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DENSITY, SMITAL PROPERTY AND QUASICONTINUITY

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Abstract

Based on the abstract version of the Smital property, we introduce an operator DS. We use it to characterise the class of semitopological abelian groups, for which addition is a quasicontinuous operation.

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1. Introduction

The starting point for our work is a well known property of the σ-field $\mathcal{L}$ of Lebesgue measurable sets on the real line $\mathbb{R}$.

Smital Lemma. *For any set $A \in \mathcal{L}$ of positive measure and for any dense set $P \subset \mathbb{R}$, the set $A + P$ is of full measure (that is, its complement is a null set).*

In the above statement, three types of structure on $\mathbb{R}$ are used: a group operation, a topology and a pair comprising a σ-field and a σ-ideal. Looking for a more general approach, we observe that the σ-field $\mathcal{L}$ and the σ-ideal of sets of measure zero can be replaced by any pair $(\mathcal{A}, \mathcal{I})$, where $\mathcal{A}$ is a field of subsets of X and $\mathcal{I} \subset \mathcal{A}$ is an ideal. In this context, the Smital property was studied in [2]. It is easily seen that no topology is needed to formulate and study the Smital property. It is enough to define a family of dense sets, as in [6] and [4].

In this paper, firstly we describe the properties of an operator D, which assigns to an arbitrary family of sets the family of dense sets with respect to the family. Then we define an operator DS related directly to the Smital property and generalising the operator D in a certain sense. The most interesting result of this paper is a characterisation of the class of semitopological abelian groups essentially bigger than a class of all topological abelian groups, namely, the class of semitopological abelian groups, for which addition is a quasicontinuous operation. This characterisation is based on the operators DS and D.

2. The operator D

Let X be a nonempty set, $\mathcal{F} \subset P(X)\backslash\{\emptyset\}$, $\mathcal{F} \neq \emptyset$. We say that the set $P \subset X$ is *dense* with respect to $\mathcal{F}$ if and only if $F \cap P \neq \emptyset$ for every $F \in \mathcal{F}$. Let $D(\mathcal{F})$ denote the family of all sets dense with respect to $\mathcal{F}$. Moreover, let $\mathcal{F}\uparrow$ denote the family of all supersets of sets from $\mathcal{F}$, that is,

$$\mathcal{F}\uparrow := \{G \subset X : (\exists F \in \mathcal{F})(F \subset G)\}.$$

We say that two families $\mathcal{F}$ and $\mathcal{G}$ are mutually coinitial if

$$\forall_{F\in\mathcal{F}} \exists_{G\in\mathcal{G}}\, G \subset F \quad \wedge \quad \forall_{G\in\mathcal{G}} \exists_{F\in\mathcal{F}}\, F \subset G.$$

PROPOSITION 2.1. *Let $\mathcal{F}, \mathcal{G}$ be arbitrary nonempty families of nonempty subsets of X. Then:*

(1) $\mathcal{G} \subset D(\mathcal{F}) \iff \mathcal{F} \subset D(\mathcal{G})$;
(2) $\mathcal{G} \subset \mathcal{F} \Rightarrow D(\mathcal{F}) \subset D(\mathcal{G})$;
(3) $(D(\mathcal{F}))\uparrow = D(\mathcal{F})$;
(4) *$\mathcal{G}$ and $\mathcal{F}$ are mutually coinitial if and only if* $D(\mathcal{G}) = D(\mathcal{F})$;
(5) $D(\mathcal{F}\uparrow) = D(\mathcal{F})$;
(6) $D^{(2)}\mathcal{F} = \mathcal{F}\uparrow$; *and*
(7) $D^{(n)}(\mathcal{F}) = \begin{cases} D(\mathcal{F}) & \text{if } n \text{ is odd,} \\ \mathcal{F}\uparrow & \text{if } n \text{ is even.} \end{cases}$

PROOF. Statements (1)–(4) can be easily obtained from the definition; (5) follows from (4) and from the fact that $\mathcal{F}\uparrow$ and $\mathcal{F}$ are mutually coinitial. Statement (7) is an immediate consequence of (6) and (5).

