline{\theta}}(x)$ is conformal for all $x \in K(\theta_0)$.*

Moreover, if $m \geq 2$, then there is a constant $C > 0$ such that given $\underline{\theta}^1, \underline{\theta}^2 \in \Sigma^-$ ending with the same letter,

$$\sup_z [|k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^2})^{-1}(z) - z| + \|D(k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^2})^{-1})(z) - I\|] \leq C \operatorname{diam}(G(\underline{\theta}^1 \wedge \underline{\theta}^2)).$$

The function $k^{\underline{\theta}}$ is called a limit geometry of K. Notice that the rate of convergence being independent of $\underline{\theta}$ implies that $D^l k^{\underline{\theta}}$, for $0 \leq l \leq [m]$, depends continuously on $\underline{\theta}$. The proof of Theorem 2.1 can also be found in [1].

For $\underline{\theta} \in \Sigma^-$, $\underline{a} \in \Sigma^{fin}$, such that $\underline{a}$ starts with the last letter of $\underline{\theta}$, define

$$G^{\underline{\theta}}(\underline{a}) = k^{\underline{\theta}}(G(\underline{a})), \quad K^{\underline{\theta}}(\underline{a}) = k^{\underline{\theta}}(K(\underline{a})), \quad c_{\underline{a}}^{\underline{\theta}} = k^{\underline{\theta}}(c_{\underline{a}})$$

$$\exp(iv_{\underline{a}}^{\underline{\theta}}) = \frac{Dk^{\underline{\theta}}(c_{\underline{a}})}{\|Dk^{\underline{\theta}}(c_{\underline{a}})\|}\exp(iv_{\underline{a}}), \quad r_{\underline{a}}^{\underline{\theta}} = \operatorname{diam}(G^{\underline{\theta}}(\underline{a})).$$

Let $F_{\underline{a}}^{\underline{\theta}}$ be the affine map determined by the equation $k^{\underline{\theta}} \circ f_{\underline{a}} = F_{\underline{a}}^{\underline{\theta}} \circ k^{\underline{\theta a}}$. It maps 0 to $c_{\underline{a}}^{\underline{\theta}}$ and can be written using $r_{\underline{a}}^{\underline{\theta}} \in \mathbb{R}^+$, $v_{\underline{a}}^{\underline{\theta}} \in \mathbb{R}/2\pi\mathbb{Z}$ as

$$F_{\underline{a}}^{\underline{\theta}}(z) = r_{\underline{a}}^{\underline{\theta}} \exp(iv_{\underline{a}}^{\underline{\theta}})z + c_{\underline{a}}^{\underline{\theta}}.$$

Definition 2.1. We will say that a C^m, $m \geq 2$, Cantor set K is *not essentially affine* if there exist $\underline{\theta}^1, \underline{\theta}^2 \in \Sigma^-$, ending in the same letter, and $z_0 \in K^{\underline{\theta}^2}(\theta_0^2)$ such that $D^2(k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^2})^{-1})(z_0) \neq 0$.

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2.3. *Mass distribution principle.* Typically, estimating the Hausdorff dimension from below is harder than from above. One usual technique is the mass distribution principle that we state below.

PROPOSITION 2.1. *Let $F \subset \mathbb{R}^l$ be a Borel measurable set, ν a Borel measure with $\nu(F) > 0$, and $a, b > 0$, $s > 0$ such that $\nu(u) \leq a \cdot \mathrm{diam}(u)^s$ for all u measurable with $\mathrm{diam}(u) < b$. Then the Hausdorff dimension of F, denoted by $HD(F)$, is bigger than s.*

The next proposition is a consequence of the mass distribution principle and it will be used to prove the desired dimension formula (Theorem B). Its proof is not difficult and can be found in [13, §1.3].

Let N be the node set of a rooted tree with the property that every node has finite index. Here, N can be described in the following way: there is a marked element $p_0 \in N$ called the root of N; for each $p \in N$, we have a finite set $Ch(p) \subset N$ called the children of p; if $p \neq q$, then $Ch(p) \cap Ch(q) = \emptyset$; for any $q \in N$, there is a sequence $q_0, q_1, \ldots, q_m$ such that $q_0 = p_0$, $q_m = q$, and $q_{i+1} \in Ch(q_i)$, $i = 0, \ldots, m-1$, such q is called an m-level node of N. Denote by $I(k)$ the set of k-level nodes. Now N can be written as the disjoint union $N = \sqcup_{k=0}^{\infty} I(k)$.

COROLLARY 2.1. *Suppose we have a set N as described above and assume that for each $p \in N$, we have a Borel measurable set $G(p) \subset \mathbb{R}^l$ with the following properties:*

(a) *if $p \in Ch(q)$, then $\overline{G(p)} \subset G(q)$;*
(b) *if $p_1, p_2 \in Ch(q)$, $p_1 \neq p_2$, then $\overline{G(p_1)} \cap \overline{G(p_2)} = \emptyset$;*
(c) *the supremum $\sup\{\mathrm{diam}(G(p)) : p \in I(k)\}$ goes to zero as k goes to infinity;*
(d) *there is a constant $\mu > 1$ such that for any $p \in Ch(q)$, we have $\mathrm{diam}(G(p)) \geq \mu^{-1}\mathrm{diam}(G(q))$;*
(e) *there is a constant $\mu > 1$ such that for any $p \in N$, the set $G(p)$ contains a ball of radius $\mu^{-1}\mathrm{diam}(G(p))$;*
(f) *there is a number $s > 0$ such that for any $q \in N$,*

$$\sum_{p \in Ch(q)} \mathrm{diam}(G(p))^s \geq \mathrm{diam}(G(q))^s.$$

Let F be the set $F = \bigcap_{k=0}^{\infty} \bigcup_{p \in I(k)} G(p)$. Then $HD(F) \geq s$.

2.4. *Not essentially real Cantor sets.* In this subsection, we will present a hypothesis in the Cantor set that will guarantee it is indeed two dimensional. We remark that for regular Cantor sets in the real line, there is no analogous definition and results to those in this section. Those are objects that start appearing at dimension two.

Definition 2.2. We will say that a Cantor set K is *essentially real* if there exists $\underline{\theta} \in \Sigma^-$ such that the limit Cantor set $K^{\underline{\theta}}(\theta_0)$ is contained in a straight line.

It is not difficult to prove that K is essentially real if and only if for every $\underline{\theta} \in \Sigma^-$, the limit Cantor set $K^{\underline{\theta}}(\theta_0)$ is contained in a straight line. Moreover, one can prove that K being essentially real is equivalent to K being contained in a C^1 one-dimensional manifold embedded in the plane. For the proof of the scale recurrence lemma, we will suppose that the Cantor set is not essentially real, in such a case, one is able to control the quantity of elements $G^{\underline{\theta}}(\underline{a})$ close to an arbitrary line; this is the content of the next lemma.

Given $c > 0, \rho > 0$, define $\Sigma(c, \rho) = \{\underline{a} \in \Sigma^{fin} : \ c^{-1}\rho \leq \operatorname{diam}(G(\underline{a})) \leq c\rho\}$. We can think of this as the set of $G(\underline{a})$ having approximate size ρ. Using standard techniques (see [10] or [13]), one can prove that there is a constant $C > 0$, depending only on c and the Cantor set K and not depending on ρ, such that

$$C^{-1}\rho^{-HD(K)} \leq \#\Sigma(c, \rho) \leq C\rho^{-HD(K)}.$$

Suppose we have fixed a constant $C_5 > 0$. Let $(a, b) \in B$. A subset $D \subset \Sigma(C_5, \rho)$ is called a *discretization* of $K(a, b)$ of order ρ if $\bigcup_{\underline{a}\in D} K(\underline{a}) = K(a, b)$.

LEMMA 2.2. *Let K be a Cantor set not essentially real. There exist an angle $\alpha \in (0, \pi/2)$ and numbers $\rho_2 > 0$, $a \in (0, 1)$, depending only on C_5 and the Cantor set K, such that for any limit geometry $k^{\underline{\theta}}$, $x \in G^{\underline{\theta}}(\theta_0)$, line L, $s \in \mathbb{A}$, D discretization of $K(\theta_0, s)$ of order less than ρ_2,*

$$\#\{\underline{a} \in D : \ G^{\underline{\theta}}(\underline{a}) \cap \operatorname{Cone}(x, L, \alpha) \neq \emptyset\} \leq a \cdot \#D,$$

where $\operatorname{Cone}(x, L, \alpha)$ *is the set of $z \in \mathbb{C}$ such that the vector $z - x$ forms an angle of measure less than α with the line L.*

Another use of the not essentially real hypothesis will be given in the next lemma. Let K be a Cantor set, for $x \in K$, consider the set

$$K_x^{dir} := \bigcap_{\delta>0} \overline{\left\{\frac{y-x}{|y-x|} : y \in B_\delta(x) \cap (K \setminus \{x\})\right\}},$$

where $B_\delta(x)$ is the open ball of radius δ centered at x. If K is not essentially real, then the set K_x^{dir} has two linearly independent vectors (over $\mathbb{R}$) and then the following lemma holds for K.

LEMMA 2.3. *Let K be a Cantor set and f a C^l function from a neighborhood of K into $\mathbb{R}^2$. Suppose that f is conformal at K, that is, $Df(x)$ is conformal for all $x \in K$, and K_x^{dir} has two linearly independent vectors (over $\mathbb{R}$) for all $x \in K$. Then, for all $x \in K$, the l-linear map $D^l f(x) : \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \to \mathbb{R}^2$ is conformal, that is, there is a complex number c_x^l such that*

$$D^l f(x)(z_1, \ldots, z_l) = c_x^l \cdot z_1 \cdot z_2 \cdot \cdots \cdot z_l.$$

The operation $\cdot$ on the right-hand side of the last equality corresponds to complex multiplication.

In particular, if a Cantor set is not essentially real and not essentially affine, then for the values $z_0 \in K$, $\underline{\theta}^1, \underline{\theta}^2 \in \Sigma^-$, given by Definition 2.1, there is a non-zero complex number d_0 such that

$$D^2(k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^2})^{-1})(z_0)(v, w) = d_0 \cdot v \cdot w.$$

2.5. *Renormalization operator.* From now on, we will be working with a finite set of conformal regular Cantor sets $K_1, \ldots, K_n$. To each of them, we have various objects

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associated, as defined in the previous subsections. We will use subscripts and superscripts to differentiate the objects from one Cantor set to the other. For example, we use $\Sigma_j(c, \rho)$ for the set $\Sigma(c, \rho)$, which was defined in the last subsection, associated to the Cantor set K_j. We will denote by d_j the Hausdorff dimension of the Cantor set K_j. In this section, we will define renormalization operators, which are operators associated to the family $K_1, \ldots, K_n$ of Cantor sets.

Define $J = \mathbb{R}^{n-1} \times \mathbb{T}^n$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The group $\mathbb{T}$ is endowed with the unique invariant distance giving diameter 2π to the whole space. For $v \in \mathbb{T}$, we denote by $\|v\|$ the distance between v and the zero element. The space J is an abelian locally compact group. We put on J the unique invariant metric such that the distance between $(t_1, \ldots, t_{n-1}, v_1, \ldots, v_n)$ and the zero element is $\max_j\{|t_j|, \|v_j\|\}$.

Lemma 3.1 justifies why we choose to work on $J = \mathbb{R}^{n-1} \times \mathbb{T}^n$. The lemma states that the 'renormalization' of h, given by $h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})$, becomes, modulo composition by an affine function, arbitrarily close to a function of the form $B \circ A(t, v) \circ (k^{\underline{\theta}^1}, \ldots, k^{\underline{\theta}^n})$. The pair (t, v), parameterizing the maps $A(\cdot, \cdot)$, is in $\mathbb{R}^{n-1} \times \mathbb{T}^n$. In fact, if the function h were not only C^1 but holomorphic, then we could reduce further the type of function A appearing and take J as $\mathbb{R}^{n-1} \times \mathbb{T}^{n-1}$. The renormalization operators are then defined observing how the expression $A(t, v) \circ (k^{\underline{\theta}^1}, \ldots, k^{\underline{\theta}^n})$ changes when one composes it by $(f_{\underline{b}^1}, \ldots, f_{\underline{b}^n})$.

For $(\underline{b}^1, \ldots, \underline{b}^n) \in \Sigma_1^{fin} \times \cdots \times \Sigma_n^{fin}$, define the operator

$$T_{\underline{b}^1,\ldots,\underline{b}^n} : \Sigma_1^- \times \cdots \times \Sigma_n^- \times J \to \Sigma_1^- \times \cdots \times \Sigma_n^- \times J,$$

given by

$$T_{\underline{b}^1,\ldots,\underline{b}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, t_1, \ldots, t_{n-1}, v_1, \ldots, v_n)$$
$$= \left(\underline{\theta}^1\underline{b}^1, \ldots, \underline{\theta}^n\underline{b}^n, t_1 + \log \frac{r_{\underline{b}^1}^{\underline{\theta}^1}}{r_{\underline{b}^n}^{\underline{\theta}^n}}, \ldots, t_{n-1} + \log \frac{r_{\underline{b}^{n-1}}^{\underline{\theta}^{n-1}}}{r_{\underline{b}^n}^{\underline{\theta}^n}}, v_1 + v_{\underline{b}^1}^{\underline{\theta}^1}, \ldots, v_n + v_{\underline{b}^n}^{\underline{\theta}^n}\right).$$

These are called renormalization operators. They will appear in the statement of the scale recurrence lemma. For $r > 0$, we also define the set

$$J_r = \{(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : |t_j| \le r, 1 \le j \le n-1\} \times \mathbb{T}^n,$$

and denote by ν the Haar measure on J giving measure $(2\pi)^n$ to the set $J_{1/2}$.

2.6. *Scale recurrence lemma.* In this subsection, we state one of the principal results in the paper. This is a multidimensional conformal version of the scale recurrence lemma of Moreira and Yoccoz [7]. The proof is technical and will be left for the end of the paper.

THEOREM A. (Scale recurrence lemma) *Let $K_1, K_2, \ldots, K_n$ be C^m conformal regular Cantor sets with $m \ge 2$. Suppose they are not essentially affine and not essentially real. Denote by d_j the Hausdorff dimension of K_j, $1 \le j \le n$. If r, c_0 are conveniently large, there exist $c_1, c_2, c_3, \rho_0 > 0$ with the following properties: given $0 < \rho < \rho_0$, and a family $F(\underline{a}^1, \ldots, \underline{a}^n)$ of subsets of J_r, $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$, such that*

$$\nu(J_r \setminus F(\underline{a}^1, \ldots, \underline{a}^n)) \le c_1 \quad \textit{for all } (\underline{a}^1, \ldots, \underline{a}^n),$$

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there is another family $F^*(\underline{a}^1, \dots, \underline{a}^n)$ of subsets of J_r satisfying the following properties.

(i) *For any $(\underline{a}^1, \dots, \underline{a}^n)$, $F^*(\underline{a}^1, \dots, \underline{a}^n)$ is contained in the $c_2\rho$-neighborhood of $F(\underline{a}^1, \dots, \underline{a}^n)$.*

(ii) *Let $(\underline{a}^1, \dots, \underline{a}^n) \in \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$, $(t, v) \in F^*(\underline{a}^1, \dots, \underline{a}^n)$; there exist at least $c_3\rho^{-(d_1+d_2+\cdots+d_n)}$ tuples $(\underline{b}^1, \dots, \underline{b}^n) \in \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$ (with $\underline{b}^1, \dots, \underline{b}^n$ starting with the last letter of $\underline{a}^1, \dots, \underline{a}^n$) such that, if $\underline{\theta}^i \in \Sigma_i^-$ ends with $\underline{a}^i$, $i = 1, \dots, n$, and*

$$T_{\underline{b}^1,\dots,\underline{b}^n}(\underline{\theta}^1, \dots, \underline{\theta}^n, t, v) = (\underline{\theta}^1\underline{b}^1, \dots, \underline{\theta}^n\underline{b}^n, \tilde{t}, \tilde{v}),$$

the ρ-neighborhood of $(\tilde{t}, \tilde{v}) \in J$ is contained in $F^*(\underline{b}^1, \dots, \underline{b}^n)$.

(iii) *$\nu(F^*(\underline{a}^1, \dots, \underline{a}^n)) \geq \nu(J_r)/2$ for at least half of the $(\underline{a}^1, \dots, \underline{a}^n) \in \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$.*

3. *Dimension formula*

In this section, we will prove the dimension formula (Theorem B), which is one of the main theorems in the paper. First, we will introduce some notation. Second, we will present the discrete Marstrand property, which will be an important tool. Finally, we give the proof of Theorem B.

The idea of the proof of Theorem B is to use the mass distribution principle in the form of Corollary 2.1. The role of the sets $G(p)$ will be played by sets of the form $h(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n))$. Thus, we will estimate the Hausdorff dimension of a subset of $h(K_1 \times \cdots \times K_n)$ by defining a nested sequence of sets of the form $h(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n))$ verifying the properties in the corollary. The main difficulty with this type of argument is to guarantee property (f). Roughly speaking, we use the Marstrand property to guarantee that for every size ρ, we have a 'positive' proportion of sets $G(\underline{a}^1) \times \cdots \times G(\underline{a}^n)$, with $\underline{a}^j \in \Sigma_j(c_0, \rho)$, such that their images by h are disjoint. This helps us get property (f) at a certain level. To be able to iterate this argument and to have property (f) at all levels, one uses the scale recurrence lemma.

We will use the notation $R(\rho) = \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$, and think of any element $Q = (\underline{a}^1, \dots, \underline{a}^n) \in R(\rho)$ as the set $G(\underline{a}^1) \times \cdots \times G(\underline{a}^n)$. Given a function φ defined on a neighborhood of $G(\underline{a}^1) \times \cdots \times G(\underline{a}^n)$, we write $\varphi(Q)$ to denote the set $\varphi(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n))$.

To each $(t, v) \in J$, we associate the linear map $A(t, v) : \mathbb{C}^n \to \mathbb{C}^n$,

$$A(t_1, \dots, t_{n-1}, v_1, \dots, v_n)(z_1, \dots, z_n) = (e^{t_1+iv_1} \cdot z_1, \dots, e^{t_{n-1}+iv_{n-1}} \cdot z_{n-1}, e^{iv_n} z_n).$$

We consider the composition $\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,t,v} := A(t, v) \circ (k^{\underline{\theta}^1}, \dots, k^{\underline{\theta}^n})$; these maps are related to the renormalization operators by the following equation:

$$\pi_{\underline{\theta}^1,\dots,\underline{\theta}^n,t,v} \circ (f_{\underline{a}^1}, \dots, f_{\underline{a}^n}) = B \circ \pi_{T_{\underline{a}^1,\dots,\underline{a}^n}(\underline{\theta}^1,\dots,\underline{\theta}^n,t,v)}, \tag{2}$$

where $B : \mathbb{C}^n \to \mathbb{C}^n$ is an affine function of the form $B(z) = \alpha \cdot z + \beta$ for $\alpha \in \mathbb{R}$. In fact, this equation is the reason why we defined the renormalization operators as we did.

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One of the main reasons to use limit geometries is that they appear naturally when one considers compositions of a C^1 function with the maps $f_{\underline{a}}$. This is explained in the next lemma.

LEMMA 3.1. *Let h be a C^1 function defined from a neighborhood of $K_1 \times \cdots \times K_n$ into $\mathbb{R}^l$, and $r > 0$. There exists a function $E : (0, \infty) \to \mathbb{R}$, depending only on h, r, and the Cantor sets, such that $\lim_{t\to 0} E(t) = 0$ and with the following property: for any $(\underline{a}^1, \ldots, \underline{a}^n)$ such that*

$$s = \left(\log \frac{r_{\underline{a}^1}}{r_{\underline{a}^n}}, \ldots, \log \frac{r_{\underline{a}^{n-1}}}{r_{\underline{a}^n}}, v_{\underline{a}^1}, \ldots, v_{\underline{a}^n} \right) \in J_r,$$

consider the affine function $L : \mathbb{R}^l \to \mathbb{R}^l$ given by $L(z) = 1/r_{\underline{a}^n}(z - h(c_{\underline{a}^1}, \ldots, c_{\underline{a}^n}))$. Then for any $\underline{\theta}^1, \ldots, \underline{\theta}^n$ ending in $\underline{a}^1, \ldots, \underline{a}^n$, the supremum distance between $L \circ h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})$ and $Dh(c_{\underline{a}^1}, \ldots, c_{\underline{a}^n}) \circ A(s) \circ (k^{\underline{\theta}^1}, \ldots, k^{\underline{\theta}^n})$ is less than $E(\max_{1\le j\le n} r_{\underline{a}^j})$.

This lemma is saying that $h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})$, modulo composition by an affine function on the left, becomes arbitrarily close to a function of the form $B \circ A(t, v) \circ (k^{\underline{\theta}^1}, \ldots, k^{\underline{\theta}^n})$ as the length of the words $\underline{a}^j$ increases.

Proof. Write $h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})$ as

$$[h \circ (\phi_{\underline{a}^1}^{-1}, \ldots, \phi_{\underline{a}^n}^{-1})] \circ (\phi_{\underline{a}^1} \circ f_{\underline{a}^1}, \ldots, \phi_{\underline{a}^n} \circ f_{\underline{a}^n}).$$

Use the fact that $(\phi_{\underline{a}^1} \circ f_{\underline{a}^1}, \ldots, \phi_{\underline{a}^n} \circ f_{\underline{a}^n})$ becomes close to a limit geometry $(k^{\underline{\theta}^1}, \ldots, k^{\underline{\theta}^n})$ and Taylor first-order approximation for $h \circ (\phi_{\underline{a}^1}^{-1}, \ldots, \phi_{\underline{a}^n}^{-1})$. □

3.1. *Discrete Marstrand property.* In this section, we present and prove the discrete Marstrand property. We first state two linear algebra results that we will need.

