FINITE PROJECTIVE PLANES WITH AFFINE SUBPLANES

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1. Introduction. A well-known theorem, due to R. H. Bruck ([4], p. 398), is the following:

If a finite projective plane of order n *has a projective subplane of order* $m < n$, *then either* $n = m^2$ *or* $n \geq m^2 + m$.

In this paper we prove an analagous theorem concerning affine subplanes of finite projective planes (Theorem 1). We then construct a number of examples; in particular we find all the finite Desarguesian projective planes containing affine subplanes of order 3 (Theorem 2).

We express our thanks to R. H. Bruck for suggesting the problem[2], and to J. F. Rigby for valuable suggestions relative to an inequality of Theorem 1.

2. Basic Definitions. A *projective plane* is a system of undefined elements called *points* and *lines*, together with a relation of *incidence*, subject to the following axioms:

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(2) The problem was raised by Bruck in his lectures at the Canadian Mathematical Congress Seminar in Saskatoon (August 1963). This paper is a direct outgrowth of that seminar.

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1. Any two distinct points are incident with one and only one line.

2. Any two distinct lines are incident with one and only one point.

3. There exist four points, no three of which are incident with a line.

The usual terminology of incidence, - "lies on", "passes through", "collinear", "concurrent", etc - is employed.

If a projective plane π is finite (i.e. contains a finite number of points and lines), it is easily shown that every line contains the same number of points. If this number is $n+1$, we say that π is <u>of order</u> n. A projective plane of order n contains n^2+n+1 points and n^2+n+1 lines, and $n+1$ lines pass through each point. Also, it is easily seen that $n \geq 2$ ([4], pp. 346-8).

A projective plane is <u>Desarguesian</u> if the Theorem of Desargues holds universally ([4], p. 351). A characteristic property of a finite Desarguesian plane is that it can be coordinatized by a field. Such a plane has order p^α (p is prime, $\alpha = 1, 2, \ldots$), is coordinatized by $GF(p^\alpha)$, and is denoted by the symbol $PG(2, p^\alpha)$. Conversely any Galois field $GF(p^\alpha)$ gives rise to a finite projective plane $PG(2, p^\alpha)$ ([2], pp. 324-327).

An <u>affine</u> plane π of order n can be derived from a projective plane π of order n by the well-known process of removing a line of π and the $n+1$ points lying on it. More formally, π satisfies the following two axioms, as well as Axiom 1 for π:

2.′ Given a line ℓ and a point P not on ℓ, there is exactly one line through P which fails to meet ℓ.

3.′ There exist three non-collinear points.

Clearly π contains n^2 points, n on each line, and n^2+n lines, $n+1$ through each point. Moreover, the lines

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of π can be divided into $n+1$ mutually exclusive <u>parallel classes</u> of n lines each, two lines belonging to the same parallel class if and only if they are parallel (i.e. have no common point). The Desarguesian affine geometry of order p^α is denoted by the symbol $EG(2, p^\alpha)$ ([2], p. 329).

A (projective or affine) <u>subplane</u> π_o of a given projective plane π is a set of points and lines of π which themselves form a (projective or affine) plane under the same incidence relation. Thus a line of π which contains two points of π_o must be a line of π_o.

3. <u>The Orders of Affine Subplanes.</u> Let π be a projective plane of order n containing an affine subplane π_o of order $m < n$.

THEOREM 1. *For each point P in π let $k = k(P)$ denote the number of lines of π_o which pass through P. Then*

(i) *If $k = 0$ for some point,* $n \geq m^2 - 1$

(ii) *If $k \geq 1$ for each point,* $n \leq m^2 - 1$

(iii) *If $k = 1$ for some point,* $n \geq m^2 - m + 1$

(iv) *If $k \geq 2$ for each point,* $n = 4$ *and* $m = 3$.

<u>Proof</u>: For any point $P \notin \pi_o$, the k lines through P which belong to π_o account for a total of km points of π_o (m on each line). The remaining $n+1-k$ lines through P each contain no more than one point of π_o. Hence

$$km + n + 1 - k \geq m^2 . \tag{3.1}$$

Setting $k = 0$ in (3.1) we have result (i).

Suppose now that $k \geq 1$ for each point. Let ℓ be a line, not in π_o, which contains exactly one point Q of π_o (such a line must exist, since there are $n+1$ lines of π through Q,

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only $m+1$ of which belong to π_o). Now for each of the other n points on ℓ, $k \geq 1$. But there are only $m^2 + m - (m+1) = m^2 - 1$ lines of π_o which do not pass through Q. Hence $n \leq m^2 - 1$. Thus we have proved result (ii).