Let us prove (6). From (1) and the fact that $D(\mathcal{F}\uparrow) \subset D(\mathcal{F}\uparrow)$, it follows that $\mathcal{F}\uparrow \subset D(D(\mathcal{F}\uparrow)) = D^{(2)}(\mathcal{F}\uparrow)$. Suppose that there exists a set $A \in D^{(2)}(\mathcal{F}\uparrow)\backslash\mathcal{F}\uparrow$. Since $A \notin \mathcal{F}\uparrow$, the complement A' of A satisfies $A' \in D(\mathcal{F}) = D(\mathcal{F}\uparrow)$, contrary to $A \in D^{(2)}(F)$. □

The operator D was recently considered in connection with Marczewski–Burstin representations in [8]. Using the property (6), we can improve [8, Theorem 1.1]. Following the ideas of Burstin and notation from [1], for an arbitrary family $\mathcal{F}$ we define

$$S^0(\mathcal{F}) := \{A \subset X : \forall_{F\in\mathcal{F}} \exists_{G\in\mathcal{F}}\, G \subset F\backslash A\}.$$

LEMMA 2.2. *Let $\mathcal{F}$ be an arbitrary family of nonempty sets. Then*

$$S^0(\mathcal{F}\uparrow) = S^0(\mathcal{F}).$$

PROOF. Let $A \in S^0(\mathcal{F}\uparrow)$ and $F \in \mathcal{F} \subset \mathcal{F}\uparrow$. Then there exists $G \in \mathcal{F}\uparrow$ such that $G \subset F\backslash A$. But, from the definition of $\mathcal{F}\uparrow$, there exists $G' \in \mathcal{F}$ with $G' \subset G$. Hence $A \in S^0(\mathcal{F})$.

Now let $A \in S^0(\mathcal{F})$. Let $F \in \mathcal{F}\uparrow$ and $F' \in \mathcal{F}$ such that $F' \subset F$. Let $G \in \mathcal{F}$ with $G \subset F'\backslash A$. Of course, $G \in \mathcal{F}\uparrow$ and $G \subset F\backslash A$. Hence $A \in S^0(\mathcal{F}\uparrow)$. □

THEOREM 2.3 (Compare [8], Theorem 1.1). *Let $\mathcal{F}$ be an arbitrary family of nonempty sets. Then*

$$S^0(D(\mathcal{F})) = S^0(\mathcal{F}).$$

PROOF. First, we show that $S^0(\mathcal{F}) \subset S^0(D(\mathcal{F}))$. Let $A \in S^0(\mathcal{F})$ and $P \in D(\mathcal{F})$. Observe that $P_2 = P \backslash A \in D(\mathcal{F})$. Indeed, let $F \in \mathcal{F}$. Since $A \in S^0(\mathcal{F})$, there exists $F_2 \in \mathcal{F}$ with $F_2 \subset F \backslash A$. Since $P \in D(\mathcal{F})$, $F_2 \cap P \neq \emptyset$. Hence $F \cap P_2 \neq \emptyset$.

From the first part of the proof and Lemma 2.2,

$$S^0(D(\mathcal{F})) \subset S^0(D^{(2)}(\mathcal{F})) = S^0(\mathcal{F}\uparrow) = S^0(\mathcal{F}). \qquad \square$$

The notion of density is strictly connected with the resolvability of a topological space and also, in a more general setting, with the resolvability of a measurable space $(X, \mathcal{A}, \mathcal{I})$ [4] and with structure resolvability [6]. If $\mathcal{F}$ is a family of nonempty subsets of X, we say that the family $(X, \mathcal{F})$ is α-resolvable if there exists a family of cardinality α of pairwise disjoint sets which are dense with respect to $\mathcal{F}$.