LEMMA 3.2. *Let $A : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map, $A \neq 0$, and denote by σ the smallest non-zero singular value of A. Then $\operatorname{dist}(x, \ker(A)) \le |Ax|/\sigma$ for all $x \in \mathbb{R}^n$.*

LEMMA 3.3. *Let $E_1, E_2 \subset \mathbb{R}^n$ be linear subspaces such that $E_1 + E_2 = \mathbb{R}^n$. Denote by θ the angle between E_1 and E_2. Define $I = E_1 \cap E_2$, $L_1 = I^\perp \cap E_1$, and $L_2 = I^\perp \cap E_2$. For $x = l_1 + v + l_2$, with $l_1 \in L_1$, $l_2 \in L_2$, $v \in I$, we have*

$$|l_1| \le \frac{\operatorname{dist}(x, E_2)}{\sin\theta} \quad \textit{and} \quad |l_2| \le \frac{\operatorname{dist}(x, E_1)}{\sin\theta}.$$

The next proposition is the main tool that will allow us to obtain the discrete Marstrand property. Given an $\mathbb{R}$-linear map $B : \mathbb{C}^n \to \mathbb{R}^l$, we will say it satisfies the transversality condition if for any set $A \subset \{1, \ldots, n\}$, we have $\dim(Im(B|_{\{z_j=0: j\notin A\}})) = \min\{l, 2 \cdot \#A\}$; in all this subsection, B will denote one such map.

PROPOSITION 3.1. *Let $r > 0$ and $B : \mathbb{C}^n \to \mathbb{R}^l$ a linear map satisfying the transversality condition. There exists a constant C, depending only on r and B, such that for any pair of*

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subsets $Q_1, Q_2 \subset \mathbb{C}^n$, *we have*

$$\nu(\{s \in J_r : B \circ A(s)(Q_1) \cap B \circ A(s)(Q_2) \neq \emptyset\}) \leq C\bigg(\frac{\max\{\mathrm{diam}(Q_1), \mathrm{diam}(Q_2)\}}{\mathrm{dist}(Q_1, Q_2)}\bigg)^l,$$

where $\mathrm{dist}(Q_1, Q_2)$ *denotes the distance between* Q_1 *and* Q_2.

Proof. Throughout the proof, we will use the notation $P = O(Q)$, meaning that there is a constant $\tilde{C}$, depending only on r and B, such that $P \leq \tilde{C} \cdot Q$.

Given a subset $A \subset \{1, 2, \ldots, n\}$, we consider the subspace $\mathbb{C}^A = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_j = 0 \text{ for all } j \notin A\}$. By the transversality condition, we can choose $\theta > 0$, only depending on B, such that the angle between $\ker(B)$ and $\mathbb{C}^A$ is bigger than θ for any non-empty subset A.

Denote $\max\{\mathrm{diam}(Q_1), \mathrm{diam}(Q_2)\}$ by ρ. Fix $c_1, c_2 \in \mathbb{C}^n$ such that

$$\mathrm{dist}(c_j, Q_j) < \rho, \quad \mathrm{dist}(Q_1, Q_2)/2 \leq |c_2 - c_1| \leq 2\,\mathrm{dist}(Q_1, Q_2)$$

and $c_2 - c_1$ has all its coordinates in $\mathbb{C}^n$ different from zero. Suppose that $s \in J_r$ is such that $BA(s)(Q_1) \cap BA(s)(Q_2) \neq \emptyset$. Then there are $\tilde{c}_1 \in Q_1$, $\tilde{c}_2 \in Q_2$ verifying $B \circ A(s)(\tilde{c}_1) = B \circ A(s)(\tilde{c}_2)$. We conclude that

$$|B \circ A(s)(c_2 - c_1)| = O(\rho).$$

Define $x = (c_2 - c_1)/|c_2 - c_1|$, and hence $|B \circ A(s)(x)| = O(\rho/\mathrm{dist}(Q_1, Q_2))$. By the first linear algebra lemma, we get that $\mathrm{dist}(A(s)(x), \ker(B)) = O(\rho/\mathrm{dist}(Q_1, Q_2))$. Up until now, we have proven that there is a constant C_1, depending only on r and B, such that

$$\mathrm{dist}(A(s)(x), \ker(B)) \leq C_1 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)}.$$

Notice that if $\rho/\mathrm{dist}(Q_1, Q_2) \geq (e^{-r} \sin\theta)/4C_1$, then the proposition follows taking $C = (2r)^{n-1}(2\pi)^n(4C_1/e^{-r} \sin\theta)^l$. This is thanks to the fact that

$$\nu(\{s \in J_r : B \circ A(s)(Q_1) \cap B \circ A(s)(Q_2) \neq \emptyset\}) \leq (2r)^{n-1}(2\pi)^n.$$

For the rest of the proof, we suppose $\rho/\mathrm{dist}(Q_1, Q_2) < (e^{-r} \sin\theta)/4C_1$. Define $a = (e^{-r} \sin\theta)/4n$ and write $x \in \mathbb{C}^n$ as $x = (e^{\chi_1 + i\phi_1}, \ldots, e^{\chi_n + i\phi_n})$. Consider the set

$$A = \{j \in \{1, \ldots, n\} : e^{\chi_j} \geq ae^{-r}\},$$

and the subspace $\mathbb{C}^A$; we will see that $\dim_{\mathbb{R}}\mathbb{C}^A \geq l$. Let $u \in \mathbb{C}^A$ be the orthogonal projection of $A(s)(x)$ in $\mathbb{C}^A$. By the definition of $\mathbb{C}^A$, we have $|A(s)x - u| < na$. Given that $\|A(s)\| \geq e^{-r}$ (remember that $s \in J_r$) and $na < e^{-r}/2$, we have

$$|u| \geq |A(s)(x)| - |A(s)x - u| > e^{-r} - n \cdot a > \tfrac{1}{2}e^{-r}.$$

If $\dim_{\mathbb{R}}\mathbb{C}^A < l$, the transversality implies $\mathbb{C}^A \cap \ker(B) = \{0\}$ and then, by the choice of θ, we would get $\mathrm{dist}(u, \ker(B)) \geq |u| \sin\theta \geq (e^{-r} \sin\theta)/2$. However, this is not possible since

$$\begin{aligned}\mathrm{dist}(u, \ker(B)) &\leq |u - A(s)(x)| + \mathrm{dist}(A(s)(x), \ker(B)) \\ &\leq na + C_1 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)} < \frac{e^{-r} \sin\theta}{2}.\end{aligned}$$

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Given that $\dim_{\mathbb{R}}\mathbb{C}^A \geq l$, the transversality condition implies $\ker(B) + \mathbb{C}^A = \mathbb{C}^n$. Let $L = \mathbb{C}^A \cap (\ker(B) \cap \mathbb{C}^A)^\perp$. Define the $\mathbb{R}$-linear function $\hat{x} : \mathbb{C}^n \to \mathbb{C}^n$ given by $\hat{x}(z_1, \ldots, z_n) = (e^{\chi_1 + i\phi_1} \cdot z_1, \ldots, e^{\chi_n + i\phi_n} \cdot z_n)$. Notice that $A(s)(x) = \hat{x}([s])$, where $[s] = (e^{t_1 + iv_1}, \ldots, e^{t_{n-1} + iv_{n-1}}, e^{iv_n})$. Write $\hat{x}([s]) = b_1 + b_2$, where $b_1 \in L$, $b_2 \in \ker(B)$. The second lemma in linear algebra implies

$$|b_1| \leq \frac{\mathrm{dist}(b_1, \ker(B))}{\sin\theta} = \frac{\mathrm{dist}(\hat{x}([s]), \ker(B))}{\sin\theta} = O\bigg(\frac{\rho}{\mathrm{dist}(Q_1, Q_2)}\bigg).$$

Given that $b_1 \in \mathbb{C}^A$, we get that $|\hat{x}^{-1}(b_1)| = O(\rho/\mathrm{dist}(Q_1, Q_2))$. Therefore, $[s] = \hat{x}^{-1}(b_1) + \hat{x}^{-1}(b_2)$ implies

$$\mathrm{dist}([s], \hat{x}^{-1}(\ker(B))) = O\bigg(\frac{\rho}{\mathrm{dist}(Q_1, Q_2)}\bigg).$$

This last inequality tells us that the vector $[s]$ is close to a $2n - l$ subspace. Moreover, the last coordinate of this vector has modulus 1. This two properties will allow us to obtain the desired estimate.

Consider the set

$$H = \{(z_1, \ldots, z_n) \in \hat{x}^{-1}(\ker(B)) : \ |z_n| = 1, \ |z_j| \in [e^{-2r}, e^{2r}], \ j = 1, \ldots, n-1\}.$$

We have proven that there is a constant $C_2 > 0$, depending only on B and r, such that $\mathrm{dist}([s], \hat{x}^{-1}(\ker(B))) \leq C_2 \cdot \rho/\mathrm{dist}(Q_1, Q_2)$. Thus, there is a constant $C_3 > 0$, depending only on B and r, such that

$$\mathrm{dist}([s], H) \leq C_3 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)}.$$

In fact, let $u = (u_1, \ldots, u_n) \in \hat{x}^{-1}(\ker(B))$ such that $|u - [s]| \leq C_2 \cdot \rho/\mathrm{dist}(Q_1, Q_2)$. We have

$$1 - C_2 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)} \leq |u_n| \leq 1 + C_2 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)},$$

and

$$e^{-r} - C_2 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)} \leq |u_j| \leq e^r + C_2 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)}$$

for $j = 1, \ldots, n-1$. Therefore, if $\rho/\mathrm{dist}(Q_1, Q_2)$ is small enough, one has that $u/|u_n|$ is in H and $|[s] - (u/|u_n|)| = O(\rho/\mathrm{dist}(Q_1, Q_2))$. If $\rho/\mathrm{dist}(Q_1, Q_2)$ is big, the proposition follows choosing C properly, as it was done before when we considered the case $\rho/\mathrm{dist}(Q_1, Q_2) \geq (e^{-r}\sin\theta)/4C_1$.

Define the function

$$\varphi : \bigg\{(z_1, \ldots, z_n) \in \mathbb{S}^{2n-1} \cap \hat{x}^{-1}(\ker(B)) : \ \frac{|z_j|}{|z_n|} \in [e^{-2r}, e^{2r}], \ j = 1, \ldots, n-1\bigg\} \to H,$$

given by $\varphi(z_1, \ldots, z_n) = (z_1/|z_n|, \ldots, z_n/|z_n|)$. Notice that φ is surjective and smooth. Moreover, one has that $\|D\varphi\|$ is bounded by a constant depending only on r and n. Since the domain of φ is contained in a $(2n - l - 1)$-dimensional unit sphere inside $\hat{x}^{-1}(\ker(B))$, there exist $w_1, \ldots, w_p \in H$ such that H is covered by the balls

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$B_{\rho/\mathrm{dist}(Q_1,Q_2)}(w_j)$, $j = 1, \ldots, p$, and

$$p = O\left(\left(\frac{\rho}{\mathrm{dist}(Q_1, Q_2)}\right)^{-(2n-l-1)}\right).$$

We conclude that $[s] \in \bigcup_{j=1}^{p} B_{C_4\cdot\rho/\mathrm{dist}(Q_1,Q_2)}(w_j)$, for a constant $C_4 > 0$ depending only on r and B.

Writing w_j as $w_j = (w_{j,1}, \ldots, w_{j,n})$, we obtain that for some $1 \le j \le p$,

$$|e^{iv_n} - w_{j,n}| \le C_4 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)},$$

$$|e^{t_q+iv_q} - w_{j,q}| \le C_4 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)},$$

$q = 1, \ldots, n-1$. Notice that $|w_{j,q}|$ is bounded below by e^{-2r}, and hence

$$|t_q - \log|w_{j,q}|| \le C_5 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)}, \quad \|v_q - \arg(w_{j,q})\| \le C_5 \cdot \frac{\rho}{\mathrm{dist}(Q_1, Q_2)},$$

$q = 1, \ldots, n$, where C_5 is a constant depending only on r and B, and $\arg(w_{j,q}) \in \mathbb{T}$ is the argument of $w_{j,q}$. This implies that the set $\{s \in J_r : B \circ A(s)(Q_1) \cap B \circ A(s)(Q_2) \neq \emptyset\}$ is contained in the union of p sets, each one with a ν-volume of order $O((\rho/\mathrm{dist}(Q_1, Q_2))^{2n-1})$. Finally, using the order of p, we conclude that

$$\nu(\{s \in J_r : B \circ A(s)(Q_1) \cap B \circ A(s)(Q_2) \neq \emptyset\}) = O\left(\left(\frac{\rho}{\mathrm{dist}(Q_1, Q_2)}\right)^{(2n-1)-(2n-l-1)}\right),$$

as we wanted. □

Proposition 3.1 implies that there is a constant $C > 0$ such that for any $\underline{\theta}^1, \ldots, \underline{\theta}^n$ and $Q_1, Q_2 \in R(\rho)$, we have

$$\nu(\{s \in J_r : B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_1) \cap B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_2) \neq \emptyset\}) \le C\left(\frac{\rho}{\mathrm{dist}(Q_1, Q_2)}\right)^l.$$

The constant $C > 0$ depends on B, r, c_0, and the Cantor sets $K_1, \ldots, K_n$, but it is independent of ρ. For the next proposition, we will also need the following fact: if p is big enough, then for any $Q_1 \in R(\rho)$ and $a \in \mathbb{Z}$, we have

$$\#\{Q_2 \in R(\rho) : p^{-a} \le \mathrm{dist}(Q_1, Q_2) < p^{-a+1}\} = O((p^{-a})^{d_1+\cdots+d_n}\rho^{-(d_1+\cdots+d_n)}),$$

where $d_j = HD(K_j)$, $j = 1, \ldots, n$. For a proof, see [13, Lemma 1.2.3].

PROPOSITION 3.2. *Assume $d_1 + \cdots + d_n < l$. Let*

$$N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) = \#\{(Q_1, Q_2) \in R(\rho)^2 : B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_1) \cap B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_2) \neq \emptyset\}.$$

Then for any $\underline{\theta}^1, \ldots, \underline{\theta}^n$, we have $\int_{J_r} N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)ds = O(\rho^{-(d_1+\cdots+d_j)})$, and the constant in the O notation is independent of $\underline{\theta}^1, \ldots, \underline{\theta}^n$.

Proof. Since the Cantor sets $K_1, \ldots, K_n$ have bounded geometries, then there is a constant C_1, independent of ρ, such that $\mathrm{dist}(Q_1, Q_2) \ge C_1\rho$ for any $Q_1, Q_2 \in R(\rho)$,

$Q_1 \cap Q_2 = \emptyset$. Let $k_0 \in \mathbb{Z}$ such that $p^{-k_0} \le C_1\rho < p^{-k_0+1}$. Using the previous lemma, we have

$$\begin{aligned}
&\int_{J_r} N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)\, ds \\
&\quad = \sum_{Q_1, Q_2 \in R(\rho)} \nu(\{s \in J_r : B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_1) \cap B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_2) \neq \emptyset\}) \\
&\quad = \sum_{Q_1 \in R(\rho)} \sum_{k=-\infty}^{k_0} \sum_{\mathrm{dist}(Q_1,Q_2) \in [p^{-k}, p^{-k+1})} O(\rho^l/[\mathrm{dist}(Q_1, Q_2)]^l) \\
&\qquad + \sum_{Q_1 \in R(\rho)} \sum_{Q_2 \cap Q_1 \neq \emptyset} (2r)^{n-1}.
\end{aligned}$$

Clearly, $\sum_{Q_1 \in R(\rho)} \sum_{Q_2 \cap Q_1 \neq \emptyset} (2r)^{n-1} = O(\#R(\rho)) = O(\rho^{-(d_1+\cdots+d_n)})$. However,

$$\begin{aligned}
&\sum_{Q_1 \in R(\rho)} \sum_{k=-\infty}^{k_0} \sum_{\mathrm{dist}(Q_1,Q_2) \in [p^{-k}, p^{-k+1})} O(\rho^l/[\mathrm{dist}(Q_1, Q_2)]^l) \\
&\quad = \sum_{Q_1 \in R(\rho)} \sum_{k=-\infty}^{k_0} O((p^k)^{l-(d_1+\cdots+d_n)} \rho^{l-(d_1+\cdots+d_n)}) \\
&\quad = \sum_{Q_1 \in R(\rho)} O((p^{k_0})^{l-(d_1+\cdots+d_n)} \rho^{l-(d_1+\cdots+d_n)}) \sum_{k=-\infty}^{0} (p^{l-(d_1+\cdots+d_n)})^k \\
&\quad = O(\rho^{-(d_1+\cdots+d_n)} \rho^{-l+(d_1+\cdots+d_l)} \rho^{l-(d_1+\cdots+d_n)}) = O(\rho^{-(d_1+\cdots+d_n)}). \qquad \square
\end{aligned}$$

PROPOSITION 3.3. *Let* $b > 0$, $F \subset R(\rho)$ *such that* $\#F \ge b\rho^{-(d_1+\cdots+d_n)}$. *Let* $(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)$ *such that* $N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \le a\rho^{-(d_1+\cdots+d_n)}$, *then there exists a subset* $T \subset F$ *with the properties that* $B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_1) \cap B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_2) = \emptyset$ *for all* $Q_1, Q_2 \in T$, $Q_1 \neq Q_2$, *and* $\#T \ge (b^2/4a)\rho^{-(d_1+\cdots+d_n)}$.

Proof. For $Q_0 \in F$, define

$$n(Q_0) = \#\{Q \in F : B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q) \cap B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_0) \neq \emptyset\}.$$

We have $\sum_{Q \in F} n(Q) \le N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \le a\rho^{-(d_1+\cdots+d_n)}$. Therefore, the set $T_0 = \{Q \in F : n(Q) \le 2a/b\}$ has at least $(1/2)\#F$ elements. Finally, it is clear that from T_0, we can extract a subset T with at least $(1/(2a/b))\#T_0$ elements and such that $B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_1) \cap B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_2) = \emptyset$ for any $Q_1, Q_2 \in T$. For this set, we have

$$\#T \ge \frac{1}{2a/b}\#T_0 \ge \frac{1}{2a/b}\frac{1}{2}b\rho^{-(d_1+\cdots+d_n)} = \frac{b^2}{4a}\rho^{-(d_1+\cdots+d_n)}. \qquad \square$$

Notice that since $\int_{J_r} N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)\, ds = O(\rho^{-(d_1+\cdots+d_m)})$, then choosing a big enough, we can guarantee that the set

$$\{s \in J_r : N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) > a\rho^{-(d_1+\cdots+d_n)}\}$$

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has measure as small as we want. Thus, for every $(\underline{\theta}^1, \ldots, \underline{\theta}^n)$, we have that most of the $s \in J_r$ verify the property of the last proposition, that is, for any family $F \subset R(\rho)$ with $\#F \geq b\rho^{-(d_1+\cdots+d_n)}$, there exist a positive proportion of F, $T \subset F$ with $\#T \geq (b^2/(4a))\rho^{-(d_1+\cdots+d_n)}$, such that elements of T project to $\mathbb{R}^l$, in the direction of s, to disjoint sets: $B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_1) \cap B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q_2) = \emptyset$ for all $Q_1, Q_2 \in T$. This is what we call the *discrete Marstrand property*.

The next lemma guarantees that the property of Proposition 3.3 still holds for small perturbations of $B \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}$; it is inspired in the presentation given by Shmerkin [**12**].

LEMMA 3.4. *Let $T \subset R(\rho)$, ϕ a function defined on a neighborhood of $\bigcup_{Q \in T} Q$ into $\mathbb{R}^l$, and $L, \tau > 0$ real numbers. Suppose that for each $Q \in T$, we have $B_{L^{-1}\rho}(c_Q) \subset \phi(Q) \subset B_{L\rho}(c_Q)$ for some $c_Q \in \mathbb{R}^l$, and $\phi(Q_1) \cap \phi(Q_2) = \emptyset$ for all $Q_1, Q_2 \in T$, $Q_1 \neq Q_2$. Then for any ψ defined on a neighborhood of $\bigcup_{Q \in T} Q$, such that the supremum norm $\|\phi - \psi\| < \tau\rho$, there exists $T' \subset T$ such that*

$$\#T' \geq [3L(L+\tau)]^{-l} \cdot \#T$$

and $\psi(Q_1) \cap \psi(Q_2) = \emptyset$ for all $Q_1, Q_2 \in T'$, $Q_1 \neq Q_2$.