We now consider case (iii). If $k = 1$ for some point, then substitution in (3.1) yields

$$n \geq m^2 - m .$$

Let us suppose that $n = m^2 - m$. Then, using result (i), we have that $k \geq 1$ for each point of π. Any two parallel lines of π_o meet in a point P of π for which $k(P) > 1$. These two lines each contain m points of π_o. There are therefore at most $m^2 - 2m$ other lines through P which contain points of π_o. But since $m > 1$ and $n = m^2 - m$, there must be at least one line through P which contains no points of π_o. Let ℓ be such a line, and let X be any point on ℓ. If $k(X) = 1$, then the lines through X fall into at least three mutually exclusive classes.

(a) the line ℓ

(b) the line of π_o through X

(c) the $m^2 - m$ distinct lines through X which contain exactly one point of π_o. Counting these lines, we have the inequality

$$n + 1 \geq 1 + 1 + m^2 - m ,$$

$$n \geq m^2 - m + 1 .$$

But $n = m^2 - m$. Therefore $k(X) \geq 2$ for each point X on ℓ. There are $m^2 - m + 1$ points on ℓ and $m^2 + m$ lines of π_o, and so

$$2(m^2 - m + 1) \le m^2 + m ,$$

$$m^2 \le 3m - 2 ,$$

$$m \le 3 - 2/m < 3 .$$

Therefore $m = 2$ and $n = m^2 - m = 2$. But this is a contradiction, since $n > m$. Therefore $n > m^2 - m$, and we have result (iii).

Finally, suppose that $k \ge 2$ for each point of π, and let ℓ be a line of π_o. There are m^2 lines in π_o which intersect ℓ in points of π_o. Excluding ℓ and these m^2 lines, we have left a set γ of $(m^2 + m) - (m^2 + 1) = m - 1$ lines of π_o to intersect ℓ in the remaining $n + 1 - m$ points of ℓ which do not belong to π_o. Since ℓ itself belongs to π_o, and $k \ge 2$, we must have

$$m - 1 \ge n + 1 - m ,$$

$$2m - 2 \ge n .$$

Substituting in (3.1), we have

$$km + (2m - 2) + 1 - k \ge m^2 ,$$

$$k(m - 1) \ge m^2 - 2m + 1 = (m - 1)^2 ,$$

$$k \ge m - 1 .$$

Let P and Q be any two distinct points of ℓ which are not points of π_o (there are at least two since $n > m$). Then, as we have just seen, $k(P) \ge m - 1$, and therefore at least $m - 2$ lines of the set γ pass through P. Thus at most one line of γ fails to meet P; since $k(Q) \ge 2$ there must be one such line, and it must meet Q. Moreover, since γ is now exhausted, P and Q are the only points of ℓ, apart from the points of π_o, i.e.

$$n = m + 1 .$$

It also follows that $k(P) = m - 1$ and $k(Q) = 2$. Reversing the roles of P and Q, we have $k(Q) = m - 1 = 2$ and $k(P) = 2 = m - 1$, i.e.

$$m = 3 .$$

Result (iv) is therefore proved.

That the case $n = 4$, $m = 3$ actually exists will be shown in the next section. We shall also show there that equality in case (ii) of Theorem 1 is attained by the example $n = 3$, $m = 2$. The example $n = 7$, $m = 3$ of the next section shows that equality is possible in case (iii). However, in case (i) we have only examples in which $n = m^2$; $EG(2, p^\alpha)$ is a subplane of $PG(2, p^{2\alpha})$ ([2], pp. 334-5). Whether strict equality can be attained in case (i) is therefore an open question.

4. Examples. Let A, B, C, D be a complete quadrangle in $PG(2, 3)$ (cf. Fig. 1). Let $E = AB.CD$, $F = AC.BD$, $G = AD.BC$, $H = AB.FG$, $I = AC.GE$, $J = AD.EF$, $K = BC.EF$, $L = BD.GE$, $M = CD.FG$. Now it is easily checked that the 7 points A, B, C, D, E, F, G are all distinct; the coincidence of any two implies that three of the four points A, B, C, D are collinear, which is not true. Moreover, the coincidence of any two of the 6 points H, I, J, K, L, M implies the coincidence of two of the 7 points $A, \ldots, G$; for example, if $J = M$, then $F = D$. Therefore the 6 points $H, \ldots, M$ are distinct. Likewise these 6 points are distinct from A, B, C and D; finally they are also distinct from E, F and G, since the diagonal points of a quadrangle are not collinear in $PG(2, 3)$ ([2], p. 341). The 13 distinct points $A, \ldots, M$ must therefore be the 13 points of $PG(2, 3)$.