Illanes [5] proved that if a topological space is n-resolvable for every n, then it is also $\aleph_0$-resolvable. The following example shows that an analogous theorem does not hold for an arbitrary family of sets.

EXAMPLE 2.4. Let $X = \mathbb{N}$. For $m, n, t \in \mathbb{N}$, let $m \sim_t n$ if and only if $m \equiv n \mod t$. Let

$$\mathcal{F} = \{[m]_{\sim_p} : m \in \mathbb{N}, p \text{ is a prime number}\}.$$

Then, for every $n \in \mathbb{N}$, there exists a disjoint subfamily of $\mathcal{F}$ with cardinality n, but there is no infinite disjoint subfamily of $\mathcal{F}$. The same is true for the family $\mathcal{F}\uparrow$. Hence the family $D(\mathcal{F})$ is n-resolvable for every finite n but it is not $\aleph_0$-resolvable.

QUESTION 2.5. *Is the thesis of Illanes true for any family $\mathcal{A} \backslash \mathcal{I}$, where $\mathcal{A}$ is an arbitrary field of sets and $\mathcal{I} \subset \mathcal{A}$ is an ideal?*

3. The operator *DS*

Let $(X, +)$ be an abelian group, let τ be a topology on X, let $\mathcal{A} \subset P(X)$ be an algebra and let $\mathcal{I} \subset \mathcal{A}$ be an ideal in $P(X)$.

We say that a triple $(\mathcal{A}, \mathcal{I}, \tau)$ has the Smital property if, for any set $A \in \mathcal{A} \backslash \mathcal{I}$ and any dense set $P \subset X$, the set $(A + P)'$ (the complement of $A + P$) belongs to $\mathcal{I}$ (see [2]).

Inspired by this idea, we consider a family of sets fulfilling a similar role to that played by the dense sets in the Smital property, but in a slightly more general setting. We assume that $(X, +)$ is an abelian group and that $\mathcal{F} \subset P(X) \backslash \{\emptyset\}$ and let $\mathcal{I} \subset P(X)$ be an arbitrary ideal of sets.

Define

$$DS_{\mathcal{I}}(\mathcal{F}) := \{P \subset X : (\forall F \in \mathcal{F})((F + P)' \in \mathcal{I})\}.$$

The operator $DS_{\mathcal{I}}$ has the following properties.

PROPOSITION 3.1. *Let $\mathcal{F}, \mathcal{G}$ be arbitrary nonempty families of nonempty subsets of X. Then:*

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(1) $\mathcal{G} \subset DS_{\mathcal{I}}(\mathcal{F}) \iff \mathcal{F} \subset DS_{\mathcal{I}}(\mathcal{G})$;
(2) $\mathcal{G} \subset \mathcal{F} \Rightarrow DS_{\mathcal{I}}(\mathcal{F}) \subset DS_{\mathcal{I}}(\mathcal{G})$; $\mathcal{I} \subset \mathcal{J} \Rightarrow DS_{\mathcal{I}}(\mathcal{F}) \subset DS_{\mathcal{J}}(\mathcal{F})$;
(3) $(DS_{\mathcal{I}}(\mathcal{F}))\!\uparrow = (DS_{\mathcal{I}}(\mathcal{F})) = (DS_{\mathcal{I}}(\mathcal{F}\!\uparrow))$;
(4) *if at least one of $\mathcal{F}$ and $\mathcal{I}$ is invariant with respect to translations, then so is $DS_{\mathcal{I}}(\mathcal{F})$; and*
(5) $DS_{\mathcal{I}}^{(n+2)}(\mathcal{F}) = DS_{\mathcal{I}}^{(n)}(\mathcal{F})$ *for* $n \geq 1$.