Proof. Here, $\|\psi - \phi\| < \tau\rho$ implies $\psi(Q) \subset B_{(L+\tau)\rho}(c_Q)$. Use the Vitali covering lemma for the family $\{B_{(L+\tau)\rho}(c_Q) : Q \in T\}$. We get $T' \subset T$ such that $\{B_{(L+\tau)\rho}(c_Q) : Q \in T'\}$ is a pairwise disjoint family and $\bigcup_{Q \in T} B_{(L+\tau)\rho}(c_Q) \subset \bigcup_{Q \in T'} B_{3(L+\tau)\rho}(c_Q)$. From this, we get

$$\begin{aligned}\#T' \cdot [3(L+\tau)\rho]^l w_l &\geq \mathrm{Vol}\bigg(\bigcup_{Q \in T} B_{(L+\tau)\rho}(c_Q)\bigg) \\ &\geq \mathrm{Vol}\bigg(\bigcup_{Q \in T} B_{L^{-1}\rho}(c_Q)\bigg) \\ &= \#T \cdot L^{-l}\rho^l w_l,\end{aligned}$$

where w_l is the volume of the l-dimensional unitary ball. Hence,

$$\#T' \geq [3L(L+\tau)]^{-l} \cdot \#T. \qquad \square$$

3.2. *Proof of the dimension formula.* In this subsection, we prove the desired dimension formula (Theorem B). Assume we have $K_1, \ldots, K_n$ satisfying the hypothesis of the scale recurrence lemma. We start by using the discrete Marstrand property and the scale recurrence lemma to obtain, for each limit geometry $(\underline{\theta}^1, \ldots, \underline{\theta}^n)$, a set of 'good' directions to project. Fix $c_0, r > 0$ big enough, and let c_1, c_2, c_3, ρ_0 be the constants given by the scale recurrence lemma. Suppose that h is a C^1 function defined from a neighborhood of $K_1 \times \cdots \times K_n$ into $\mathbb{R}^l$ such that there exists a point $x_0 \in K_1 \times \cdots \times K_n$, where $B = Dh(x_0)$ verifies the transversality hypotheses. By the results in §3.1, we can fix $a > 0$ big enough such that

$$\nu(J_r \setminus \{s \in J_r : N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \leq a\rho^{-(d_1+\cdots+d_n)}\}) < c_1$$

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for all $(\underline{\theta}^1, \ldots, \underline{\theta}^n)$. Define

$$F(\underline{\theta}^1, \ldots, \underline{\theta}^n) = \{s \in J_r : N_\rho(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \leq a\rho^{-(d_1+\cdots+d_n)}\},$$

and for $(\underline{a}^1, \ldots, \underline{a}^n) \in \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$,

$$F(\underline{a}^1, \ldots, \underline{a}^n) = \bigcup_{\underline{\theta}^1,\ldots,\underline{\theta}^n} F(\underline{\theta}^1, \ldots, \underline{\theta}^n),$$

where the union is over all $\underline{\theta}^1, \ldots, \underline{\theta}^n$ ending in $\underline{a}^1, \ldots, \underline{a}^n$, respectively. We clearly have $\nu(J_r \setminus F(\underline{a}^1, \ldots, \underline{a}^n)) < c_1$, and thus we can apply the scale recurrence lemma (from now on, we assume $\rho < \rho_0$) to obtain sets $F^*(\underline{a}^1, \ldots, \underline{a}^n)$ with the three properties in Theorem A (see §2.6).

THEOREM 3.1. *Suppose that $d_1 + \cdots + d_n < l$ and for any $(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in \Sigma_1^- \times \cdots \times \Sigma_n^- \times J_r$, there exists $(\underline{c}^1, \ldots, \underline{c}^n) \in \Sigma_1^{fin} \times \cdots \times \Sigma_n^{fin}$ such that $T_{\underline{c}^1,\ldots,\underline{c}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) = (\underline{\theta}^1\underline{c}^1, \ldots, \underline{\theta}^n\underline{c}^n, \tilde{s})$ and $\tilde{s} \in F^*(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n)$ for some $(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n)$ for which $(\underline{\theta}^1\underline{c}^1, \ldots, \underline{\theta}^n\underline{c}^n)$ ends in it. Then $HD(h(K_1 \times \cdots \times K_n)) = d_1 + \cdots + d_n$.*

Proof. Since h is Lipschitz in a neighborhood of $K_1 \times \cdots \times K_n$ and $HD(K_1 \times \cdots \times K_n) = d_1 + \cdots + d_n$, we have $HD(h(K_1 \times \cdots \times K_n)) \leq d_1 + \cdots + d_n$. Thus, we only need to show $HD(h(K_1 \times \cdots \times K_n)) \geq d_1 + \cdots + d_n$. Let $\eta > 0$ arbitrary, we will prove that $HD(h(K_1 \times \cdots \times K_n)) \geq d_1 + \cdots + d_n - \eta$, this will finish the proof of the theorem.

Since $\Sigma_1^- \times \cdots \times \Sigma_n^- \times J_r$ is compact and $\pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}$ depends continuously in $(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)$, then we can choose a constant $L > 0$ (only depending on r, $K_1, \ldots, K_n$, and $Dh(x_0)$) such that

$$B_{L^{-1}\rho}(c_Q) \subset Dh(x_0) \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}(Q) \subset B_{L\rho}(c_Q) \quad \text{for all } Q \in R(\rho), \tag{3}$$

for some $c_Q \in \mathbb{R}^l$, which depends on $\underline{\theta}^1, \ldots, \underline{\theta}^n$, s and Q.

Choose $\tau > 0$ big enough such that

$$\|Dh(x_0) \circ \pi_{\underline{\tilde{\theta}}^1,\ldots,\underline{\tilde{\theta}}^n,\tilde{s}} - Dh(x_0) \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}\| \leq \tfrac{1}{3}\tau\rho \tag{4}$$

for all $\underline{\theta}^j, \underline{\tilde{\theta}}^j \in \Sigma_j^-$, $j = 1, \ldots, n$, $s, \tilde{s} \in J_r$ with $|s - \tilde{s}| < c_2\rho$, $\underline{\theta}^j \wedge \underline{\tilde{\theta}}^j \in \Sigma_j(c_0, \rho)$, $j = 1, \ldots, n$.

Until now, all the statements where ρ appeared were true for any value small enough. For the rest of the proof, we are going to fix a particular value, which we call ρ_1 to distinguish it from the 'variable' ρ. It is chosen such that $\rho_1^\eta \leq C_3[3L(L + \tau)]^{-l}c_3^2/4a$, where $C_3 > 0$ is a constant, independent of ρ, which will be fixed later (equation (5)).

We can choose $\underline{a}_0^1, \ldots, \underline{a}_0^n$ and $\underline{\theta}_0^1, \ldots, \underline{\theta}_0^n$ ending in it, respectively, with the following properties.

(a) The element $s_0 = (\log(r_{\underline{a}_0^1}/r_{\underline{a}_0^n}), \ldots, \log(r_{\underline{a}_0^{n-1}}/r_{\underline{a}_0^n}), v_{\underline{a}_0^1}, \ldots, v_{\underline{a}_0^n})$ is in J_r.

(b) $x_0 \in G(\underline{a}_0^1) \times \cdots \times G(\underline{a}_0^n)$.

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(c) For any $\underline{b}^1, \ldots, \underline{b}^n$, consider $\tilde{s} = (\log(r_{\underline{a}_0^1\underline{b}^1}/r_{\underline{a}_0^n\underline{b}^n}), \ldots, \log(r_{\underline{a}_0^{n-1}\underline{b}^{n-1}}/r_{\underline{a}_0^n\underline{b}^n}), v_{\underline{a}_0^1\underline{b}^1}, \ldots, v_{\underline{a}_0^n\underline{b}^n})$. Then

$$\|Dh(c_{\underline{a}_0^1\underline{b}^1}, \ldots, c_{\underline{a}_0^n\underline{b}^n}) \circ \pi_{\underline{\theta}_0^1\underline{b}^1,\ldots,\underline{\theta}_0^n\underline{b}^n,\tilde{s}} - Dh(x_0) \circ \pi_{T_{\underline{b}^1,\ldots,\underline{b}^n}(\underline{\theta}_0^1,\ldots,\underline{\theta}_0^n,s)}\| \leq \tfrac{1}{3}\tau\rho_1.$$

This is achieved by choosing very long words for $\underline{a}_0^1, \ldots, \underline{a}_0^n$. If the number of symbols in $\underline{a}_0^j$, for $1 \leq j \leq n$, increases, then $|x_0 - (c_{\underline{a}_0^1\underline{b}^1}, \ldots, c_{\underline{a}_0^n\underline{b}^n})|$ goes to zero; the same happens for the distance between the last coordinate of $T_{\underline{b}^1,\ldots,\underline{b}^n}(\underline{\theta}_0^1, \ldots, \underline{\theta}_0^n, s)$ and $\tilde{s}$. This a consequence of item (b) and the fact that $r_{\underline{a}_0^j} r_{\underline{b}^j}^{\underline{\theta}_0^j}/r_{\underline{a}_0^j\underline{b}^j} \to 1$, $\|v_{\underline{a}_0^j\underline{b}^j} - (v_{\underline{a}_0^j} + v_{\underline{b}^j}^{\underline{\theta}_0^j})\| \to 0$ as the number of symbols in $\underline{a}_0^j$ goes to infinity.

(d) For any $\underline{b}^1, \ldots, \underline{b}^n$ such that $\tilde{s} \in J_r$, there is an affine function L such that

$$\|L \circ h \circ (f_{\underline{a}_0^1\underline{b}^1}, \ldots, f_{\underline{a}_0^n\underline{b}^n}) - Dh(c_{\underline{a}_0^1\underline{b}^1}, \ldots, c_{\underline{a}_0^n\underline{b}^n}) \circ \pi_{\underline{\theta}_0^1\underline{b}^1,\ldots,\underline{\theta}_0^n\underline{b}^n,\tilde{s}}\| \leq \tfrac{1}{3}\tau\rho_1.$$

This is a consequence of Lemma 3.1.

By hypothesis, there exists $(\underline{c}^1, \ldots, \underline{c}^n) \in \Sigma_1^{fin} \times \cdots \times \Sigma_n^{fin}$ such that

$$T_{\underline{c}^1,\ldots,\underline{c}^n}(\underline{\theta}_0^1, \ldots, \underline{\theta}_0^n, s_0) = (\underline{\theta}_0^1\underline{c}^1, \ldots, \underline{\theta}_0^n\underline{c}^n, \tilde{s}_0)$$

and $\tilde{s}_0 \in F^*(\tilde{\underline{a}}_0^1, \ldots, \tilde{\underline{a}}_0^n)$ for some $(\tilde{\underline{a}}_0^1, \ldots, \tilde{\underline{a}}_0^n)$ for which $(\underline{\theta}_0^1\underline{c}^1, \ldots, \underline{\theta}_0^n\underline{c}^n)$ ends in it.

We will define inductively a set $N \subset \Sigma_1^{fin} \times \cdots \times \Sigma_n^{fin} \times \Sigma_1^- \times \cdots \times \Sigma_n^- \times J_r$. Every $p = (\underline{a}^1, \ldots, \underline{a}^n, \underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in N$ should verify:

(i) $s \in F^*(\tilde{\underline{a}}^1, \ldots, \tilde{\underline{a}}^n)$ for some $(\tilde{\underline{a}}^1, \ldots, \tilde{\underline{a}}^n)$ such that $(\underline{\theta}^1, \ldots, \underline{\theta}^n)$ ends in $(\tilde{\underline{a}}^1, \ldots, \tilde{\underline{a}}^n)$;

(ii) $(\underline{a}^1, \ldots, \underline{a}^n, \underline{\theta}^1, \ldots, \underline{\theta}^n, s) = (\underline{a}_0^1\underline{b}^1, \ldots, \underline{a}_0^n\underline{b}^n, T_{\underline{b}^1,\ldots,\underline{b}^n}(\underline{\theta}_0^1, \ldots, \underline{\theta}_0^n, s_0))$ for some $(\underline{b}^1, \ldots, \underline{b}^n) \in \Sigma_1^{fin} \times \cdots \times \Sigma_n^{fin}$.

For $p = (\underline{a}^1, \ldots, \underline{a}^n, \underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in N$, we will define a set $T'(p) \subset R(\rho_1)$ verifying:

(iii) $\#T'(p) \geq C_3^{-1}\rho_1^{\eta-(d_1+\cdots+d_n)}$;

(iv) $h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})(Q_1) \cap h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})(Q_2) = \emptyset$ for all $Q_1, Q_2 \in T'(p)$, $Q_1 \neq Q_2$;

(v) for all $(\underline{b}^1, \ldots, \underline{b}^n) \in T'(p)$, we have $T_{\underline{b}^1,\ldots,\underline{b}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) = (\underline{\theta}^1\underline{b}^1, \ldots, \underline{\theta}^n\underline{b}^n, \tilde{s})$ and $\tilde{s} \in F^*(\underline{b}^1, \ldots, \underline{b}^n)$.

Elements of N are defined inductively, that is, every element already defined as $p \in N$ generates new elements, which we call the children of p and denote by $Ch(p)$. Thus, N has the structure of a rooted tree. The root of the tree is $p_0 = (\underline{a}_0^1\underline{c}^1, \ldots, \underline{a}_0^n\underline{c}^n, \underline{\theta}_0^1\underline{c}^1, \ldots, \underline{\theta}_0^n\underline{c}^n, \tilde{s}_0)$, the set $T'(p_0)$ is defined as described below.

Given $p = (\underline{a}^1, \ldots, \underline{a}^n, \underline{\theta}^1, \ldots, \underline{\theta}^n, s)$ verifying items (i), (ii) (as p_0 does), define $T'(p)$ in the following way.

By item (i), we know that $s \in F^*(\tilde{\underline{a}}^1, \ldots, \tilde{\underline{a}}^n)$, and hence the scale recurrence lemma implies that there exists a set $F \subset R(\rho_1)$ with $\#F \geq c_3\rho_1^{-(d_1+\cdots+d_n)}$ and such that item (v) holds for F.

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Since $F^*(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n) \subset V_{c_2\rho_1}(F(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n))$, then there exist $s' \in F(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n)$ with $|s - s'| \leq c_2\rho_1$. By the definition of $F(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n)$, we have $N_{\rho_1}(\underline{\tilde{\theta}}^1, \ldots, \underline{\tilde{\theta}}^n, s') \leq a\rho_1^{-(d_1+\cdots+d_n)}$ for some $(\underline{\tilde{\theta}}^1, \ldots, \underline{\tilde{\theta}}^n)$ that ends in $(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n)$.

Using Proposition 3.3, we obtain a set $T \subset F$ such that

$$Dh(x_0) \circ \pi_{\underline{\tilde{\theta}}^1,\ldots,\underline{\tilde{\theta}}^n,s'}(Q_1) \cap Dh(x_0) \circ \pi_{\underline{\tilde{\theta}}^1,\ldots,\underline{\tilde{\theta}}^n,s'}(Q_2) = \emptyset$$
$$\text{for all } Q_1, Q_2 \in T, \quad Q_1 \neq Q_2,$$

and $\#T \geq (c_3^2/(4a))\rho_1^{-(d_1+\cdots+d_n)}$.

We want to use Lemma 3.4 for $\phi = Dh(x_0) \circ \pi_{\underline{\tilde{\theta}}^1,\ldots,\underline{\tilde{\theta}}^n,s'}$, $\psi = L \circ h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})$, where L is some affine function, and the set T. Note first that $\underline{\theta}^j \wedge \underline{\tilde{\theta}}^j \in \Sigma_j(c_0, \rho_1)$, $1 \leq j \leq n$, since both $\underline{\theta}^j$ and $\underline{\tilde{\theta}}^j$ end in $\underline{\tilde{a}}^j$. Equation (4) implies then

$$\|Dh(x_0) \circ \pi_{\underline{\tilde{\theta}}^1,\ldots,\underline{\tilde{\theta}}^n,s'} - Dh(x_0) \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}\| \leq \tfrac{1}{3}\tau\rho_1.$$

However, item (ii) together with items (c) and (d) imply

$$\|L \circ h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n}) - Dh(x_0) \circ \pi_{\underline{\theta}^1,\ldots,\underline{\theta}^n,s}\| \leq \tfrac{2}{3}\tau\rho_1.$$

We conclude

$$\|L \circ h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n}) - Dh(x_0) \circ \pi_{\underline{\tilde{\theta}}^1,\ldots,\underline{\tilde{\theta}}^n,s'}\| \leq \tau\rho_1,$$

which together with equation (3) shows that we can use Lemma 3.4. Hence, there is a subset $T'(p) \subset T \subset F$ such that

$$h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})(Q_1) \cap h \circ (f_{\underline{a}^1}, \ldots, f_{\underline{a}^n})(Q_1) = \emptyset$$
$$\text{for all } Q_1, Q_2 \in T'(p), Q_1 \neq Q_2,$$

and

$$\#T'(p) \geq [3L(L+\tau)]^{-l} \cdot \frac{c_3^2}{4a}\rho_1^{-(d_1+\cdots+d_n)} \geq C_3^{-1}\rho_1^{\eta-(d_1+\cdots+d_n)}.$$

In the way we have defined $T'(p)$, it clearly verifies items (iii), (iv), and (v).

Given $p = (\underline{a}^1, \ldots, \underline{a}^n, \underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in N$, the children of p are defined by

$$Ch(p) = \{(\underline{a}^1\underline{b}^1, \ldots, \underline{a}^n\underline{b}^n, T_{\underline{b}^1,\ldots,\underline{b}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s)) : (\underline{b}^1, \ldots, \underline{b}^n) \in T'(p)\}.$$

The children of p clearly satisfy items (i) and (ii).

Now that we have defined N, we can finish the proof. For each non-negative integer k, consider the set $I(k)$ of elements $p \in N$ generated in the k-step of the inductive process. (They are children of elements generated in the $(k-1)$-step, and the only element in the 0-step is p_0.) For each $p = (\underline{a}^1, \ldots, \underline{a}^n, \underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in N$, define the set

$$G(p) = h(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n)) \subset \mathbb{R}^l.$$

We clearly have

$$\bigcap_{k\geq 0} \bigcup_{p\in I(k)} G(p) \subset h(K_1 \times \cdots \times K_n).$$

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The desired result, $HD(h(K_1 \times \cdots \times K_n)) \geq d_1 + \cdots + d_n - \eta$, follows from Corollary 2.1 if we can prove that

$$\sum_{q \in Ch(p)} \left(\frac{\operatorname{diam}(G(q))}{\operatorname{diam}(G(p))}\right)^{d_1+\cdots+d_n-\eta} \geq 1,$$

and each set $G(p)$ contains a ball with radius proportional to its diameter. All other requirements in the corollary are obviously verified.

Given $p = (\underline{a}^1, \ldots, \underline{a}^n, \underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in N$, properties (i), (ii) and the observation in property (c) imply that $s \in J_r$ and $s(\underline{a}^1, \ldots, \underline{a}^n) := (\log(r_{\underline{a}^1}/r_{\underline{a}^n}), \ldots, \log(r_{\underline{a}^{n-1}}/r_{\underline{a}^n}), v_{\underline{a}^1}, \ldots, v_{\underline{a}^n})$ is very close to s. Thus, we can assume $s(\underline{a}^1, \ldots, \underline{a}^n) \in J_{2r}$ and then we can think of $G(\underline{a}^1) \times \cdots \times G(\underline{a}^n)$ as being a square; indeed we have $\operatorname{diam}(G(\underline{a}^j)) \leq e^{4r}\operatorname{diam}(G(\underline{a}^m))$ for any j, m. This together with the fact that $Dh(x_0)$ verify that the transversality hypotheses allow us to conclude that $h(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n))$ contains a ball with radius proportional to its diameter. Moreover,

$$\begin{aligned}(C_3')^{-1}\operatorname{diam}(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n)) &\leq \operatorname{diam}(h(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n))) \\ &\leq C_3'\operatorname{diam}(G(\underline{a}^1) \times \cdots \times G(\underline{a}^n))\end{aligned}$$

for a constant $C_3' > 0$, depending only on r, h, and the Cantor sets. However, we can choose $C_4' > 0$, independent of ρ, such that $\operatorname{diam}(G(\underline{ab})) \geq C_4'\rho \cdot \operatorname{diam}(G(\underline{a}))$ for any $\underline{a} \in \Sigma_j^{fin}$ and $\underline{b} \in \Sigma_j(c_0, \rho)$, $0 \leq j \leq n$. Therefore, we can choose $C_3 > 0$ that does not depend on ρ_1, such that

$$\left(\frac{\operatorname{diam}(G(q))}{\operatorname{diam}(G(p))}\right)^{d_1+\cdots+d_n-\eta} \geq C_3\rho_1^{d_1+\cdots+d_n-\eta} \tag{5}$$

for any $q \in Ch(p)$. Now that C_3 has been chosen, we get

$$\begin{aligned}\sum_{q \in Ch(p)} \left(\frac{\operatorname{diam}(G(q))}{\operatorname{diam}(G(p))}\right)^{d_1+\cdots+d_n-\eta} &\geq \sum_{q \in Ch(p)} C_3\rho_1^{d_1+\cdots+d_n-\eta} \\ &= C_3\rho_1^{d_1+\cdots+d_n-\eta} \cdot \#T'(p) \geq 1. \qquad \square\end{aligned}$$

THEOREM B. (Dimension formula) *Let $K_1, \ldots, K_n$ be C^m, $m \geq 2$, conformal regular Cantor sets generated by expanding maps $g_1, \ldots, g_n$, respectively. Suppose all of them are not essentially affine. Assume that there exist periodic points $p_j \in K_j$, with period n_j for $1 \leq j \leq n$, such that if we write $Dg_j^{n_j}(p_j) = \frac{1}{r_j}R_{-v_j}$, where R_v is the rotation matrix by an angle $v \in \mathbb{T}$, then*

$$\begin{gathered}(\log r_1, 0, \ldots, 0; v_1, 0, \ldots, 0), \\ \vdots \\ (0, \ldots, \log r_{n-1}; 0, \ldots, v_{n-1}, 0), \\ (-\log r_n, \ldots, -\log r_n; 0, \ldots, 0, v_n)\end{gathered}$$

generate a dense subgroup of J. Let h be any C^1 function defined on a neighborhood of $K_1 \times \cdots \times K_n$ into $\mathbb{R}^l$ such that there exists a point $x_0 \in K_1 \times \cdots \times K_n$ where $Dh(x_0)$

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verifies the transversality hypotheses, then

$$HD(h(K_1 \times \cdots \times K_n)) = \min\{l, HD(K_1) + \cdots + HD(K_n)\}.$$

Proof. We first treat the case $HD(K_1) + \cdots + HD(K_n) < l$. Notice that $K_1, \ldots, K_n$ verify the hypotheses of the scale recurrence lemma; the existence of the periodic points p_j imply that all of the Cantor sets are not essentially real. The desired result follows from the preceding theorem if we show that for any $(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) \in \Sigma_1^- \times \cdots \times \Sigma_n^- \times J_r$, there exists $(\underline{c}^1, \ldots, \underline{c}^n) \in \Sigma_1^{fin} \times \cdots \times \Sigma_n^{fin}$ such that $T_{\underline{c}^1,\ldots,\underline{c}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) = (\underline{\theta}^1\underline{c}^1, \ldots, \underline{\theta}^n\underline{c}^n, \tilde{s})$ and $\tilde{s} \in F^*(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n)$ for some $(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^n)$ for which $(\underline{\theta}^1\underline{c}^1, \ldots, \underline{\theta}^n\underline{c}^n)$ ends in it.