The 4 vertices and 6 sides of the complete quadrangle A, B, C, D may be interpreted as the 4 points and 6 lines of an affine subplane $EG(2, 2)$ in $PG(2, 3)$. The above considerations show that $k(P) \geq 1$ for each point P in $PG(2, 3)$. Thus we have an example of equality in case (ii) of Theorem 1.

With the aid of Fig. 1 we easily find 9 of the 13 lines in $PG(2, 3)$. Denoted by the 4 points which they contain, they are ABEH, ACFI, ADGJ, BCGK, BDFL, CDEM, EFJK, FGHM,

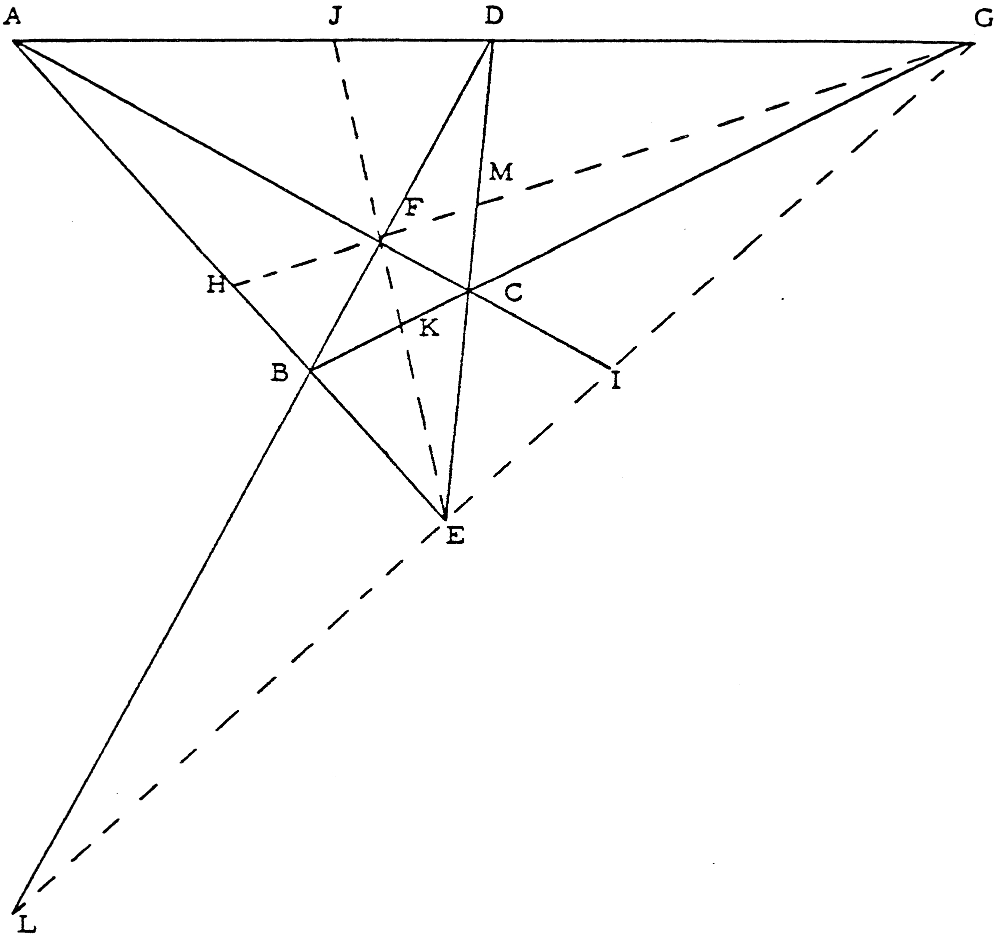

Fig. 1

and GEIL. One soon picks out the other four lines; they must be AKLM, BIJM, CHJL, and DHIK.

If we remove the line GEIL and the points on it, we have the 9 points and 12 lines of EG(2, 3) (cf. Fig. 2). The configuration formed by the points and lines of EG(2, 3) is the famous configuration of the 9 inflexion points of a cubic curve ([1], p. 19). Thus EG(2, 3) is a subplane of the complex projective plane. We now prove a theorem which determines all the finite Desarguesian projective planes which contain EG(2, 3).