PROOF. Statements (1)–(4) are straightforward consequences of the definition of $DS_{\mathcal{I}}$. For (5), observe that from $DS_{\mathcal{I}}(\mathcal{F}) = DS_{\mathcal{I}}(\mathcal{F})$ and (1) we obtain $\mathcal{F} \subset DS_{\mathcal{I}}^{(2)}(\mathcal{F})$. Hence, by (2), $DS_{\mathcal{I}}^{(1)}(\mathcal{F}) \supset DS_{\mathcal{I}}^{(3)}(\mathcal{F})$. But, again from $DS_{\mathcal{I}}^{(2)}(\mathcal{F}) = DS_{\mathcal{I}}^{(2)}(\mathcal{F})$ and (1), it follows that $DS_{\mathcal{I}}^{(1)}(\mathcal{F}) \subset DS_{\mathcal{I}}^{(3)}(\mathcal{F})$. □

The following proposition shows that the operator DS is, in some sense, a generalisation of the operator D.

PROPOSITION 3.2. *Let $(X, +)$ be an abelian group and let $\mathcal{F}$ be a family of sets invariant with respect to $(X, +)$ and $\mathcal{E} = \{\emptyset\}$. Then*

$$DS_{\mathcal{E}}(\mathcal{F}) = D(\mathcal{F}).$$

PROOF. Let $P \subset X$. Suppose that $P + F \neq X$ for some $F \in \mathcal{F}$. Since $\mathcal{F}$ is invariant, we can assume that $0 \notin P + F$. Then $P \cap (-F) = \emptyset$. But $(-F) \in \mathcal{F}$. As a result, $P \notin D(\mathcal{F})$.

Let $P \notin D(\mathcal{F})$. Then $P \cap F = \emptyset$ for some $F \in \mathcal{F}$. Hence $0 \notin P + (-F)$, and so $P \notin DS_{\mathcal{E}}(\mathcal{F})$. □

In [2], it is shown that the Smital property implies the Steinhaus property, understood in the following way. Let $(X, +)$ be an abelian group, let τ be a topology on X, let $\mathcal{A} \subset P(X)$ be an algebra and let $\mathcal{I} \subset \mathcal{A}$ be an ideal in $P(X)$. We say that the structure $(\mathcal{A}, \mathcal{I}, \tau)$ has the Steinhaus property if, for any sets $A, B \in \mathcal{A} \backslash \mathcal{I}$, the set $A - B$ has an interior point.

The next proposition shows the connection between the Steinhaus and Smital properties in our setting. We need the following notation: $\widetilde{\mathcal{F}} = \{F_1 - F_2 : F_1, F_2 \in \mathcal{F}\}$.

PROPOSITION 3.3. *Let X be an abelian group, let $\mathcal{A}$ be a σ-field of subsets of X and let $\mathcal{I} \subset \mathcal{A}$ be a σ-ideal. Let $P \subset X$-countable. Then*

$$P \in DS_{\mathcal{I}}(\mathcal{A} \backslash \mathcal{I}) \quad \textit{if and only if } P \in D(\widetilde{\mathcal{A} \backslash \mathcal{I}}).$$

PROOF. '$\Rightarrow$' Suppose $P \notin D(\widetilde{\mathcal{A} \backslash \mathcal{I}})$, so that there exist sets $A, B \in \mathcal{A} \backslash \mathcal{I}$ such that $(A - B) \cap P = \emptyset$. Then the set $B + P$ is disjoint from A, and hence $P \notin DS_{\mathcal{I}}(\mathcal{A} \backslash \mathcal{I})$.

'$\Leftarrow$' Observe that, for every $A \in \mathcal{A} \backslash \mathcal{I}$, the set $B := (A + P)'$ is measurable. Suppose that $B \notin \mathcal{I}$. Then $(B - A) \cap P = \emptyset$ and $P \notin D(\widetilde{\mathcal{A} \backslash \mathcal{I}})$. □

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4. The case of semitopological groups

Let $(X, +)$ be an abelian group equipped with a topology τ. The structure $(X, +, \tau)$ is called a semitopological group if τ is invariant with respect to the group operations, and it is called a topological group if the group operations are continuous with respect to τ. It is easy to observe that a sufficient condition for a semitopological group to be topological is the continuity of the operation '+'.