Let $\underline{a}_j \in \Sigma_j^{fin}$ be the word of length n_j such that the periodic point p_j corresponds to the sequence $\underline{a}_j\underline{a}_j\underline{a}_j \cdots$. For a finite sequence $\underline{a} \in \Sigma_j^{fin}$ and $k \in \mathbb{Z}^+$, we are going to use the notation $\underline{a}^k = \underbrace{\underline{a}\underline{a}\cdots\underline{a}}_{k-times}$.

Choosing c_0 big enough and assuming ρ is small, we can find $k_j \in \mathbb{Z}^+$ such that $\underline{\tilde{a}}_j := \underline{a}_j^{k_j} \in \Sigma_j(c_0, \rho)$. Define $\underline{\theta}_j = \cdots \underline{\tilde{a}}_j \cdots \underline{\tilde{a}}_j \in \Sigma_j^-$.

By properties (ii) and (iii) of the scale recurrence lemma, we know that there are $s_0, s_1 \in J$, $\underline{b}_j \in \Sigma_j(c_0, \rho)$ and $\underline{c}_j \in \Sigma_j^{fin}$, $1 \le j \le n$, such that $T_{\underline{c}_1\underline{b}_1,\ldots,\underline{c}_n\underline{b}_n}(\underline{\theta}_1, \ldots, \underline{\theta}_n, s_0) = (\underline{\theta}_1\underline{c}_1\underline{b}_1, \ldots, \underline{\theta}_n\underline{c}_n\underline{b}_n, s_1)$, and the ρ-neighborhood of s_1 is contained in $F^*(\underline{b}_1, \ldots, \underline{b}_n)$.

Thanks to the continuity of the map $\underline{\theta} \to k^{\underline{\theta}}$, we can choose positive integers $l_1, \ldots, l_n$, depending on ρ, such that for any $m_j > l_j$, any $\underline{\theta}^j \in \Sigma_j^-$, $1 \le j \le n$, and x in the $\rho/2$-neighborhood of s_0, we have

$$T_{\underline{c}_1\underline{b}_1,\ldots,\underline{c}_n\underline{b}_n}(\underline{\theta}^1\underline{\tilde{a}}_1^{m_1}, \ldots, \underline{\theta}^n\underline{\tilde{a}}_n^{m_n}, x) = (\underline{\theta}^1\underline{\tilde{a}}_1^{m_1}\underline{c}_1\underline{b}_1, \ldots, \underline{\theta}^n\underline{\tilde{a}}_n^{m_n}\underline{c}_n\underline{b}_n, \tilde{x})$$

and $\tilde{x} \in F^*(\underline{b}_1, \ldots, \underline{b}_n)$. (We assume $\underline{\theta}^j$, $1 \le j \le n$, ends with the letters in which $\underline{a}_j$, $1 \le j \le n$, starts, otherwise we consider $\underline{d}^j\underline{\tilde{a}}_j^m$ instead of $\underline{\tilde{a}}_j^m$ for some $\underline{d}^j$, and the proof follows in the same way.)

Now notice that $r^{\underline{\theta}^j}_{\underline{\tilde{a}}^{l_j+m_j}} = r^{\underline{\theta}^j}_{\underline{\tilde{a}}^{l_j}} \cdot r^{\underline{\theta}^j\underline{\tilde{a}}^{l_j}}_{\underline{\tilde{a}}^{m_j}}$, and if l_j is big enough, we have

$$\left| \log r^{\underline{\theta}^j\underline{\tilde{a}}_j^{l_j}}_{\underline{\tilde{a}}_j^{m_j}} - \log r^{\underline{\theta}_j}_{\underline{\tilde{a}}_j^{m_j}} \right| < \frac{\rho}{8(2n-1)}$$

for any $m_j \in \mathbb{Z}^+$. Similar formulas hold for $v^{\underline{\theta}^j\underline{\tilde{a}}_j^{l_j}}_{\underline{\tilde{a}}_j^{m_j}}$.

For $m_j \in \mathbb{Z}^+$, $1 \le j \le n$, consider $T_{\underline{\tilde{a}}_1^{l_1+m_1},\ldots,\underline{\tilde{a}}_j^{l_j+m_j}}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) = (\underline{\tilde{\theta}}^1, \ldots, \underline{\tilde{\theta}}^n, \tilde{s})$. We have

$$\begin{aligned}
\tilde{s} &= s + \Big(\log r^{\underline{\theta}^1}_{\underline{\tilde{a}}_1^{l_1+m_1}} - \log r^{\underline{\theta}^n}_{\underline{\tilde{a}}_n^{l_n+m_n}}, \ldots, \log r^{\underline{\theta}^{n-1}}_{\underline{\tilde{a}}_{n-1}^{l_{n-1}+m_{n-1}}} - \log r^{\underline{\theta}^n}_{\underline{\tilde{a}}_n^{l_n+m_n}}; v^{\underline{\theta}^1}_{\underline{\tilde{a}}_1^{l_1+m_1}}, \ldots, v^{\underline{\theta}^n}_{\underline{\tilde{a}}_n^{l_n+m_n}}\Big) \\
&= s + \Big(\log r^{\underline{\theta}^1}_{\underline{\tilde{a}}_1^{l_1}} - \log r^{\underline{\theta}^n}_{\underline{\tilde{a}}_n^{l_n}}, \ldots, \log r^{\underline{\theta}^{n-1}}_{\underline{\tilde{a}}_{n-1}^{l_{n-1}}} - \log r^{\underline{\theta}^n}_{\underline{\tilde{a}}_n^{l_n}}; v^{\underline{\theta}^1}_{\underline{\tilde{a}}_1^{l_1}}, \ldots, v^{\underline{\theta}^n}_{\underline{\tilde{a}}_n^{l_n}}\Big) \\
&\quad + \Big(\log r^{\underline{\theta}^1\underline{\tilde{a}}_1^{l_1}}_{\underline{\tilde{a}}_1^{m_1}} - \log r^{\underline{\theta}^n\underline{\tilde{a}}_n^{l_n}}_{\underline{\tilde{a}}_n^{m_n}}, \ldots, \log r^{\underline{\theta}^{n-1}\underline{\tilde{a}}_{n-1}^{l_{n-1}}}_{\underline{\tilde{a}}_{n-1}^{m_{n-1}}} - \log r^{\underline{\theta}^n\underline{\tilde{a}}_n^{l_n}}_{\underline{\tilde{a}}_n^{m_n}}; v^{\underline{\theta}^1\underline{\tilde{a}}_1^{l_1}}_{\underline{\tilde{a}}_1^{m_1}}, \ldots, v^{\underline{\theta}^n\underline{\tilde{a}}_n^{l_n}}_{\underline{\tilde{a}}_n^{m_n}}\Big).
\end{aligned}$$

Now notice that

$$\left(\log r_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}^1\underline{\tilde{a}}_1^{l_1}} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}^n\underline{\tilde{a}}_n^{l_n}}, \ldots, \log r_{\underline{\tilde{a}}_{n-1}^{m_{n-1}}}^{\underline{\theta}^{n-1}\underline{\tilde{a}}_{n-1}^{l_{n-1}}} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}^n\underline{\tilde{a}}_n^{l_n}}; v_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}^1\underline{\tilde{a}}_1^{l_1}}, \ldots, v_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}^n\underline{\tilde{a}}_n^{l_n}}\right)$$

is in the $\rho/4$ neighborhood of

$$\left(\log r_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}_1} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}, \ldots, \log r_{\underline{\tilde{a}}_{n-1}^{m_{n-1}}}^{\underline{\theta}_{n-1}} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}; v_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}_1}, \ldots, v_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}\right).$$

However,

$$\begin{aligned}
&\left(\log r_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}_1} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}, \ldots, \log r_{\underline{\tilde{a}}_{n-1}^{m_{n-1}}}^{\underline{\theta}_{n-1}} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}; v_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}_1}, \ldots, v_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}\right)\\
&\quad = m_1\left(\log r_{\underline{\tilde{a}}_1}^{\underline{\theta}_1}, 0, \ldots, 0; v_{\underline{\tilde{a}}_1}^{\underline{\theta}_1}, 0, \ldots, 0\right)\\
&\qquad + \cdots + m_{n-1}\left(0, \ldots, \log r_{\underline{\tilde{a}}_{n-1}}^{\underline{\theta}_{n-1}}; 0, \ldots, v_{\underline{\tilde{a}}_{n-1}}^{\underline{\theta}_{n-1}}, 0\right)\\
&\qquad + m_n\left(-\log r_{\underline{\tilde{a}}_n}^{\underline{\theta}_n}, \ldots, -\log r_{\underline{\tilde{a}}_n}^{\underline{\theta}_n}; 0, \ldots, 0, v_{\underline{\tilde{a}}_n}^{\underline{\theta}_n}\right).
\end{aligned}$$

Moreover, $r_{\underline{\tilde{a}}_j}^{\underline{\theta}_j} R_{v_{\underline{\tilde{a}}_j}^{\underline{\theta}_j}} = [Dg^{k_j n_j}(p_j)]^{-1} = r_j^{k_j} R_{k_j v_j}$. Therefore, the density hypothesis in the theorem, together with the next lemma, implies that there exist $m_1, \ldots, m_n \in \mathbb{Z}^+$ such that

$$(\log r_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}_1} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}, \ldots, \log r_{\underline{\tilde{a}}_{n-1}^{m_{n-1}}}^{\underline{\theta}_{n-1}} - \log r_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n}; v_{\underline{\tilde{a}}_1^{m_1}}^{\underline{\theta}_1}, \ldots, v_{\underline{\tilde{a}}_n^{m_n}}^{\underline{\theta}_n})$$

is in the $\rho/4$-neighborhood of

$$s_0 - s - (\log r_{\underline{\tilde{a}}_1^{l_1}}^{\underline{\theta}^1} - \log r_{\underline{\tilde{a}}_n^{l_n}}^{\underline{\theta}^n}, \ldots, \log r_{\underline{\tilde{a}}_{n-1}^{l_{n-1}}}^{\underline{\theta}^{n-1}} - \log r_{\underline{\tilde{a}}_n^{l_n}}^{\underline{\theta}^n}; v_{\underline{\tilde{a}}_1^{l_1}}^{\underline{\theta}^1}, \ldots, v_{\underline{\tilde{a}}_n^{l_n}}^{\underline{\theta}^n}).$$

Hence, $\tilde{s}$ is in the $\rho/2$-neighborhood of s_0 and from this, we get

$$\begin{aligned}
T_{\underline{\tilde{a}}_1^{l_1+m_1}\underline{c}_1\underline{b}_1,\ldots,\underline{\tilde{a}}_n^{l_n+m_n}\underline{c}_n\underline{b}_n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, s) &= T_{\underline{c}_1\underline{b}_1,\ldots,\underline{c}_n\underline{b}_n}(\underline{\theta}^1\underline{\tilde{a}}_1^{l_1+m_1}, \ldots, \underline{\theta}^n\underline{\tilde{a}}_n^{l_n+m_n}, \tilde{s})\\
&= (\underline{\theta}^1\underline{\tilde{a}}_1^{l_1+m_1}\underline{c}_1\underline{b}_1, \ldots, \underline{\theta}^n\underline{\tilde{a}}_n^{l_n+m_n}\underline{c}_n\underline{b}_n, \tilde{s}'),
\end{aligned}$$

and $\tilde{s}' \in F^*(\underline{b}_1, \ldots, \underline{b}_n)$, as we wanted.

If $HD(K_1) + \cdots + HD(K_n) \geq l$, fix $\epsilon > 0$ and find conformal regular Cantor sets $\tilde{K}_j \subset K_j$, $1 \leq j \leq n$ such that $l - \epsilon < HD(\tilde{K}_1) + \cdots + HD(\tilde{K}_n) < l$, $p_j \in \tilde{K}_j$, and the expanding map of $\tilde{K}_j$ is given by a power of g_j, $1 \leq j \leq n$ (see [8, Lemma in p. 16]). We get

$$\begin{aligned}
l \geq HD(h(K_1 \times \cdots \times K_n)) &\geq HD(h(\tilde{K}_1 \times \cdots \times \tilde{K}_n))\\
&= HD(\tilde{K}_1) + \cdots + HD(\tilde{K}_n) > l - \epsilon.
\end{aligned}$$

Since ϵ can be arbitrarily small, we obtain $HD(h(K \times \cdots \times K_n)) = l$ as we wanted. □

The following lemma was used in the previous theorem. It also implies that the hypothesis needed for the dimension formula is generic. Its proof is based on the well-known Kronecker theorem.

LEMMA 3.5. *Let $\lambda_j < 0$, $v_j \in \mathbb{T}$, $1 \leq j \leq n$, and consider the set $E(\lambda_1, \ldots, \lambda_n, v_1, \ldots, v_n) \subset \mathbb{R}^{n-1} \times \mathbb{T}^n$ given by the vectors*

$$(\lambda_1, 0, \ldots, 0; v_1, 0, \ldots, 0),$$
$$\vdots$$
$$(0, \ldots, \lambda_{n-1}; 0, \ldots, v_{n-1}, 0),$$
$$(-\lambda_n, \ldots, -\lambda_n; 0, \ldots, 0, v_n).$$

We have the following properties.

(a) *If $E(\lambda_1, \ldots, \lambda_n, v_1, \ldots, v_n)$ generates a dense subgroup of $\mathbb{R}^{n-1} \times \mathbb{T}^n$, then $E(k_1\lambda_1, \ldots, k_n\lambda_n, k_1v_1, \ldots, k_nv_n)$ also generates a dense subgroup for all $k_1, \ldots, k_n \in \mathbb{Z} \setminus \{0\}$.*

(b) *If $E(\lambda_1, \ldots, \lambda_n, v_1, \ldots, v_n)$ generates a dense subgroup of $\mathbb{R}^{n-1} \times \mathbb{T}^n$, then it also generates a dense semigroup, that is, the set of linear combinations of vectors in $E(\lambda_1, \ldots, \lambda_n, v_1, \ldots, v_n)$ with coefficients in $\mathbb{N}$ is dense in $\mathbb{R}^{n-1} \times \mathbb{T}^n$.*

(c) *The set of values $(\lambda_1, \ldots, \lambda_n, v_1, \ldots, v_n) \in \mathbb{R}^n_{<0} \times \mathbb{T}^n$, for which the set $E(\lambda_1, \ldots, \lambda_n, v_1, \ldots, v_n)$ generates a dense subgroup, is a countable intersection of open and dense subsets.*

4. *Proof of the scale recurrence lemma*

In this section, we will present the proof of the scale recurrence lemma, it follows the ideas in [7] with some modifications. One of the main new features is the use of the not essentially real hypotheses. In §4.3, we use this hypothesis to estimate the norm $|x - y|$ from an inner product $\langle \xi, x - y \rangle$, for some pairs x, y in a Cantor set. Roughly speaking, the not essentially real hypothesis allows us to choose many x, y such that the vector $x - y$ is far from being orthogonal to ξ.

We also introduced new objects that were not present in the logical structure of [7]; there was a minor flaw in that paper, exactly in the proof of the scale recurrence lemma. We defined these objects to deal with this problem.

The flaw in [7] is in the proof of lemma 6.6, exactly in the part where one tries to estimate the measure of $\hat{\hat{E}}_n(\lambda)$ outside the interval $[-r + 2\log c_0 + \Delta_1\rho, r - 2\log c_0 - \Delta_1\rho]$, which would correspond to estimating the measure of $\tilde{E}_m(\lambda)$ outside of $J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho}$ in our paper. We considered, apart from the parameter c_0, another parameter $\tilde{c}_0$, which is smaller than c_0. We then created a statement, similar to the scale recurrence lemma, involving both parameters, the sets $\Sigma_j(\tilde{c}_0, \rho)$ used to parameterize sets $E(\underline{a}^1, \ldots, \underline{a}^n)$ and the sets $\Sigma_j(c_0, \rho)$ used to parameterize renormalization operators. We prove that the statement implies the original lemma. Finally, we prove the statement using an analogous logical structure to that presented in [7]. A similar approach can be used in [7].

4.1. *General setting.* We proceed as in [7, 6.1] using Fourier analysis in the group J instead of $\mathbb{R}$. Let A be a set of indices, Λ a finite set, and maps $\alpha : \Lambda \to A$, $\omega : \Lambda \to A$. Define $\Lambda_i = \alpha^{-1}(i)$, $\Lambda^j = \omega^{-1}(j)$, $\Lambda_i^j = \Lambda_i \cap \Lambda^j$, $N_i = \#\Lambda_i$, $N_i^j = \#\Lambda_i^j$,

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$p_i^j = N_i^j/N_i$. The numbers (p_i^j) define a stochastic matrix, it has a probability vector (p^i) verifying $\sum_{i\in A} p^i p_i^j = p^j$, $\sum_{i\in A} p^i = 1$. Set

$$p_\lambda^{\lambda'} = \begin{cases} 0 & \text{if } \omega(\lambda) \neq \alpha(\lambda'), \\ 1/N_{\omega(\lambda)} & \text{if } \omega(\lambda) = \alpha(\lambda'), \end{cases}$$

and $p^\lambda = \frac{p^{\alpha(\lambda)}}{N_{\alpha(\lambda)}}$. It is easily proved that $(p_\lambda^{\lambda'})$ is a stochastic matrix with probability vector (p^λ). Let $J^* = \mathbb{R}^{n-1} \times \mathbb{Z}^n$ denote the Pontryagin dual of J. Elements $\xi = (\mu_1, \ldots, \mu_{n-1}, m_1, \ldots, m_n) \in J^*$ are homomorphisms from J to $\mathbb{S}^1$, given by

$$\xi(t_1, \ldots, t_{n-1}, v_1, \ldots, v_n) = e^{(\sum_{j=1}^{n-1} t_j\mu_j + \sum_{j=1}^{n} m_j v_j)i}.$$

Now suppose that for each $(\lambda, \lambda') \in \Lambda^2$, there is an element $a_\lambda^{\lambda'} \in J$. Using this, we define for each $\xi \in J^*$, a linear operator $T_\xi : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$ given by $T_\xi((z_\lambda)_{\lambda\in\Lambda}) = (w_\lambda)_{\lambda\in\Lambda}$, where $w_\lambda = \sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'} \xi(a_\lambda^{\lambda'}) z_{\lambda'}$.

We endow the space $\mathbb{C}^\Lambda$ with the norm $\|(z_\lambda)_{\lambda\in\Lambda}\|^2 = \sum_{\lambda\in\Lambda} p^\lambda |z_\lambda|^2$. In a similar way to [7], a short computation shows that $\|T_\xi\| \le 1$ for all $\xi \in J^*$.

Assume that we have a family $\{E(\lambda)\}_{\lambda\in\Lambda}$ of bounded measurable subsets of J, and consider the function

$$n_\lambda(x) = \frac{1}{N_{\omega(\lambda)}} \cdot \#\{\lambda' \in \Lambda_{\omega(\lambda)} : B_\rho(x + a_\lambda^{\lambda'}) \subset E(\lambda')\}.$$

Let $0 < \tau < 1$, and denote by $E^*(\lambda)$ the set $E^*(\lambda) = \{x \in J : n_\lambda(x) > \tau\}$.