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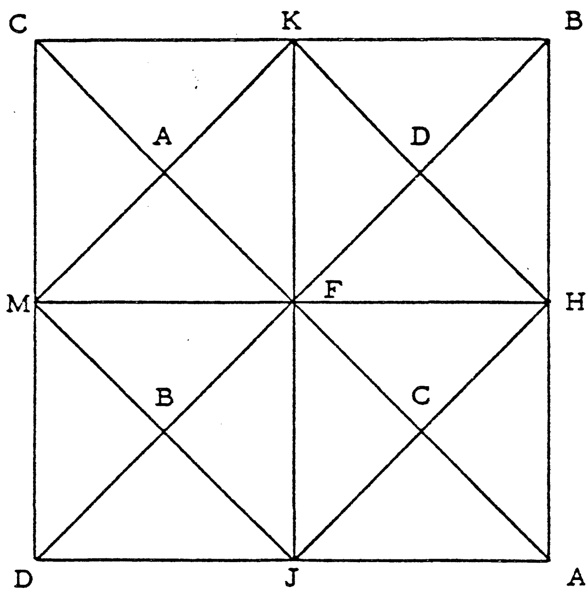

Fig. 2

THEOREM 2. Let p be a prime and α a positive integer. Then $PG(2, p^\alpha)$ contains a subplane $EG(2, 3)$ if and only if either $p = 3$ or $p^\alpha \equiv 1 \pmod 3$.

Proof: (cf [3], p. 237, ex. 3). Suppose that $\pi = PG(2, p^\alpha)$ contains $\pi_o = EG(2, 3)$. Using the notation of Fig. 2, we introduce homogeneous coordinates into π in such a way that $F = (0, 0, 1)$, $H = (1, 0, 1)$, $K = (0, 1, 1)$, and $B = (1, 1, 1)$. Then D, being the intersection of FB and HK, has coordinates $(1, 1, 2)$.

We assign the coordinates $(1, 0, -t)$ to M, noting that $t \neq -1$. Taking collinearity relations into account, we can now give coordinates to the three remaining points of π_o:

C, being the intersection of BK and DM, must have coordinates $(1 + t, t, t)$. (Note that if π_o is to exist, C must be distinct from M; thus $t \neq 0$).

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$A = FC.BH$ has coordinates $(1+t, t, 1+t)$.

$J = FK.DA$ has coordinates $(0, 1, 1+t)$.

In assigning coordinates to the 9 points of π_o we have taken into account the collinearity of 9 of the 12 different triples of collinear points. The collinearity of a tenth triple, $B(1, 1, 1)$, $M(1, 0, -t)$, and $J(0, 1, 1+t)$ follows automatically, since the value of the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -t \\ 0 & 1 & 1+t \end{vmatrix}$$

is zero. We have yet to consider the collinearity of A, K, and M, and of C, J, and H. The first yields the equation

$$\begin{vmatrix} 1+t & t & 1+t \\ 0 & 1 & 1 \\ 1 & 0 & -t \end{vmatrix} = 0$$

which reduces to

$$t^2 + t + 1 = 0 . \tag{4.1}$$

The second collinearity relation also yields (4.1). Thus a necessary condition that π contains π_o is that the field $GF(p^\alpha)$ must contain the roots of equation (4.1). A reversal of the above argument shows that this condition is also sufficient.

We observe first that (4.1) is satisfied by $t = 1$ if and only if $p = 3$. If $p \neq 3$, then the roots of (4.1) are the primitive cube roots of unity in $GF(p^\alpha)$. Now $GF(p^\alpha)$ contains an element of multiplicative order 3 if and only if the order of its multiplicative group, which is cyclic ([2], p. 248), is divisible by 3. But this order is $p^\alpha - 1$. This completes the proof of Theorem 2.

The first two cases of planes $PG(2, p^\alpha)$ having subplanes $EG(2, 3)$ are $PG(2, 4)$ and $PG(2, 7)$; they illustrate cases (iv) and (iii) respectively of Theorem 1. It is also interesting to note that $PG(2, p^{2r})$ $(r = 1, 2, \ldots)$ contains $EG(2, 3)$ regardless of the prime p involved. For if $p \neq 3$, then $p \equiv \pm 1 \pmod 3$, and therefore $p^2 \equiv 1 \pmod 3$. On the other hand, $PG(2, p^{2r+1})$ $(p \neq 3)$ can contain $EG(2,3)$ when and only when $p \equiv 1 \pmod 3$.

5. Irregular Subplanes. We shall say that a subplane of a given finite projective plane is irregular if its order does not divide the order of the whole plane.

The only known irregular projective subplanes are subplanes of order two embedded in non-Desarguesian planes of odd order (cf, for example, [5], p. 39). Every plane contains an affine subplane of order two, namely the vertices and sides of any quadrangle.

We have exhibited irregular affine subplanes of order 3 embedded in Desarguesian planes. It is well known that a Desarguesian plane cannot contain an irregular projective subplane. Now consider a non-Desarguesian plane π which contains a Desarguesian subplane π_1, where π_1 in turn contains $EG(2,3)$ as an irregular subplane. We then have $EG(2,3)$ embedded in π. This leads naturally to the following questions:

Are there (non-Desarguesian) planes which contain irregular projective subplanes of order 3?

Are there irregular (affine or projective) subplanes whose order is greater than 3?

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