PROPOSITION 4.1. *Suppose there exist $\Delta_0 > 0$ and $k_0 \in (0, 1)$ such that $\|T_\xi\| < k_0$ for all $\xi = (\mu_1, \ldots, \mu_{n-1}, m_1, \ldots, m_n)$, with $|\xi| = \max_j\{|\mu_j|, |m_j|\} \in [1, \Delta_0\rho^{-1}]$. Then there exist $k_1 \in (0, 1)$, $\epsilon > 0$, and $\tau \in (0, 1)$ depending only on Δ_0, k_0 (and not on ρ) such that if $\nu(E(\lambda)) \le \epsilon$ for all $\lambda \in \Lambda$, then*

$$\sum_{\lambda\in\Lambda} p^\lambda \nu(E^*(\lambda)) \le k_1 \sum_{\lambda\in\Lambda} p^\lambda \nu(E(\lambda)).$$

Proof. Consider the functions $X_\lambda = 1_{E(\lambda)}$, $Y_\lambda(x) = \sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'} X_{\lambda'}(x + a_\lambda^{\lambda'})$,

$$Z_\lambda(x) = \frac{1}{\nu(B_\rho(0))} \cdot \int_{B_\rho(0)} Y_\lambda(x - t)\, d\nu(t) = \frac{1}{\nu(B_\rho(0))} 1_{B_\rho(0)} * Y_\lambda(x).$$

Note that $Z_\lambda(x) \ge n_\lambda(x)$, then $\|Z_\lambda\|_{L^2}^2 \ge \tau^2 \nu(E^*(\lambda))$, which implies

$$\sum_{\lambda\in\Lambda} p^\lambda \nu(E^*(\lambda)) \le \tau^{-2} \sum_{\lambda\in\Lambda} p^\lambda \|Z_\lambda\|_{L^2}^2. \tag{6}$$

The Fourier transforms of X_λ, Y_λ, Z_λ are

$$\hat{X}_\lambda(\xi) = \int_J X_\lambda(x)\bar{\xi}(x)\, d\nu(x), \quad \hat{Y}_\lambda(\xi) = \sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'} \xi(a_\lambda^{\lambda'}) \hat{X}_{\lambda'}(\xi),$$

$$\hat{Z}_\lambda(\xi) = \frac{1}{\nu(B_\rho(0))} \hat{1}_{B_\rho(0)} \cdot \hat{Y}_\lambda(\xi) = \prod_{j=1}^{n-1} \frac{\sin(\mu_j\rho)}{\mu_j\rho} \prod_{j=1}^{n} \frac{\sin(m_j\rho)}{m_j\rho} \cdot \hat{Y}_\lambda(\xi),$$

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where $\xi = (\mu_1, \ldots, \mu_{n-1}, m_1, \ldots, m_n)$. Hence, $|\hat{Z}_\lambda(\xi)| \le |\hat{Y}_\lambda(\xi)|$, and there exists $\tilde{k}_1 \in (0, 1)$, depending only on Δ_0, such that $|\hat{Z}_\lambda(\xi)| \le \tilde{k}_1 |\hat{Y}_\lambda(\xi)|$ if $|\xi| > \Delta_0 \rho^{-1}$. We estimate $\sum_{\lambda \in \Lambda} p^\lambda |\hat{Z}_\lambda(\xi)|^2$ in various ways depending on $|\xi|$.

If $|\xi| < 1$, then

$$|\hat{Z}_\lambda(\xi)| \le |\hat{Y}_\lambda(\xi)| \le \int_J |Y_\lambda(x)| d\nu(x) \le \sum_{\lambda' \in \Lambda} p_\lambda^{\lambda'} \nu(E(\lambda')),$$

therefore,

$$\sum_{\lambda \in \Lambda} p^\lambda |\hat{Z}_\lambda(\xi)|^2 \le \sum_{\lambda' \in \Lambda} \left(\sum_{\lambda \in \lambda} p^\lambda p_\lambda^{\lambda'} \right) \nu(E(\lambda'))^2 = \sum_{\lambda \in \Lambda} p^\lambda \nu(E(\lambda))^2.$$

If $1 \le |\xi| \le \Delta_0 \rho^{-1}$,

$$\sum_{\lambda \in \Lambda} p^\lambda |\hat{Z}_\lambda(\xi)|^2 \le \sum_{\lambda \in \Lambda} p^\lambda |\hat{Y}_\lambda(\xi)|^2 \le k_0^2 \sum_{\lambda \in \Lambda} p^\lambda |\hat{X}_\lambda(\xi)|^2,$$

note that we used the fact that $(\hat{Y}_\lambda(\xi))_{\lambda \in \Lambda} = T_\xi((\hat{X}_\lambda(\xi))_{\lambda \in \Lambda})$.

If $|\xi| > \Delta_0 \rho^{-1}$,

$$\sum_{\lambda \in \Lambda} p^\lambda |\hat{Z}_\lambda(\xi)|^2 \le \tilde{k}_1^2 \sum_{\lambda \in \Lambda} p^\lambda |\hat{Y}_\lambda(\xi)|^2 \le \tilde{k}_1^2 \sum_{\lambda \in \Lambda} p^\lambda |\hat{X}_\lambda(\xi)|^2.$$

Combining all three inequalities, we get

$$\begin{aligned}
&\int_{J^*} \sum_{\lambda \in \Lambda} p^\lambda |\hat{Z}_\lambda(\xi)|^2 d\hat{\nu}(\xi) \\
&\quad \le \int_{|\xi|<1} \sum_{\lambda \in \Lambda} p^\lambda \nu(E(\lambda))^2 d\hat{\nu}(\xi) + k_0^2 \int_{1 \le |\xi| \le \Delta_0 \rho^{-1}} \sum_{\lambda \in \Lambda} p^\lambda |\hat{X}_\lambda(\xi)|^2 d\hat{\nu}(\xi) \\
&\qquad + \tilde{k}_1^2 \int_{|\xi| > \Delta_0 \rho^{-1}} \sum_{\lambda \in \Lambda} p^\lambda |\hat{X}_\lambda(\xi)|^2 d\hat{\nu}(\xi).
\end{aligned}$$

Hence,

$$\sum_{\lambda \in \Lambda} p^\lambda \|\hat{Z}_\lambda\|_{L^2}^2 \le \hat{\nu}(\{|\xi| < 1\}) \cdot \epsilon \sum_{\lambda \in \Lambda} p^\lambda \nu(E(\lambda)) + \max\{k_0^2, \tilde{k}_1^2\} \sum_{\lambda \in \Lambda} p^\lambda \|\hat{X}_\lambda\|_{L^2}^2.$$

Using Plancherel theorem and the fact that $\nu(E(\lambda)) = \|X_\lambda\|_{L^2}^2$, we get

$$\sum_{\lambda \in \Lambda} p^\lambda \|Z_\lambda\|_{L^2}^2 \le [\hat{\nu}(\{|\xi| < 1\}) \cdot \epsilon + \max\{k_0^2, \tilde{k}_1^2\}] \sum_{\lambda \in \Lambda} p^\lambda \nu(E(\lambda)).$$

This together with equation (6) imply

$$\sum_{\lambda \in \Lambda} p^\lambda \nu(E^*(\lambda)) \le \tau^{-2} [\hat{\nu}(\{|\xi| < 1\}) \cdot \epsilon + \max\{k_0^2, \tilde{k}_1^2\}] \sum_{\lambda \in \Lambda} p^\lambda \nu(E(\lambda)).$$

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Finally, we get the desired inequality setting

$$\tau = [\hat{\nu}(\{|\xi| < 1\}) \cdot \epsilon + \max\{k_0^2, \tilde{k}_1^2\}]^{1/3}, k_1 = \tau^{-2}[\hat{\nu}(\{|\xi| < 1\}) \cdot \epsilon + \max\{k_0^2, \tilde{k}_1^2\}],$$

and taking ϵ small enough such that $\hat{\nu}(\{|\xi| < 1\}) \cdot \epsilon + \max\{k_0^2, \tilde{k}_1^2\} < 1$. □

Let $\Delta_1 > 0$ be any positive number and consider

$$\hat{n}_\lambda(x) = \frac{1}{\#\Lambda_{\omega(\lambda)}} \cdot \#\{\lambda' \in \Lambda_{\omega(\lambda)} : B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \cap E(\lambda') \neq \emptyset\};$$

define $\hat{E}(\lambda) = \{x \in J : \hat{n}_\lambda(x) > \tau\}$. For a set $E \subset J$, denote by $V_\delta(E)$ the δ-neighborhood of E.

COROLLARY 4.1. *Under the same hypothesis as Proposition 4.1, let $k_4 > 0$ such that $k_1 < k_4 < 1$; if we choose $\Delta > 0$ big enough and $\epsilon_1 > 0$ small enough such that*

$$k_1\left(1 + \frac{1+\Delta_1}{\Delta}\right)^{2n-1} < k_4, \quad \epsilon_1\left(1 + \frac{1+\Delta_1}{\Delta}\right)^{2n-1} < \epsilon,$$

then $\nu(V_{\Delta\rho}(E(\lambda))) \le \epsilon_1$, for all $\lambda \in \Lambda$, implies that

$$\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(\hat{E}(\lambda))) \le k_4 \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E(\lambda))).$$

Proof. First observe that (see [5])

$$\nu(V_{\rho+\Delta\rho+\Delta_1\rho}(E(\lambda))) \le \left(1 + \frac{1+\Delta_1}{\Delta}\right)^{2n-1} \nu(V_{\Delta\rho}(E(\lambda))).$$

Now consider the family $A(\lambda) = V_{\rho+\Delta\rho+\Delta_1\rho}(E(\lambda))$; by our choice of ϵ_1, we can apply Proposition 4.1 to $A(\lambda)$. Notice also that if $x \in V_{\Delta\rho}(\hat{E}(\lambda))$, then there exist $y \in B_{\Delta\rho}(x)$ and $\tau \cdot \#\Lambda_{\omega(\lambda)}$ elements $\lambda' \in \Lambda_{\omega(\lambda)}$ such that $B_{\Delta_1\rho}(y + a_\lambda^{\lambda'}) \cap E(\lambda') \neq \emptyset$, and thus $B_\rho(x + a_\lambda^{\lambda'}) \subset V_{\rho+\Delta\rho+\Delta_1\rho}(E(\lambda')) = A(\lambda')$.

This shows that $V_{\Delta\rho}(\hat{E}(\lambda)) \subset A^*(\lambda)$. Applying Proposition 4.1 to $A(\lambda)$ gives

$$\begin{aligned}\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(\hat{E}(\lambda))) &\le \sum_{\lambda\in\Lambda} p^\lambda \nu(A^*(\lambda)) \le k_1 \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\rho+\Delta\rho+\Delta_1\rho}(E(\lambda)))\\ &\le k_1\left(1 + \frac{1+\Delta_1}{\Delta}\right)^{2n-1} \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E(\lambda)))\\ &\le k_4 \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E(\lambda))).\end{aligned}$$ □

4.2. *Proof of Theorem A.* In this subsection, we will prove the multidimensional conformal scale recurrence lemma. First we will fix the values of the main parameters playing a role in the proof. This is done to make it clear that there are no contradictions between their values.

Start by choosing a positive constant μ such that $-\log r_{\underline{c}}^{\underline{\theta}} < \mu$ for any $\underline{\theta} \in \Sigma_j^-$, and $\underline{c} = (c_0, c_1) \in \Sigma_j^{fin}$ a finite sequence with only two symbols. The choice of μ and the

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equation $r^{\theta}_{\underline{bc}} = r^{\theta}_{\underline{b}} \cdot r^{\theta b}_{\underline{c}}$ imply that $\log r^{\theta}_{\underline{bc}} > \log r^{\theta}_{\underline{b}} - \mu$. This is saying that as the length of $\underline{b}$ increases, the number $\log r^{\theta}_{\underline{b}}$ decreases by steps no bigger than μ.

Now choose $c > 0$ such that $c^{-1}\mathrm{diam}(G(\underline{a}^j)) \le \mathrm{diam}(G^{\underline{\theta}^j}(\underline{a}^j)) \le c\,\mathrm{diam}(G(\underline{a}^j))$ for $\underline{a}^j \in \Sigma_j^{fin}$, $\underline{\theta}^j \in \Sigma_j^-$. These constants only depend on $K_1, \ldots, K_n$. Fix $\tilde{c}_0 > 0$ such that

$$2\log(c\tilde{c}_0) > \mu. \tag{7}$$

We will use Proposition 4.1 with the following data:

$$\begin{aligned}
\Lambda &= \Sigma_1(\tilde{c}_0, \rho) \times \cdots \times \Sigma_n(\tilde{c}_0, \rho),\\
A &= \mathbb{A}_1 \times \cdots \times \mathbb{A}_n,\\
\alpha(\underline{a}^1, \ldots, \underline{a}^n) &= (a_0^1, \ldots, a_0^n),\\
\omega(\underline{a}^1, \ldots, \underline{a}^n) &= (a_{m_1}^1, \ldots, a_{m_n}^n),\\
a_\lambda^{\lambda'} &= \left(\log \frac{r^{\underline{\theta}^1}_{\underline{b}^1}}{r^{\underline{\theta}^n}_{\underline{b}^n}}, \ldots, \log \frac{r^{\underline{\theta}^{n-1}}_{\underline{b}^{n-1}}}{r^{\underline{\theta}^n}_{\underline{b}^n}}, v^{\underline{\theta}^1}_{\underline{b}^1}, \ldots, v^{\underline{\theta}^n}_{\underline{b}^n}\right),
\end{aligned}$$

where $\underline{a}^j = (a_0^j, \ldots, a_{m_j}^j)$, $\lambda = (\underline{a}^1, \ldots, \underline{a}^n)$, $\lambda' = (\underline{b}^1, \ldots, \underline{b}^n)$, and $\underline{\theta}^j \in \Sigma_j^-$ finishes in $\underline{a}^j$, $1 \le j \le n$. (For every $\underline{a}^j \in \Sigma_j^{fin}$, we choose, arbitrarily, an element $\underline{\theta}^j \in \Sigma_j^-$ that ends in $\underline{a}^j$. Using this, we define $a_\lambda^{\lambda'}$.) Assume that the hypothesis of Proposition 4.1 holds, namely that there exist Δ_0, k_0 such that $\|T_\xi\| \le k_0$ for all $|\xi| \in [1, \Delta_0\rho^{-1}]$; this will be verified in the next subsection. Applying the proposition in this setting gives constants k_1, τ, ϵ.

Now fix $k_4, k_5 > 0$ such that $k_1 < k_4 < k_5 < 1$ and $\delta > 0$ such that

$$k_4 + 2 \cdot 3^n k_1^{-1} L C_2^{-1} C_4 \delta < k_5, \tag{8}$$

where $C_2 > 0$, $L > 0$ are constants such that

$$L^{-1}\rho^{-(d_1+\cdots+d_n)} \le \#\Lambda_i \le L\rho^{-(d_1+\cdots+d_n)}, \tag{9}$$

$$C_2\rho^{d_1+\cdots+d_n} \le p^\lambda, \tag{10}$$

for any $i \in A$, $\lambda \in \Lambda$, and $C_4 > 0$ is defined by equation (19). All these constants depend only on $\tilde{c}_0$.

Fix $r > 0$ such that

$$2r > \delta^{-1}\left(9 + \frac{1}{4}\right)\log(c\tilde{c}_0). \tag{11}$$

The choice of μ and $\tilde{c}_0$ allow us to find $\rho_1 > 0$, small enough, such that for any family of intervals $I_1, \ldots, I_{n-1}$, with $\mathrm{diam}(I_j) \ge 2\log(c\tilde{c}_0)$, any $x = (t, v) \in J$ with $\mathrm{dist}(t_j, I_j) \le \delta^{-1}(9 + \frac{1}{4})\log(c\tilde{c}_0)$, and any $\lambda \in \Lambda$, there exists $\lambda_0 = (\underline{b}^1, \ldots, \underline{b}^n) \in \Sigma_1^{fin} \times \cdots \times \Sigma_n^{fin}$ with the property $x + a_\lambda^{\lambda_0} \in I_1 \times \cdots \times I_{n-1} \times \mathbb{T}^n$, and $\mathrm{diam}(G(\underline{b}^j)) > \rho_1$, $1 \le j \le n$. Choose $c_0 > \tilde{c}_0$ big enough such that

$$\lambda_0\lambda \in \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$$

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for all $\lambda_0 = (\underline{b}^1, \ldots, \underline{b}^n)$, such that $\mathrm{diam}(G(\underline{b}^j)) > \rho_1$, $1 \le j \le n$, and any $\lambda \in \Lambda$ with $\omega(\lambda_0) = \alpha(\lambda)$. (If $\lambda = (\underline{a}^1, \ldots, \underline{a}^n)$, $\lambda_0 = (\underline{b}^1, \ldots, \underline{b}^n)$, with $\omega(\lambda_0) = \alpha(\lambda)$, then $\lambda_0\lambda = (\underline{b}^1\underline{a}^1, \ldots, \underline{b}^n\underline{a}^n)$.)

We also fix a constant $\Delta_1 > 0$ that should be big enough to verify equation (15); this is a condition that only depends on c_0. Finally, we choose Δ, ϵ_1 as in Corollary 4.1.

Notice that $\Lambda' = \Sigma_1(c_0, \rho) \times \cdots \times \Sigma_n(c_0, \rho)$ contains Λ. Choose a function $\varphi : \Lambda' \to \Lambda$ such that we have the following properties.

(a) If $\lambda = (\underline{a}^1, \ldots, \underline{a}^n) \in \Lambda'$ and $\varphi(\lambda) = (\underline{b}^1, \ldots, \underline{b}^n)$, then either $\underline{a}^j$ ends with $\underline{b}^j$ or $\underline{b}^j$ ends with $\underline{a}^j$ for every $1 \le j \le n$.

(b) $\varphi(\lambda) = \lambda$ for all $\lambda \in \Lambda$.

Thanks to properties (a),(b) of φ, there are constants T_1, T_2 depending only on c_0, $\tilde{c}_0$, and not on ρ, such that $1 \le T_1 \le \#\varphi^{-1}(\lambda) \le T_2$ for all $\lambda \in \Lambda$.

We show that we can suppose $F(\lambda) = F(\varphi(\lambda))$. Assume that for the given values of c_0 and r, there exist $c_1, c_2, c_3, \rho_0 > 0$ such that the scale recurrence lemma is verified in the special case when $F(\lambda) = F(\varphi(\lambda))$ for all $\lambda \in \Lambda'$. We find new values for c_1, c_2, c_3, $\rho_0 > 0$ verifying the lemma in the general case. In fact, we do not need to change $c_2, c_3, \rho_0 > 0$, just redefine c_1 as c_1/T_2. Given a family $\{F(\lambda)\}_{\lambda\in\Lambda'}$ with $\nu(J_r \setminus F(\lambda)) \le c_1/T_2$, consider $\tilde{F}(\lambda) = \bigcap_{\lambda' \in \varphi^{-1}(\varphi(\lambda))} F(\lambda')$. This new family verifies $\tilde{F}(\lambda) = \tilde{F}(\varphi(\lambda))$, moreover, $\nu(J_r \setminus \tilde{F}(\lambda)) \le c_1$, then there exists $\tilde{F}^*(\lambda)$ with the properties of the scale recurrence lemma. Taking $F^*(\lambda) = \tilde{F}^*(\lambda)$ gives the lemma in the general case.

For the scale recurrence lemma to hold, it is enough to prove the following statement.

STATEMENT 4.1. *For the given values of c_0 and r, there exist $c_1, c_2, c_3, \rho_0 > 0$ with the following properties: given $0 < \rho < \rho_0$ and a family $F(\lambda)$ of subsets of J_r, $\lambda \in \Lambda = \Sigma_1(\tilde{c}_0, \rho) \times \cdots \times \Sigma_n(\tilde{c}_0, \rho)$ such that $\nu(J_r \setminus F(\lambda)) \le c_1$ for all λ, there is another family $F^*(\lambda)$ of subsets of J_r satisfying the following properties.*

(i) *For any $\lambda \in \Lambda$, $F^*(\lambda)$ is contained in the $c_2\rho$-neighborhood of $F(\lambda)$.*

(ii) *Let $\lambda = (\underline{a}^1, \ldots, \underline{a}^n) \in \Lambda$, $(t, v) \in F^*(\lambda)$; there exist at least $c_3\rho^{-(d_1+\cdots+d_n)}$ elements $\lambda' = (\underline{b}^1, \ldots, \underline{b}^n) \in \Lambda'$ (with $\underline{b}^j$ starting with the last letter of $\underline{a}^j$) such that if $\underline{\theta}^j \in \Sigma_j^-$, $1 \le j \le n$, verify $\underline{\theta}^j \wedge \underline{a}^j \in \Sigma_j(c_0, \rho)$, $1 \le j \le n$ and*

$$T_{\underline{b}^1,\ldots,\underline{b}^n}(\underline{\theta}^1, \ldots, \underline{\theta}^n, t, v) = (\underline{\theta}^1\underline{b}^1, \ldots, \underline{\theta}^n\underline{b}^n, \tilde{t}, \tilde{v})$$

the ρ-neighborhood of $(\tilde{t}, \tilde{v}) \in J$ is contained in $F^*(\varphi(\lambda'))$.

(iii) *$\nu(F^*(\lambda)) \ge \nu(J_r)/2$ for at least $T_2/(T_2 + T_1)$ of the $\lambda \in \Lambda$.*

The difference between Statement 4.1 and the scale recurrence lemma is that Λ is parameterizing the sets $F(\lambda)$ instead of Λ'; however, Λ', which is much bigger than Λ, still parameterizes the set of renormalization operators.

Let $\{F(\lambda)\}_{\lambda\in\Lambda'}$ be a family of sets as in the scale recurrence lemma. We can suppose that $F(\lambda) = F(\varphi(\lambda))$. Now assume that Statement 4.1 holds, then we can apply it to the restricted family $\{F(\lambda)\}_{\lambda\in\Lambda}$, which produces another family $\{F^*(\lambda)\}_{\lambda\in\Lambda}$. We extend it to $\lambda \in \Lambda'$ by $F^*(\lambda) = F^*(\varphi(\lambda))$. It is easily seen that $\{F^*(\lambda)\}_{\lambda\in\Lambda'}$ verifies the desired properties on the scale recurrence lemma. From now on, we will focus in the proof of Statement 4.1.

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Suppose we have a family of sets $\{F(\lambda)\}_{\lambda\in\Lambda}$. Define $E_0(\lambda) = J_r \setminus V_{\Delta\rho}(F(\lambda))$, $\lambda \in \Lambda$. Now we define recursively two families of sets $\{E_m(\lambda)\}_{\lambda\in\Lambda}$ and $\{\tilde{E}_m(\lambda)\}_{\lambda\in\Lambda}$. The set $\tilde{E}_m(\lambda)$ is given by the $x \in J_r$ such that (when $\lambda' \in \Lambda' \setminus \Lambda$, the element $a_\lambda^{\lambda'}$ is defined in the same way as when $\lambda, \lambda' \in \Lambda$)

$$\#\{\lambda' \in \Lambda' : \alpha(\lambda') = \omega(\lambda);\ B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J_r \setminus E_m(\varphi(\lambda'))\} \le c_3\rho^{-(d_1+\cdots+d_n)},$$

and $E_{m+1}(\lambda) = E_0(\lambda) \cup \tilde{E}_m(\lambda)$. (In fact, since $\tilde{E}_j(\lambda) \subset \tilde{E}_{j+1}(\lambda)$, we have $E_m(\lambda) = E_0(\lambda) \cup \tilde{E}_0(\lambda) \cup \cdots \cup \tilde{E}_{m-1}(\lambda)$.) The value of c_3 will be fixed during the proof of the next lemma. Note that for $x \in \tilde{E}_0(\lambda)$, one has

$$\{\lambda' \in \Lambda' :\ B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J_r \setminus E_1(\varphi(\lambda'))\} \subset \{\lambda' \in \Lambda' :\ B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J_r \setminus E_0(\varphi(\lambda'))\},$$

and hence $\tilde{E}_0(\lambda) \subset \tilde{E}_1(\lambda)$. Analogously, one proves that $\tilde{E}_m(\lambda) \subset \tilde{E}_{m+1}(\lambda)$, $E_m(\lambda) \subset E_{m+1}(\lambda)$ for all m.

LEMMA 4.1. *If c_1, c_3, ρ_0 are sufficiently small, then*

$$\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(\tilde{E}_m(\lambda))) \le k_5 \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))), \tag{12}$$

$$\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))) \le \frac{\nu(J_{r+\Delta\rho} \setminus J_r) + c_1}{1 - k_5}. \tag{(13)}$$

Before proving the lemma, we will use it to prove Statement 4.1. Consider $E_\infty(\lambda) = \bigcup_{m\ge 0} E_m(\lambda)$; thanks to equation (13), we have

$$\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_\infty(\lambda))) \le \frac{\nu(J_{r+\Delta\rho} \setminus J_r) + c_1}{1 - k_5}. \tag{14}$$

Now define $F^*(\lambda) = J_r \setminus E_\infty(\lambda)$. We will prove that this family of sets has the desired properties.

(i) Since $E_0(\lambda) \subset E_\infty(\lambda)$ then $F^*(\lambda) \subset J_r \setminus E_0(\lambda) = V_{\Delta\rho}(F(\lambda))$, choosing $c_2 = \Delta$ gives the first property.

(ii) Let $x \in F^*(\lambda)$, then $x \notin \tilde{E}_m(\lambda)$ and the set

$$A_m = \{\lambda' \in \Lambda' : \alpha(\lambda') = \omega(\lambda);\quad B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J_r \setminus E_m(\varphi(\lambda'))\}$$

has more than $c_3\rho^{-(d_1+\cdots+d_n)}$ elements for all m. Moreover, since $E_m(\varphi(\lambda')) \subset E_{m+1}(\varphi(\lambda'))$, one has $A_{m+1} \subset A_m$ and then $\#(\bigcap_{m\ge 0} A_m) \ge c_3\rho^{-(d_1+\cdots+d_n)}$. Therefore,

$$\#\{\lambda' \in \Lambda' : \alpha(\lambda') = \omega(\lambda);\quad B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset F^*(\varphi(\lambda'))\} \ge c_3\rho^{-(d_1+\cdots+d_n)}.$$

To finish, it is enough to prove that for any $\underline{\tilde{\theta}}^1, \ldots, \underline{\tilde{\theta}}^n$ such that $\underline{\tilde{\theta}}^j \wedge \underline{a}^j \in \Sigma_j(c_0, \rho)$, $1 \le j \le n$, one has

$$B_\rho\left(x + \left(\log \frac{r_{\underline{b}^1}^{\underline{\tilde{\theta}}^1}}{r_{\underline{b}^n}^{\underline{\tilde{\theta}}^n}}, \ldots, \log \frac{r_{\underline{b}^{n-1}}^{\underline{\tilde{\theta}}^{n-1}}}{r_{\underline{b}^n}^{\underline{\tilde{\theta}}^n}}, v_{\underline{b}^1}^{\underline{\tilde{\theta}}^1}, \ldots, v_{\underline{b}^n}^{\underline{\tilde{\theta}}^n}\right)\right) \subset B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}),$$

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where $\lambda' = (\underline{b}^1, \ldots, \underline{b}^n)$, $\lambda = (\underline{a}^1, \ldots, \underline{a}^n)$, $a_\lambda^{\lambda'} = (\log(r_{\underline{b}^1}^{\underline{\theta}^1}/r_{\underline{b}^n}^{\underline{\theta}^n}), \ldots, \log(r_{\underline{b}^{n-1}}^{\underline{\theta}^{n-1}}/r_{\underline{b}^n}^{\underline{\theta}^n}),$ $v_{\underline{b}^1}^{\underline{\theta}^1}, \ldots, v_{\underline{b}^n}^{\underline{\theta}^n})$ for some $\underline{\theta}^j$ ending in $\underline{a}^j$, $1 \leq j \leq n$. This is accomplished by taking Δ_1 big. More precisely, since

$$\|D(k^{\underline{\theta}^j} \circ (k^{\underline{\tilde{\theta}}^j})^{-1})(z) - I\| \leq C \operatorname{diam}(G(\underline{\theta}^j \wedge \underline{\tilde{\theta}}^j)) \leq \tilde{C}\rho$$

for all $1 \leq j \leq n$, for some constant $\tilde{C}$ only depending on c_0, we can conclude that $|\log r_{\underline{b}}^{\underline{\theta}^j} - \log r_{\underline{b}}^{\underline{\tilde{\theta}}^j}| \leq \tilde{C}_1\rho$, $|v_{\underline{b}}^{\underline{\theta}^j} - v_{\underline{b}}^{\underline{\tilde{\theta}}^j}| \leq \tilde{C}_1\rho$ for all $1 \leq j \leq n$, for some constant $\tilde{C}_1$ only depending on c_0. Therefore, imposing

$$1 + 2(2n-1)\tilde{C}_1 < \Delta_1 \tag{15}$$

would be sufficient to guarantee the second property.

(iii) By equation (14), choosing c_1, ρ_0 small such that

$$\frac{\nu(J_{r+\Delta\rho} \setminus J_r) + c_1}{1 - k_5} < \frac{C_1 T_1}{2(T_1 + T_2)} \nu(J_r),$$

where C_1 is a constant such that $p^\lambda \geq C_1(\#\Lambda)^{-1}$ for all $\lambda \in \Lambda$, implies that

$$\sum_{\lambda \in \Lambda} p^\lambda \nu(F^*(\lambda)) \geq \left(1 - \frac{C_1 T_1}{2(T_1 + T_2)}\right) \nu(J_r).$$

Let $A = \{\lambda : \nu(F^*(\lambda)) \geq \nu(J_r)/2\}$, and hence,

$$\begin{aligned}\left(1 - \frac{C_1 T_1}{2(T_1 + T_2)}\right) \nu(J_r) &\leq \sum_{\lambda \in A} p^\lambda \nu(F^*(\lambda)) + \sum_{\lambda \in \Lambda \setminus A} p^\lambda \nu(F^*(\lambda)) \\ &\leq \nu(J_r) \sum_{\lambda \in A} p^\lambda + \frac{\nu(J_r)}{2} \sum_{\lambda \in \Lambda \setminus A} p^\lambda \\ &= \nu(J_r) - \frac{\nu(J_r)}{2} \sum_{\lambda \in \Lambda \setminus A} p^\lambda.\end{aligned}$$

From this inequality, we get $(C_1/2)(\#(\Lambda \setminus A)/\#\Lambda) \leq \frac{1}{2} \sum_{\lambda \in \Lambda \setminus A} p^\lambda \leq C_1 T_1/2(T_1 + T_2)$. Finally, this implies $\#A > (T_2/(T_1 + T_2))\#\Lambda$, as we wanted.

Proof of Lemma 4.1. Choose $c_3 > 0$, $\epsilon_2 > 0$ small such that

$$\left(c_3 + C_2^{-1} \frac{\epsilon_2}{\epsilon_1}\right) \rho^{-(d_1 + \cdots + d_n)} < (1 - \tau) N_{\omega(\lambda)} \quad \text{for all } \lambda \in \Lambda, \tag{16}$$

where C_2 is a constant such that $p^\lambda \geq C_2 \rho^{d_1 + \cdots + d_n}$ for all $\lambda \in \Lambda$. We suppose that c_1, ρ_0 are small enough such that

$$\frac{\nu(J_{r+\Delta\rho} \setminus J_r) + c_1}{1 - k_5} < \epsilon_2. \tag{17}$$

We will proceed by induction following the scheme

equation (13) for m $\Rightarrow$ equation (12) for m $\Rightarrow$ equation (13) for $m + 1$.

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For the base case, equation (13) for $m = 0$, notice that $V_{\Delta\rho}(E_0(\lambda)) \subset J_{r+\Delta\rho} \setminus F(\lambda)$. Therefore,

$$\nu(V_{\Delta\rho}(E_0(\lambda))) \le \nu(J_{r+\Delta\rho} \setminus J_r) + \nu(J_r \setminus F(\lambda)) \le \frac{\nu(J_{r+\Delta\rho} \setminus J_r) + c_1}{1 - k_5}.$$

Now we prove '*equation (13) for m* $\Rightarrow$ *equation (12) for m*': define $\Lambda_b = \{\lambda \in \Lambda : \nu(V_{\Delta\rho}(E_m(\lambda))) < \epsilon_1\}$, $A(\lambda) = E_m(\lambda)$ for $\lambda \in \Lambda_b$ and $A(\lambda) = \emptyset$ otherwise. We will show that $\tilde{E}_m(\lambda) \cap J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho} \subset \hat{A}(\lambda)$. Here $\hat{A}(\lambda)$ is the set from Corollary 4.1, that is, the $x \in J$ such that

$$\frac{1}{\#\Lambda_{\omega(\lambda)}} \cdot \#\{\lambda' \in \Lambda_{\omega(\lambda)} : B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J \setminus A(\lambda')\} \le 1 - \tau.$$

Using equation (17) gives $\epsilon_2 > \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))) \ge C_2\rho^{d_1+\cdots+d_n}\epsilon_1\#(\Lambda \setminus \Lambda_b)$, and hence,

$$\#(\Lambda \setminus \Lambda_b) \le C_2^{-1}\frac{\epsilon_2}{\epsilon_1}\rho^{-(d_1+\cdots+d_n)}.$$

Let $x \in \tilde{E}_m(\lambda) \cap J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho}$, using the last inequality, equation (16), and the fact that $a_\lambda^{\lambda'} \in J_{2\log(c\tilde{c}_0)}$ for all $\lambda, \lambda' \in \Lambda$, we have

$$\begin{aligned}
&\#\{\lambda' \in \Lambda_{\omega(\lambda)} : B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J \setminus A(\lambda')\} \\
&\qquad \le \#\{\lambda' \in \Lambda_b : B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J \setminus E_m(\varphi(\lambda'))\} \\
&\qquad\quad + \#(\Lambda \setminus \Lambda_b) \\
&\qquad \le \#\{\lambda' \in \Lambda' : B_{\Delta_1\rho}(x + a_\lambda^{\lambda'}) \subset J_r \setminus E_m(\varphi(\lambda'))\} \\
&\qquad\quad + C_2^{-1}\frac{\epsilon_2}{\epsilon_1}\rho^{-(d_1+\cdots+d_n)} \\
&\qquad \le \Big(c_3 + C_2^{-1}\frac{\epsilon_2}{\epsilon_1}\Big)\rho^{-(d_1+\cdots+d_n)} \\
&\qquad < (1-\tau)N_{\omega(\lambda)}.
\end{aligned}$$

Hence, $x \in \hat{A}(\lambda)$, and we have shown $\tilde{E}_m(\lambda) \cap J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho} \subset \hat{A}(\lambda)$. Clearly the family $A(\lambda)$ satisfies the hypothesis of Corollary 4.1, that is, $\nu(V_{\Delta\rho}(A(\lambda))) < \epsilon_1$. Using the corollary gives

$$\begin{aligned}
\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(\tilde{E}_m(\lambda) \cap J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho})) &\le \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(\hat{A}(\lambda))) \\
&\le k_4 \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(A(\lambda))) \\
&\le k_4 \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))). \qquad (18)
\end{aligned}$$

Now we estimate $\tilde{E}_m(\lambda)$ outside of $J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho}$. Assume that ρ_0 is small enough such that $2\Delta\rho < \frac{1}{4}\log(c\tilde{c}_0)$, $\Delta_1\rho < \frac{1}{4}\log(c\tilde{c}_0)$. By equation (11), for every $j = 1, \ldots, n$, there exist intervals

$$I_j^- \subset [-r, -r + \delta^{-1}(9\log(c\tilde{c}_0) + 2\Delta\rho)] \subset [-r, r],$$
$$I_j^+ \subset [r - \delta^{-1}(9\log(c\tilde{c}_0) + 2\Delta\rho), r] \subset [-r, r],$$

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with $\operatorname{diam}(I_j^-) = \operatorname{diam}(I_j^+) = 9\log(c\tilde{c}_0)$ such that

$$\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda)\cap J_j^-)) \le \delta \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))),$$
$$\sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda)\cap J_j^+)) \le \delta \sum_{\lambda\in\Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))),$$

where $J_j^{+(-)} = \{(t_1,\ldots,t_{n-1},v)\in J_r : t_j \in I_j^{+(-)}\}$.

For every pair (A, B) such that $A, B\subset\{1,\ldots,n-1\}$ and $A\cap B=\emptyset$, consider $y_{A,B} = (t_1(A,B),\ldots,t_{n-1}(A,B),0)\in J$ given by

$$t_j(A,B) = \begin{cases} -r+\log(c\tilde{c}_0) & \text{if } j\in A,\\ r-\log(c\tilde{c}_0) & \text{if } j\in B,\\ 0 & \text{otherwise.}\end{cases}$$

We also define sets $\tilde{J}_{A,B}, J_{A,B}\subset J$ in the following way:

$$J_{A,B} = \{(t_1,\ldots,t_{n-1},v)\in J_r : t_j\in I_j^+ \text{ if } j\in B, t_j\in I_j^- \text{ if } j\in A\},$$
$$\tilde{J}_{A,B} = \{(t_1,\ldots,t_{n-1},v)\in J_r : t_j\in \tilde{I}_j^+ \text{ if } j\in B, t_j\in \tilde{I}_j^- \text{ if } j\in A\},$$

where the interval $\tilde{I}_j^u$ has the same center as I_j^u and has length $2\log(c\tilde{c}_0)$; here $u=+$ or $-$.

Fix $\lambda\in\Lambda$. From the choice of ρ_1 and c_0, we know that there is $\lambda_{A,B}\in\Sigma_1^{fin}\times\cdots\times\Sigma_n^{fin}$ such that $y_{A,B}+a_\lambda^{\lambda_{A,B}}\in\tilde{J}_{A,B}$, and $\lambda_{A,B}\lambda'\in\Lambda'$ for all $\lambda'\in\Lambda_{\omega(\lambda_{A,B})}$.

Let $x\in V_{\Delta\rho}(\tilde{E}_m(\lambda)\cap J_r\setminus J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho})$, then there is $y\in\tilde{E}_m(\lambda)\cap J_r\setminus J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho}$, $y\in B_{\Delta\rho}(x)$, and for y, we have

$$\#\{\lambda'\in\Lambda' : B_{\Delta_1\rho}(y+a_\lambda^{\lambda'})\subset J_r\setminus E_m(\varphi(\lambda'))\} < c_3\rho^{-(d_1+\cdots+d_n)}.$$

Write $y=(t_1,\ldots,t_{n-1},v)$ and consider the sets

$$A = \{j\in[1,n-1]\cap\mathbb{Z} : t_j < -r+2\log(c\tilde{c}_0)+\Delta_1\rho\},$$
$$B = \{j\in[1,n-1]\cap\mathbb{Z} : t_j > r-2\log(c\tilde{c}_0)-\Delta_1\rho\}.$$

Since $y\notin J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho}$, we know that $A\cup B\neq\emptyset$ and we can consider $\lambda_{A,B}$. Given that $\#\Lambda_{\omega(\lambda_{A,B})}\ge L^{-1}\rho^{-(d_1+\cdots+d_n)}$, we conclude that

$$\#\{\lambda'\in\Lambda : B_{\Delta_1\rho}(y+a_\lambda^{\lambda_{A,B}\lambda'})\cap E_m(\varphi(\lambda_{A,B}\lambda'))\neq\emptyset\} \ge (L^{-1}-c_3)\rho^{-(d_1+\cdots+d_n)} > \frac{L^{-1}}{2}\rho^{-(d_1+\cdots+d_n)},$$

where we are assuming that $c_3 < L^{-1}/2$. Notice that $B_{\Delta_1\rho}(y+a_\lambda^{\lambda_{A,B}\lambda'})\cap J_r\subset J_{A,B}$ for all $\lambda'\in\Lambda_{\omega(\lambda_0)}$; therefore,

$$\#\{\lambda'\in\Lambda_{\omega(\lambda_{A,B})} : x+a_\lambda^{\lambda_{A,B}\lambda'}\in V_{\Delta\rho+\Delta_1\rho}(E_m(\varphi(\lambda_{A,B}\lambda'))\cap J_{A,B})\} > \frac{L^{-1}}{2}\rho^{-(d_1+\cdots+d_n)}.$$

Hence, $V_{\Delta\rho}(\tilde{E}_n(\lambda) \cap J_r \setminus J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho})$ is contained in

$$\bigg\{x : \sum_{(A,B)} \sum_{\lambda' \in \Lambda} 1_{V_{\Delta\rho+\Delta_1\rho}(E_m(\varphi(\lambda_{A,B}\lambda')) \cap J_{A,B}) - a_\lambda^{\lambda_{A,B}\lambda'}}(x) > \frac{L^{-1}}{2}\rho^{-(d_1+\cdots+d_n)}\bigg\},$$

where the first sum is over all pairs (A, B) such that $A, B \subset \{1, \ldots, n\}$, $A \cap B = \emptyset$, $A \cup B \neq \emptyset$. Now using Chebyshev's inequality, we get

$$\begin{aligned}
&\nu(V_{\Delta\rho}(\tilde{E}_m(\lambda) \cap J_r \setminus J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho}))\\
&\quad\leq \frac{1}{(L^{-1}/2)\rho^{-(d_1+\cdots+d_n)}} \sum_{(A,B)} \sum_{\lambda' \in \Lambda} \nu(V_{\Delta\rho+\Delta_1\rho}(E_m(\varphi(\lambda_{A,B}\lambda')) \cap J_{A,B}))\\
&\quad\leq 2\bigg(1+\frac{\Delta_1}{\Delta}\bigg)^{2n-1} L\rho^{d_1+\cdots+d_n} \sum_{(A,B)} \sum_{\lambda' \in \Lambda} \nu(V_{\Delta\rho}(E_m(\varphi(\lambda_{A,B}\lambda')) \cap J_{A,B}))\\
&\quad\leq 2\bigg(1+\frac{\Delta_1}{\Delta}\bigg)^{2n-1} C_4 L\rho^{d_1+\cdots+d_n} \sum_{(A,B)} \sum_{\lambda' \in \Lambda} \nu(V_{\Delta\rho}(E_m(\lambda') \cap J_{A,B})),
\end{aligned}$$

where $C_4 > 0$ is a constant, only depending on $\tilde{c}_0$, such that

$$\#\{\lambda' \in \Lambda_{\omega(\lambda_{A,B})} : \varphi(\lambda_{A,B}\lambda') = \lambda_1\} < C_4 \quad \text{for all } \lambda_1 \in \Lambda. \tag{19}$$

Now we will sum over λ. By the definition of Δ, we know that $(1 + \Delta_1/\Delta)^{2n-1} < k_1^{-1}$, and using $p^{\lambda'} \geq C_2\rho^{d_1+\cdots+d_n}$, we get

$$\begin{aligned}
&\sum_{\lambda \in \Lambda} p^\lambda \nu(V_{\Delta\rho}(\tilde{E}_m(\lambda) \cap J_r \setminus J_{r-2\log(c\tilde{c}_0)-\Delta_1\rho}))\\
&\quad\leq 2\bigg(1+\frac{\Delta_1}{\Delta}\bigg)^{2n-1} C_4 L\rho^{d_1+\cdots+d_n} \sum_{(A,B)} \sum_{\lambda' \in \Lambda} \nu(V_{\Delta\rho}(E_m(\lambda') \cap J_{A,B}))\\
&\quad\leq 2k_1^{-1} C_4 L\rho^{d_1+\cdots+d_n} \frac{1}{C_2\rho^{d_1+\cdots+d_n'}} \sum_{(A,B)} \sum_{\lambda' \in \Lambda} p^{\lambda'} \nu(V_{\Delta\rho}(E_m(\lambda') \cap J_{A,B}))\\
&\quad\leq 2 \cdot 3^n k_1^{-1} L C_2^{-1} C_4 \delta \sum_{\lambda \in \Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))).
\end{aligned}$$

Putting this inequality together with (18) gives

$$\begin{aligned}
\sum_{\lambda \in \Lambda} p^\lambda \nu(V_{\Delta\rho}(\tilde{E}_m(\lambda))) &\leq (k_4 + 2 \cdot 3^n k_1^{-1} L C_2^{-1} C_4 \delta) \sum_{\lambda \in \Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda)))\\
&\leq k_5 \sum_{\lambda \in \Lambda} p^\lambda \nu(V_{\Delta\rho}(E_m(\lambda))),
\end{aligned}$$

which is equation (12) for m. Here we have used equation (8) where δ was chosen.

To finish, we prove '*equation (13) for m and equation (12) for m* $\Rightarrow$ *equation (13) for* $m + 1$'. Since $E_{m+1}(\lambda) = E_0(\lambda) \cup \tilde{E}_m(\lambda)$ and $\nu(V_{\Delta\rho}(E_0(\lambda))) \leq \nu(J_{r+\Delta\rho} \setminus J_r) + c_1$, we get

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$$\begin{aligned}\sum_{\lambda\in\Lambda} p^{\lambda}\nu(V_{\Delta\rho}(E_{m+1}(\lambda))) &\le \sum_{\lambda\in\Lambda} p^{\lambda}\nu(V_{\Delta\rho}(E_0(\lambda))) + \sum_{\lambda\in\Lambda} p^{\lambda}\nu(V_{\Delta\rho}(\tilde{E}_m(\lambda)))\\ &\le \nu(J_{r+\Delta\rho}\setminus J_r) + c_1 + k_5\sum_{\lambda\in\Lambda} p^{\lambda}\nu(V_{\Delta\rho}(E_m(\lambda)))\\ &\le \nu(J_{r+\Delta\rho}\setminus J_r) + c_1 + k_5\frac{\nu(J_{r+\Delta\rho}\setminus J_r)+c_1}{1-k_5}\\ &= \frac{\nu(J_{r+\Delta\rho}\setminus J_r)+c_1}{1-k_5}. \qquad \square\end{aligned}$$

4.3. *Proof of hypothesis on Proposition 4.1.* In this subsection, we prove that there exist $0 < k_0 < 1$, $\Delta_0 > 0$ such that $\|T_\xi\| \le k_0$ for all $|\xi| \in [1, \Delta_0\rho^{-1}]$, and these constants does not depend on ρ. Remember that the operator $T_\xi : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$ is given by $T_\xi((z_\lambda)_{\lambda\in\Lambda}) = (w_\lambda)_{\lambda\in\Lambda}$, where $w_\lambda = \sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'}\xi(a_\lambda^{\lambda'})z_{\lambda'}$. Notice that $\mathbb{C}^\Lambda$ can be decomposed in two ways:

$$\mathbb{C}^\Lambda = \bigoplus_{i\in A}\mathbb{C}^{\Lambda_i}, \quad \mathbb{C}^\Lambda = \bigoplus_{j\in A}\mathbb{C}^{\Lambda^j},$$

and the operator T_ξ sends $\mathbb{C}^{\Lambda_i}$ into $\mathbb{C}^{\Lambda^i}$. Let $\|\cdot\|_i$, $\|\cdot\|^j$ be the restriction to $\mathbb{C}^{\Lambda_i}$, $\mathbb{C}^{\Lambda^j}$, respectively, of the norm $\|\cdot\|$ on Λ. Note that

$$\|z\|^2 = \sum_{i\in A}\|\pi_i(z)\|_i^2 = \sum_{j\in A}(\|\pi^j(z)\|^j)^2 \text{ for all } z \in \mathbb{C}^\Lambda,$$

where $\pi_i : \mathbb{C}^\Lambda \to \mathbb{C}^{\Lambda_i}$, $\pi^j : \mathbb{C}^\Lambda \to \mathbb{C}^{\Lambda^j}$ are the projections given by the decompositions. This implies that $\|T_\xi\| \le k_0$ if and only if $\|T_\xi|_{\mathbb{C}^{\Lambda_i}}\| \le k_0$ for all $i \in A$, where $T_\xi|_{\mathbb{C}^{\Lambda_i}}$ is the restriction $T_\xi|_{\mathbb{C}^{\Lambda_i}} : (\mathbb{C}^{\Lambda_i}, \|\cdot\|_i) \to (\mathbb{C}^{\Lambda^i}, \|\cdot\|^i)$. We start by supposing that there exist $\rho > 0$, $|\xi| \in [1, \Delta_0\rho^{-1}]$, $i \in A$ such that

$$\|T_\xi|_{\mathbb{C}^{\Lambda_i}}\| \ge (1-\eta_0)^{1/2}.$$

From this, we will derive a series of inequalities depending on parameters $\eta_0, \eta_1, \eta_2, \ldots$, each new parameter η_{j+1} will depend on η_j, not on ρ, and $\lim_{\eta_j\to 0}\eta_{j+1} = 0$. Finally, we will see that with the appropriate value of Δ_0, the last η_j will be bounded away from zero and then also η_0, and this will complete the proof.

By our assumption, there is $z = (z_\lambda) \in \mathbb{C}^{\Lambda_i}$ such that $\sum_{\lambda\in\Lambda_i} p^\lambda|z_\lambda|^2 = 1$ and for $w = T_\xi(z)$, we have $\|w\|^2 = \sum_{\lambda\in\Lambda^i} p^\lambda|w_\lambda|^2 \ge 1-\eta_0$. Note that

$$|w_\lambda|^2 = \bigg|\sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'}\xi(a_\lambda^{\lambda'})z_{\lambda'}\bigg|^2 \le \bigg(\frac{1}{\#\Lambda_i}\sum_{\lambda'\in\Lambda_i}|z_{\lambda'}|\bigg)^2 \le \frac{1}{\#\Lambda_i}\sum_{\lambda'\in\Lambda_i}|z_{\lambda'}|^2.$$

Consider the set

$$\tilde{\Lambda} = \bigg\{\lambda \in \Lambda^i : |w_\lambda|^2 \ge (1-\eta_1)\frac{1}{\#\Lambda_i}\sum_{\lambda'\in\Lambda_i}|z_{\lambda'}|^2\bigg\},$$

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where $\eta_1 = \eta_0^{1/2}$. Hence,

$$\begin{aligned}1-\eta_0 \le \sum_{\lambda\in\Lambda^i} p^\lambda |w_\lambda|^2 &= \sum_{\lambda\in\tilde\Lambda} p^\lambda |w_\lambda|^2 + \sum_{\lambda\in\Lambda^i\setminus\tilde\Lambda} p^\lambda |w_\lambda|^2 \\ &< \sum_{\lambda\in\tilde\Lambda} p^\lambda\bigg(\sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'}|z_{\lambda'}|^2\bigg) + \sum_{\lambda\in\Lambda^i\setminus\tilde\Lambda} p^\lambda\bigg((1-\eta_1)\sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'}|z_{\lambda'}|^2\bigg) \\ &= \sum_{\lambda,\lambda'\in\Lambda} p^\lambda p_\lambda^{\lambda'}|z_{\lambda'}|^2 - \eta_1 \sum_{\lambda\in\Lambda^i\setminus\tilde\Lambda} p^\lambda\bigg(\sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'}|z_{\lambda'}|^2\bigg).\end{aligned}$$

Since $\sum_{\lambda,\lambda'\in\Lambda} p^\lambda p_\lambda^{\lambda'}|z_{\lambda'}|^2 = \sum_{\lambda'\in\Lambda} p^{\lambda'}|z_{\lambda'}|^2 = 1$ and $\eta_1^2 = \eta_0$, we get

$$\begin{aligned}\eta_1 > \sum_{\lambda\in\Lambda^i\setminus\tilde\Lambda} p^\lambda\bigg(\sum_{\lambda'\in\Lambda} p_\lambda^{\lambda'}|z_{\lambda'}|^2\bigg) &= \sum_{\lambda\in\Lambda^i\setminus\tilde\Lambda} \frac{p^\lambda}{p^i}\sum_{\lambda'\in\Lambda} p^{\lambda'}|z_{\lambda'}|^2 \\ &= \sum_{\lambda\in\Lambda^i\setminus\tilde\Lambda} \frac{p^\lambda}{p^i} \ge C_2\rho^{d_1+\cdots+d_n}\#(\Lambda^i\setminus\tilde\Lambda).\end{aligned}$$

Putting $\eta_2 = \eta_1/C_2$, we obtain $\#(\Lambda^i\setminus\tilde\Lambda) \le \eta_2\rho^{-(d_1+\cdots+d_n)}$. Proceeding as in [7], define $Z_\lambda^{\lambda'} = \xi(a_\lambda^{\lambda'})z_{\lambda'}$, then

$$\frac{1}{2}\sum_{\lambda_0'\in\Lambda_i}\sum_{\lambda_1'\in\Lambda_i} p_\lambda^{\lambda_0'}p_\lambda^{\lambda_1'}|Z_\lambda^{\lambda_0'} - Z_\lambda^{\lambda_1'}|^2 = \sum_{\lambda'\in\Lambda_i} p_\lambda^{\lambda'}|z_{\lambda'}|^2 - |w_\lambda|^2.$$

If $\lambda\in\tilde\Lambda$,

$$\frac{1}{2}\sum_{\lambda_0'\in\Lambda_i}\sum_{\lambda_1'\in\Lambda_i} p_\lambda^{\lambda_0'}p_\lambda^{\lambda_1'}|Z_\lambda^{\lambda_0'} - Z_\lambda^{\lambda_1'}|^2 \le \frac{\eta_1}{\#\Lambda_i}\sum_{\lambda'\in\Lambda_i}|z_{\lambda'}|^2 = \frac{\eta_1}{p^i},$$

and hence $\sum_{\lambda_0'\in\Lambda_i}\sum_{\lambda_1'\in\Lambda_i}|Z_\lambda^{\lambda_0'} - Z_\lambda^{\lambda_1'}|^2 \le (2(\#\Lambda_i)^2/p^i)\eta_1$. Now set $\tilde Z_\lambda^{\lambda'} = z_{\lambda'} - \xi(-a_\lambda^{\lambda'})w_\lambda$, then

$$\begin{aligned}|\tilde Z_\lambda^{\lambda'}|^2 = |\xi(a_\lambda^{\lambda'})z_{\lambda'} - w_\lambda|^2 &= \bigg|\xi(a_\lambda^{\lambda'})z_{\lambda'} - \frac{1}{\#\Lambda_i}\sum_{\lambda_0'\in\Lambda_i} Z_\lambda^{\lambda_0'}\bigg|^2 \\ &= \bigg|\frac{1}{\#\Lambda_i}\sum_{\lambda_0'\in\Lambda_i}(Z_\lambda^{\lambda'} - Z_\lambda^{\lambda_0'})\bigg|^2 \le \frac{1}{\#\Lambda_i}\sum_{\lambda_0'\in\Lambda_i}|Z_\lambda^{\lambda'} - Z_\lambda^{\lambda_0'}|^2.\end{aligned}$$

Summing over λ' gives

$$\sum_{\lambda'\in\Lambda_i}|\tilde Z_\lambda^{\lambda'}|^2 \le \frac{1}{\#\Lambda_i}\sum_{\lambda'\in\Lambda_i}\sum_{\lambda_0'\in\Lambda_i}|Z_\lambda^{\lambda'} - Z_\lambda^{\lambda_0'}|^2 \le \frac{2\#\Lambda_i}{p^i}\eta_1 \le \eta_3\rho^{-(d_1+\cdots+d_n)},$$

for η_3 a constant multiple of η_1. Pick $\lambda_0, \lambda_1 \in \tilde{\Lambda}$, then $z_{\lambda'} = \xi(-a^{\lambda'}_{\lambda_0})w_{\lambda_0} + \tilde{Z}^{\lambda'}_{\lambda_0} = \xi(-a^{\lambda'}_{\lambda_1})w_{\lambda_1} + \tilde{Z}^{\lambda'}_{\lambda_1}$, and from this (redefine η_3 as $4\eta_3$),

$$\sum_{\lambda' \in \Lambda_i} |\xi(-a^{\lambda'}_{\lambda_0})w_{\lambda_0} - \xi(-a^{\lambda'}_{\lambda_1})w_{\lambda_1}|^2 = \sum_{\lambda' \in \Lambda_i} |\tilde{Z}^{\lambda'}_{\lambda_1} - \tilde{Z}^{\lambda'}_{\lambda_0}|^2 \le \eta_3 \rho^{-(d_1+\cdots+d_n)}.$$

Observe that

$$\begin{aligned}|\xi(-a^{\lambda'}_{\lambda_0})w_{\lambda_0} - \xi(-a^{\lambda'}_{\lambda_1})w_{\lambda_1}| &= |\xi(a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0})w_{\lambda_0} - w_{\lambda_1}| \\ &\ge \min\{|w_{\lambda_0}|, |w_{\lambda_1}|\} \cdot 2 \sin\left(\frac{\langle \xi, a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0}\rangle + \phi}{2}\right),\end{aligned}$$

where $\langle \xi, (t, v)\rangle = \sum_{j=1}^{n-1} \mu_j t_j + \sum_{j=1}^{n} m_j v_j \in \mathbb{T}$ for $\xi = (\mu, m) \in \mathbb{R}^{n-1} \times \mathbb{Z}^n$, and ϕ is the argument of the complex number $w_{\lambda_0}/w_{\lambda_1}$. Using this inequality together with (here we assume $\eta_1 < 3/4$, which can be assumed without loss of generality)

$$|w_\lambda|^2 \ge (1 - \eta_1)\frac{1}{\#\Lambda_i} \sum_{\lambda' \in \Lambda_i} |z_{\lambda'}|^2 = \frac{1 - \eta_1}{p^i} \ge \frac{1}{4} \quad \text{for all } \lambda \in \tilde{\Lambda},$$

we see that

$$\sum_{\lambda' \in \Lambda_i} \sin^2\left(\frac{\langle \xi, a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0}\rangle + \phi}{2}\right) \le \eta_3 \rho^{-(d_1+\cdots+d_n)}.$$

Let $\eta_4 = \eta_3^{1/3}$. The previous inequality implies that

$$\sin\left(\frac{\langle \xi, a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0}\rangle + \phi}{2}\right) \le \eta_4$$

for all $\lambda' \in \Lambda_i$, but $\eta_4 \rho^{-(d_1+\cdots+d_n)}$ λ'. From this, we get $\|\langle \xi, a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0}\rangle + \phi\| \le \eta_5$ for all $\lambda' \in \Lambda_i$, but $\eta_5 \rho^{-(d_1+\cdots+d_n)}$ λ', where η_5 is a constant multiple of η_4.

Let j_0 such that $|\xi| = |\mu_{j_0}|$ or $|\xi| = |m_{j_0}|$. We will fix some specific $\lambda_0, \lambda_1 \in \tilde{\Lambda}$ of the form

$$\begin{aligned}\lambda_0 &= (\underline{d}^0, \ldots, \underline{d}^{j_0-1}, \underline{a}^0, \underline{d}^{j_0+1}, \ldots, \underline{d}^n), \\ \lambda_1 &= (\underline{d}^0, \ldots, \underline{d}^{j_0-1}, \underline{a}^1, \underline{d}^{j_0+1}, \ldots, \underline{d}^n).\end{aligned}$$

Notice that λ_0, λ_1 only differ on the j_0 coordinate. Moreover, if $j_0 \ne n$, we have $a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0} = (0, \ldots, 0, \log(r^{\underline{\theta}^1}_{\underline{b}^{j_0}}/r^{\underline{\theta}^0}_{\underline{b}^{j_0}}), 0, \ldots, 0, v^{\underline{\theta}^1}_{\underline{b}^{j_0}} - v^{\underline{\theta}^0}_{\underline{b}^{j_0}}, 0, \ldots, 0)$, where $\lambda' = (\underline{b}^1, \ldots, \underline{b}^n)$, and $\underline{\theta}^0$, $\underline{\theta}^1 \in \Sigma^-_{j_0}$ end with $\underline{a}^0, \underline{a}^1$, respectively. If $j_0 = n$, then $a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0} = (\log(r^{\underline{\theta}^0}_{\underline{b}^n}/r^{\underline{\theta}^1}_{\underline{b}^n}), \ldots, \log(r^{\underline{\theta}^0}_{\underline{b}^n}/r^{\underline{\theta}^1}_{\underline{b}^n}), 0, \ldots, 0, v^{\underline{\theta}^1}_{\underline{b}^n} - v^{\underline{\theta}^0}_{\underline{b}^n})$. We remark that $a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0}$ only depends on the j_0 Cantor set K_{j_0}.

Given that K_{j_0} is not essentially affine, there is $\underline{\tilde{\theta}}^0, \underline{\tilde{\theta}}^1 \in \Sigma^-_{j_0}$ and $x_0 \in K^{\underline{\tilde{\theta}}^0}_{j_0}$ such that

$$D^2[k^{\underline{\tilde{\theta}}^1} \circ (k^{\underline{\tilde{\theta}}^0})^{-1}](x_0) \ne 0.$$

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For any $\underline{\theta}^0, \underline{\theta}^1 \in \Sigma^-_{j_0}$, we define $F_{\underline{\theta}^0,\underline{\theta}^1} := k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^0})^{-1}$. Since $x_0 \in K^{\tilde{\underline{\theta}}^0}_{j_0}$, then $DF_{\tilde{\underline{\theta}}^0,\tilde{\underline{\theta}}^1}(x_0)$ is a conformal matrix. Denote by $C \subset GL(2,\mathbb{R})$ the set of 2×2 conformal matrices. Let $P : U \to C$ be a smooth function from a neighborhood $U \subset GL(2,\mathbb{R})$ of $DF_{\tilde{\underline{\theta}}^0,\tilde{\underline{\theta}}^1}(x_0)$ into C, such that $P(A) = A$ for all $A \in C \cap U$. We will use the notation $\mathbb{D}F_{\underline{\theta}^0,\underline{\theta}^1}(x) = P(DF_{\underline{\theta}^0,\underline{\theta}^1}(x))$. The properties of P and the fact that K_{j_0} is not essentially real allow us to conclude that $D\mathbb{D}F_{\tilde{\underline{\theta}}^0,\tilde{\underline{\theta}}^1}(x_0) = D^2F_{\tilde{\underline{\theta}}^0,\tilde{\underline{\theta}}^1}(x_0)$. Now notice that C can be naturally identified with $\mathbb{C}^*$ and in this sense, we can choose a branch of logarithm log defined in $P(U)$ (for U small). Then Lemma 2.3 will imply that $\beta := D \log \mathbb{D}F_{\tilde{\underline{\theta}}^0,\tilde{\underline{\theta}}^1}(x_0) \neq 0$ is a conformal matrix.

In the rest of the proof, we will make an abuse of notation, where $v^{\underline{\theta}^1}_{\underline{b}^{j_0}} - v^{\underline{\theta}^0}_{\underline{b}^{j_0}}$ will not represent an element of $\mathbb{T}$ but the imaginary part of $\log \exp[(v^{\underline{\theta}^1}_{\underline{b}^{j_0}} - v^{\underline{\theta}^0}_{\underline{b}^{j_0}})i]$. In this way, we have chosen a representative in the class defined by $v^{\underline{\theta}^1}_{\underline{b}^{j_0}} - v^{\underline{\theta}^0}_{\underline{b}^{j_0}}$. Define the following vectors in $\mathbb{R}^2$:

$$d^{\lambda'}_{\lambda_1,\lambda_0} = \left(\log \frac{r^{\underline{\theta}^1}_{\underline{b}^{j_0}}}{r^{\underline{\theta}^0}_{\underline{b}^{j_0}}},\, v^{\underline{\theta}^1}_{\underline{b}^{j_0}} - v^{\underline{\theta}^0}_{\underline{b}^{j_0}}\right),$$

and

$$\tilde{\xi} = \begin{cases} (\mu_{j_0}, m_{j_0}) & \text{if } j_0 \neq n, \\ (-(\mu_1 + \cdots + \mu_{n-1}), m_n) & \text{if } j_0 = n. \end{cases}$$

Notice that $1 \le |\tilde{\xi}| \le n\Delta_0\rho^{-1}$ and $\langle \xi, a^{\lambda'}_{\lambda_1} - a^{\lambda'}_{\lambda_0}\rangle = \langle \tilde{\xi}, d^{\lambda'}_{\lambda_1,\lambda_0}\rangle \bmod 2\pi\mathbb{Z}$, where the $\langle\cdot,\cdot\rangle$ in the right-hand side of the equation refers to the usual inner product on $\mathbb{R}^2$.

Since $k^{\underline{\theta}}$ depends continuously on $\underline{\theta}$, we get that

$$|D \log \mathbb{D}F_{\underline{\theta}^0,\underline{\theta}^1}(x) - \beta| \le \delta_1 \tag{20}$$

for all $\underline{\theta}^0$, $\underline{\theta}^1$, x close enough to $\tilde{\underline{\theta}}^0$, $\tilde{\underline{\theta}}^1$, x_0; the value of δ_1 will be fixed later. We assume that η_2 is small such that the proportion of $\tilde{\Lambda}$ inside Λ^i is big enough to exist $\lambda_0, \lambda_1 \in \tilde{\Lambda}$, with the form specified before, verifying that $\underline{\theta}^0$, $\underline{\theta}^1$ are close to $\tilde{\underline{\theta}}^0$, $\tilde{\underline{\theta}}^1$ so that equation (20) holds. (Here we also need to suppose that ρ_0 is small enough.) From now on, λ_0 and λ_1 are fixed as these values.

Now fix $\underline{c}^0 \in \Sigma^{fin}_{j_0}$ such that any x in the convex hull of $G^{\underline{\theta}^0}(\underline{c}^0)$ is close enough to x_0 to have equation (20). Denote by $\Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$ all the elements of $\Sigma_{j_0}(\tilde{c}_0, \rho)$ starting with $\underline{c}^0$. This is a positive proportion of $\Sigma_{j_0}(\tilde{c}_0, \rho)$ (independent of ρ). Then, if we assume η_5 small enough, we can guarantee that for a proportion of $\underline{b} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$, as big as we want, there exist $\underline{b}^j \in \Sigma_j(\tilde{c}_0, \rho)$, $j \neq j_0$, such that $\lambda' = (\underline{b}^1, \ldots, \underline{b}^{j_0-1}, \underline{b}, \underline{b}^{j_0+1}, \ldots, \underline{b}^n)$ verifies

$$|\langle \tilde{\xi}, d^{\lambda'}_{\lambda_1,\lambda_0}\rangle + \phi - 2m(\underline{b})\pi| \le \eta_5, \tag{21}$$

where $m(\underline{b})$ is an integer depending on $\underline{b}$. Denote the set of such $\underline{b}$ by $\tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$.

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For simplicity, write $F = k^{\underline{\theta}^1} \circ (k^{\underline{\theta}^0})^{-1}$, instead of $F_{\underline{\theta}^0,\underline{\theta}^1}$, then we have

$$F(c^{\underline{\theta}^0}_{\underline{b}}) = c^{\underline{\theta}^1}_{\underline{b}}, \quad DF(c^{\underline{\theta}^0}_{\underline{b}}) = \|DF(c^{\underline{\theta}^0}_{\underline{b}})\| \cdot R_{v^{\underline{\theta}^1}_{\underline{b}} - v^{\underline{\theta}^0}_{\underline{b}}}. \tag{22}$$

We will show that the distance between $\log \mathbb{D}F(c^{\underline{\theta}^0}_{\underline{b}})$ and $d^{\lambda'}_{\lambda_1,\lambda_0}$ is of order ρ for any $\lambda' \in \Lambda$. Let $z_0, z_1 \in G^{\underline{\theta}^0}(\underline{b})$ such that $r^{\underline{\theta}^0}_{\underline{b}} = |z_0 - z_1|$. Using Taylor expansion at $c^{\underline{\theta}^0}_{\underline{b}}$, we get $F(z_0) - F(z_1) = DF(c^{\underline{\theta}^0}_{\underline{b}})(z_0 - z_1) + O(|z_0 - z_1|^2)$, where the constant in the O notation does not depend on ρ, $\underline{b}$, $\underline{\theta}^1$, or $\underline{\theta}^0$. Hence,

$$\begin{aligned} r^{\underline{\theta}^0}_{\underline{b}} \|DF(c^{\underline{\theta}^0}_{\underline{b}})\| &= |z_0 - z_1| \|DF(c^{\underline{\theta}^0}_{\underline{b}})\| \\ &\le |F(z_0) - F(z_1)| + O(|z_0 - z_1|^2) \le r^{\underline{\theta}^1}_{\underline{b}} + O((r^{\underline{\theta}^0}_{\underline{b}})^2). \end{aligned}$$

From this and the fact that $r^{\underline{\theta}^0}_{\underline{b}}$ is of order ρ, we get $\|DF(c^{\underline{\theta}^0}_{\underline{b}})\| - r^{\underline{\theta}^1}_{\underline{b}}/r^{\underline{\theta}^0}_{\underline{b}} \le O(\rho)$. A similar argument gives $r^{\underline{\theta}^1}_{\underline{b}}/r^{\underline{\theta}^0}_{\underline{b}} - \|DF(c^{\underline{\theta}^0}_{\underline{b}})\| \le O(\rho)$. Therefore, $|\|DF(c^{\underline{\theta}^0}_{\underline{b}})\| - r^{\underline{\theta}^1}_{\underline{b}}/r^{\underline{\theta}^0}_{\underline{b}}| \le O(\rho)$.

Now, given the fact that $\|DF(c^{\underline{\theta}^0}_{\underline{b}})\|$ and $r^{\underline{\theta}^1}_{\underline{b}}/r^{\underline{\theta}^0}_{\underline{b}}$ are uniformly bounded away from zero, we obtain that there is a constant $C_3 > 0$, independent of ρ, such that

$$|\log \|DF(c^{\underline{\theta}^0}_{\underline{b}})\| - (\log r^{\underline{\theta}^1}_{\underline{b}} - \log r^{\underline{\theta}^0}_{\underline{b}})| \le C_3 \rho. \tag{23}$$

Given $\underline{b}^1, \underline{b}^2 \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$, by the choice of $\underline{c}^0$, using Taylor approximation and equation (20), we will have that

$$\log \mathbb{D}F(c^{\underline{\theta}^0}_{\underline{b}^1}) - \log \mathbb{D}F(c^{\underline{\theta}^0}_{\underline{b}^2}) = \beta_1(c^{\underline{\theta}^0}_{\underline{b}^1} - c^{\underline{\theta}^0}_{\underline{b}^2}) \tag{24}$$

for some β_1 such that $\|\beta_1 - \beta\| \le \delta_1$.

The idea to finish the proof is the following: we use equation (21) to see that the set of $d^{\lambda'}_{\lambda_1,\lambda_0}$ projected to the line generated by $\tilde{\xi}$ is close to an arithmetic progression, then two points will be either very close or very far from each other. Equations (22), (23), (24) allow to translate this fact about $d^{\lambda'}_{\lambda_1,\lambda_0}$ to the analogous one about the set of $c^{\underline{\theta}^0}_{\underline{b}}$. Finally, we will use the fact that K_{j_0} is not essentially real to estimate $|c^{\underline{\theta}^0}_{\underline{b}^1} - c^{\underline{\theta}^0}_{\underline{b}^2}|$ from $\langle \tilde{\xi}, \beta_1(c^{\underline{\theta}^0}_{\underline{b}^1} - c^{\underline{\theta}^0}_{\underline{b}^2})\rangle$, and thus it will happen that $|c^{\underline{\theta}^0}_{\underline{b}^1} - c^{\underline{\theta}^0}_{\underline{b}^2}|$ is either too big or too small which will bring us into a contradiction with the boundedness of the geometry of the Cantor set.

Any pair $\underline{b}_1, \underline{b}_2 \in \tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$ should verify one of two options.

(i) If $m(\underline{b}_1) = m(\underline{b}_2)$, using equation (21) for λ'_1, λ'_2 associated to $\underline{b}_1, \underline{b}_2$, respectively, we get $|\langle \tilde{\xi}, d^{\lambda'_1}_{\lambda_1,\lambda_0}\rangle - \langle \tilde{\xi}, d^{\lambda'_2}_{\lambda_1,\lambda_0}\rangle| \le 2\eta_5$. This together with equations (22), (23) give

$$|\langle \tilde{\xi}, \log \mathbb{D}F(c^{\underline{\theta}^0}_{\underline{b}_1}) - \log \mathbb{D}F(c^{\underline{\theta}^0}_{\underline{b}_2})\rangle| \le 2\eta_5 + 2C_3|\tilde{\xi}|\rho.$$

Considering equation (24) leads to

$$|\langle \beta_1^T \tilde{\xi}, c^{\underline{\theta}^0}_{\underline{b}_1} - c^{\underline{\theta}^0}_{\underline{b}_2}\rangle| = |\langle \tilde{\xi}, \beta_1(c^{\underline{\theta}^0}_{\underline{b}_1} - c^{\underline{\theta}^0}_{\underline{b}_2})\rangle| \le 2\eta_5 + 2C_3|\tilde{\xi}|\rho,$$

where β_1^T is the transpose of β_1.

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(ii) If $m(\underline{b}_1) \neq m(\underline{b}_2)$, a similar process arrives to

$$|\langle \beta_1^T \tilde{\xi}, c_{\underline{b}_1}^{\theta^0} - c_{\underline{b}_2}^{\theta^0} \rangle| \geq \pi - 2C_3|\tilde{\xi}|\rho.$$

Now we use the hypothesis that K_{j_0} is not essentially real. First, we choose a constant $C_5 > 0$, depending only on $\tilde{c}_0$ and the Cantor set K_{j_0}, such that for any $\underline{a} \in \Sigma_{j_0}^{fin}$, one has

$$\{f_{\underline{a}}^{-1}(G(\underline{b})) : \underline{b} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{a})\} \subset \{G(\underline{b}) : \underline{b} \in \Sigma_{j_0}(C_5, \tilde{\rho})\}$$

for some $\tilde{\rho} > 0$, which depends on ρ and $\underline{a}$. Lemma 2.2 proves that there is an angle $\alpha \in (0, \pi/2)$ and numbers $\rho_2 > 0$, $a \in (0, 1)$ such that for any limit geometry $k^{\underline{\theta}}$, $x \in G^{\underline{\theta}}(\theta_0)$, line L, $s \in \mathbb{A}_{j_0}$, D discretization of $K_{j_0}(\theta_0, s)$ of order less than ρ_2,

$$\#\{\underline{a} \in D : \quad G^{\underline{\theta}}(\underline{a}) \cap \text{Cone}(x, L, \alpha) \neq \emptyset\} \leq a \cdot \#D.$$

Remember that a discretization D of $K_{j_0}(\theta_0, s)$ of order ρ is a subset of $\Sigma_{j_0}(C_5, \rho)$ such that $\bigcup_{\underline{a} \in D} K_{j_0}(\underline{a}) = K_{j_0}(\theta_0, s)$ for some pre-fixed constant C_5.

Fix δ_1 by requiring that $\|\beta_1 - \beta\| < \delta_1$ implies that

$$m(\beta)/2 \leq m(\beta_1^T) \leq \|\beta_1^T\| \leq 2\|\beta\|$$

and the angle between $\beta^T w$ and $\beta_1^T w$ is less than $\alpha/2$ for any $w \in \mathbb{R} \setminus \{0\}$. Remember that $m(A) = \inf_{w \neq 0} \frac{|Aw|}{|w|}$ and that β is conformal, and hence $m(\beta^T) = m(\beta) = \|\beta\| = \|\beta^T\|$. Fix $\tilde{a} \in (a, 1)$. Assuming η_5 small enough, we can guarantee that

$$\#\tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0) > \tilde{a} \cdot \#\Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^0).$$

This allows us to find a finite sequence $\underline{c}^0, \ldots, \underline{c}^m$ of elements of Σ^{fin} such that:

- $\underline{c}^{j+1}$ starts with $\underline{c}^j$ and has one more letter;
- $\#(\Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^j) \cap \tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)) > \tilde{a} \cdot \#\Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^j)$;
- $\Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^j) \cap \tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0) \not\subset \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^{j+1})$ (actually, for this property to be true, we take $\tilde{a}$ big such that $\#\Sigma_{\tilde{c}_0}(\rho, \underline{c}^{j+1}) < \tilde{a} \cdot \Sigma_{\tilde{c}_0}(\rho, \underline{c}^j)$);
- $\underline{c}^j \in \Sigma_{j_0}(\tilde{c}_0, \rho)$ only for $j = m$.

Fix an integer $m_0 < m$ such that for any $\underline{b} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^j)$ and $j < m_0$, we have $f_{\underline{c}^j}^{-1}(G(\underline{b})) = G(\underline{b}')$ for $\underline{b}' \in \Sigma_{j_0}(C_5, \tilde{\rho})$, for $\tilde{\rho} < \rho_2$ (this only requires that $m - m_0$ is big enough). For each $\underline{c}^j$, $j < m$, we will choose two elements $\underline{a}^{1,j}, \underline{a}^{2,j} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^j) \cap \tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$ in the following way.

- First, we choose any $\underline{a}^{1,j} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^j s') \cap \tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$, where s' is a letter in $\mathbb{A}_{j_0}$ such that $\underline{c}^{j+1} \neq \underline{c}^j s'$.
- If $j \geq m_0$, then we choose any $\underline{a}^{2,j} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^{j+1}) \cap \tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$.
- Suppose $j < m_0$. Given $\underline{b} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^{j+1})$, it can be written as

$$\underline{b} = (c_{-k}^j, \ldots, c_{-1}^j, c_0^j, c_0^{j+1}, b_1, \ldots, b_p),$$

where c_0^j, c_0^{j+1} are the last letters of $\underline{c}^j, \underline{c}^{j+1}$, respectively. Using this notation, we can define the set

$$D = \{(c_0^j, c_0^{j+1}, b_1, \ldots, b_p) \in \Sigma^{fin} : (c_{-k}^j, \ldots, c_{-1}^j, c_0^j, c_0^{j+1}, b_1, \ldots, b_p)$$

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$$\in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^{j+1})\}.$$

This set is a discretization of $K_{j_0}(c_0^j, c_0^{j+1})$ and by our choice of m_0, it has order less than ρ_2. Now use Lemma 2.2 for the limit geometry $K_{j_0}^{\underline{\theta}^0 \underline{c}^j}$, point $x = (F_{\underline{c}^j}^{\underline{\theta}^0})^{-1}(c_{\underline{a}^{1,j}}^{\underline{\theta}^0})$, and line L such that $F_{\underline{c}^j}^{\underline{\theta}^0}(L)$ is orthogonal to the line generated by $\beta^T \tilde{\xi}$. Hence,

$$\#\{\underline{a} \in D : \quad G^{\underline{\theta}^0 \underline{c}^j}(\underline{a}) \cap \mathrm{Cone}(x, L, \alpha) \neq \emptyset\} \leq a \cdot \#D,$$

and then

$$\begin{aligned}&\#\{\underline{b} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^{j+1}) : G^{\underline{\theta}^0}(\underline{b}) \cap \mathrm{Cone}(c_{\underline{a}^{1,j}}^{\underline{\theta}^0}, F_{\underline{c}^j}^{\underline{\theta}^0}(L), \alpha) \neq \emptyset\} \\ &\quad \leq a \cdot \#\Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^{j+1}).\end{aligned}$$

Since $\tilde{a} > a$, then there are elements $\underline{a} \in \Sigma_{j_0}(\tilde{c}_0, \rho, \underline{c}^{j+1}) \cap \tilde{\Sigma}_{j_0}(\tilde{c}_0, \rho, \underline{c}^0)$ such that

$$G^{\underline{\theta}^0}(\underline{a}) \cap \mathrm{Cone}(c_{\underline{a}^{1,j}}^{\underline{\theta}^0}, F_{\underline{c}^j}^{\underline{\theta}^0}(L), \alpha) = \emptyset.$$

We choose $\underline{a}^{2,j}$ as any such element. It easily follows from the choice of $\underline{a}^{2,j}$ that

$$\measuredangle(c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}, F_{\underline{c}^j}^{\underline{\theta}^0}(L)) > \alpha.$$

Let $\tilde{L}$ be the line orthogonal to the vector $\beta_1^T \tilde{\xi}$. The previous inequality and the choice of δ_1 implies that

$$\measuredangle(c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}, \tilde{L}) > \alpha/2. \tag{25}$$

Now we have all the ingredients to finish the proof. For any j, the pair $\underline{a}^{1,j}, \underline{a}^{2,j}$ verifies either option (i) or (ii), note that option (ii) implies

$$\begin{aligned}|c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}| \geq \frac{|\langle \beta_1^T \tilde{\xi}, c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}\rangle|}{\|\beta_1^T\| \cdot |\tilde{\xi}|} &\geq \pi \|\beta_1\|^{-1} |\tilde{\xi}|^{-1} - 2C_3 \|\beta_1\|^{-1} \rho \\ &\geq \frac{\pi \|\beta\|^{-1}}{2} |\tilde{\xi}|^{-1} - 4C_3 \|\beta\|^{-1} \rho \\ &\geq \bigg(\frac{\pi \|\beta\|^{-1}}{2n} \Delta_0^{-1} - 4C_3 \|\beta\|^{-1}\bigg) \rho.\end{aligned}$$

Hence, choosing Δ_0 small enough, we can guarantee that option (ii) is not verified for $j = n - 1$. However, if $\underline{a}^{1,j}, \underline{a}^{2,j}$ verifies option (i), then using equation (25), we get

$$\begin{aligned}|c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}| \cdot \sin(\alpha/2) &\leq |c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}| \cdot \sin \measuredangle(c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}, \tilde{L}) \\ &= |c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}| \cdot \cos \measuredangle(c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}, \mathbb{R}\beta_1^T \tilde{\xi}) \\ &= \frac{|\langle \beta_1^T \tilde{\xi}, c_{\underline{a}^{1,j}}^{\underline{\theta}^0} - c_{\underline{a}^{2,j}}^{\underline{\theta}^0}\rangle|}{|\beta_1^T \tilde{\xi}|}\end{aligned}$$

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$$\leq \frac{2\eta_5 + 2C_3|\tilde{\xi}|\rho}{1/2\|\beta\| \cdot |\tilde{\xi}|}$$
$$\leq 4\eta_5|\tilde{\xi}|^{-1}\|\beta\|^{-1} + 4C_3\rho\|\beta\|^{-1},$$

and we obtain

$$|c^{\theta^0}_{\underline{a}^{1,j}} - c^{\theta^0}_{\underline{a}^{2,j}}| \leq (\sin(\alpha/2))^{-1}(4\eta_5|\tilde{\xi}|^{-1}\|\beta\|^{-1} + 4C_3\rho\|\beta\|^{-1}).$$

Hence, assuming η_5 and ρ small enough, we can guarantee that option (i) is not verified for $j = 0$. Therefore, there exists j such that $\underline{a}^{1,j}$, $\underline{a}^{2,j}$ verifies option (ii) and $\underline{a}^{1,j+1}$, $\underline{a}^{2,j+1}$ verifies option (i). From the inequalities obtained, we see that

$$\frac{|c^{\theta^0}_{\underline{a}^{1,j+1}} - c^{\theta^0}_{\underline{a}^{2,j+1}}|}{|c^{\theta^0}_{\underline{a}^{1,j}} - c^{\theta^0}_{\underline{a}^{2,j}}|} \leq \left(\sin\frac{\alpha}{2}\right)^{-1} \frac{4\eta_5|\tilde{\xi}|^{-1}\|\beta\|^{-1} + 4C_3\rho\|\beta\|^{-1}}{\frac{\pi\|\beta\|^{-1}}{2}|\tilde{\xi}|^{-1} - 4C_3\|\beta\|^{-1}\rho}$$
$$= \left(\sin\frac{\alpha}{2}\right)^{-1} \frac{4\eta_5 + 4C_3\rho|\tilde{\xi}|}{(\pi/2) - 4C_3\rho|\tilde{\xi}|}$$
$$\leq \left(\sin\frac{\alpha}{2}\right)^{-1} \frac{4\eta_5 + 4nC_3\Delta_0}{(\pi/2) - 4nC_3\Delta_0},$$

where we used $|\tilde{\xi}| \in [1, n\Delta_0\rho^{-1}]$. We obtain

$$\frac{|c^{\theta^0}_{\underline{a}^{1,j+1}} - c^{\theta^0}_{\underline{a}^{2,j+1}}|}{|c^{\theta^0}_{\underline{a}^{1,j}} - c^{\theta^0}_{\underline{a}^{2,j}}|} \leq \left(\sin\frac{\alpha}{2}\right)^{-1} \frac{4\eta_5 + 4nC_3\Delta_0}{(\pi/2) - 4nC_3\Delta_0},$$

notice that the right-hand side of the inequality goes to zero as Δ_0 and η_5 go to zero; however, the left-hand side is bounded away from zero thanks to the bounded geometry of the Cantor set K_{j_0}. We conclude that for Δ_0 small enough, η_5 is bounded away from zero, as we wanted to prove.

5. *Final remarks*

In this final section, we want to comment, very briefly and without details, about our work in progress, together with Araujo, about a complex version of Palis's conjecture on the arithmetic difference of Cantor sets.

More precisely, given a pair of conformal Cantor sets (K, K') with $HD(K) + HD(K') > 2$, we are close to prove that arbitrarily close to this pair, we can find another pair $(\tilde{K}, \tilde{K}')$ such that $int(\tilde{K} - \tilde{K}') \neq \emptyset$.

To make sense of the previous statement, we need to explain the topology in the space of Cantor sets. Fix a set of *letters* $\mathbb{A}$ and *admissible* pairs $B \subset \mathbb{A} \times \mathbb{A}$, so that they generate a subshift of finite type Σ^+. Denote by $\Omega^m_{\Sigma^+}$ the set of all conformal C^m Cantor sets with the same associated subshift Σ^+. We introduce a topology in this space by defining a basis of neighborhoods. For each $K \in \Omega^m_{\Sigma^+}$ and $\delta > 0$, we define the δ-neighborhood around K as the set of C^m conformal Cantor sets K' defined by an expanding map $g' : V' \to \mathbb{C}$, $\sqcup_a G'(a) \subset V'$ such that $G'(a) \subset V_\delta(G(a))$ for every $a \in \mathbb{A}$ (where $V_\delta(G(a))$ refers to the

δ-neighborhood of $G(a)$) and the restrictions to $V \cap V'$ of the functions g, g' are δ close in the C^m topology.

The definition of this topology is similar to that for regular Cantor sets in the real line; moreover, it verifies that if we have regular conformal Cantor sets associated to horseshoes for automorphisms of $\mathbb{C}^2$ (as considered in [1]), then those Cantor sets depend continuously on the automorphisms.

Our proof follows the strategy of [7] adapted to conformal Cantor sets. We prove the existence of a *recurrent compact set* for a pair $(\tilde{K}, \tilde{K}')$ close to the original pair. The perturbed pair $(\tilde{K}, \tilde{K}')$ is selected, within a family of possible perturbations, using a probabilistic argument. To be able to perturb properly, we use the fact that in the definition of conformal Cantor sets, the derivative Dg is conformal at the Cantor set and this is only required at these points. The not essentially real hypothesis also plays an important role in this part.

For the construction of the recurrent compact set, we first define a set of 'good' scales, and for this, we use a complex version of Marstrand's projection theorem together with the scale recurrence lemma we proved in these pages. The conformal version of the scale recurrence lemma is a key tool for our proof of a complex version of Palis's conjecture.

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