We recall that a function $f : X \to Y$ between topological spaces is quasicontinuous at a point $x \in X$ if, for any neighbourhood $O_x \subset X$ of x and any neighbourhood $O_{f(x)} \subset Y$ of $f(x)$, there exists a nonempty open set $U \subset O_x$ such that $f(U) \subset O_{f(x)}$. A function $f : X \to Y$ is quasicontinuous if it is quasicontinuous at each point $x \in X$. Formally, quasicontinuous functions were introduced by Kempisty [7] and then they were studied in many settings.

Let $\tau^* = \tau \backslash \{\emptyset\}$. Then $D(\tau^*)$ is the family of all dense sets with respect to τ.

Theorem 4.1. *Let $(X, +, \tau)$ be a semitopological group. Let $\mathcal{ND}$ denote the ideal of nowhere dense sets. Then the following statements are equivalent.*

(1) *The operation ' + ' : $X \times X \to X$ is quasicontinuous.*
(2) *The families $\widetilde{\tau^*}$ and τ^* are mutually coinitial.*
(3) $DS_{\mathcal{ND}}(\tau^*) = D(\tau^*)$.

Proof. (1) ⇒ (2). Let $G \in \tau^*$ and $x_0 \in G$. Let V_1, V_2 be open neighbourhoods of x_0 and zero, respectively. By the quasicontinuity of '+', there exist open sets U_1, U_2 such that $U_1 \times U_2 \subset V_1 \times V_2$ and $U_1 + U_2 \subset G$.

(2) ⇒ (1). Let $a_1, a_2 \in X$, let G be an open neighbourhood of $a_1 + a_2$ and let V_1, V_2 be open neighbourhoods of a_1 and a_2, respectively. Let $H = G \cap (V_1 + V_2)$. Let U_1, U_2 be open sets such that $U_1 + U_2 \subset H$. Let $b \in U_1 + U_2$. Then there exist $b_1 \in U_1, b_2 \in U_2, c_1 \in V_1, c_2 \in V_2$ such that $b_1 + b_2 = c_1 + c_2 = b$ and

$$(V_1 \cap (U_1 - b_1 + c_1)) + (V_2 \cap (U_2 + b_2 - c_2)) \subset H \subset G.$$

(2) ⇒ (3). The inclusion $DS_{\mathcal{ND}}(\tau^*) \supset D(\tau^*)$ is obvious. Assume that $P \notin D(\tau^*)$. Then there exists $G \in \tau^*$ such that $G \cap P = \emptyset$. From (2), there exist $H_1, H_2 \in \tau^*$ such that $H_1 + H_2 \subset G$. Hence $(P + (-H_2)) \cap H_1 = \emptyset$. Since $(-H_2) \in \tau^*$, we obtain $P \notin DS_{\mathcal{ND}}(\tau^*)$.

(3) ⇒ (2). Let $V \in \tau^*$. Then $P = X \backslash V$ is not dense, and hence $P \notin DS_{\mathcal{ND}}(\tau^*)$. There exists $G \in \tau^*$ such that $(G + P)' \notin \mathcal{ND}$. But the set $(G + P)'$ is closed, so there exists $H \in \tau^*$ with $H \subset (G + P)'$. Hence $H + (-G) \subset V$. □

Corollary 4.2. *For every topological abelian group $(X, +, \tau)$,*

$$DS_{\mathcal{ND}}(\tau^*) = D(\tau^*).$$

The next example shows that there exists a semitopological abelian group with a quasicontinuous operation '+' which is not a topological group.

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EXAMPLE 4.3. Let $X = \mathbb{R}^2$. Let

$$MC(r) = \{(x, y) \in \mathbb{R}^2 : (|y| < |x/2| < r) \vee (|x| < |y/2| < r) \vee (x = y = 0)\}.$$

The topology τ_{MC} generated by the family $\{MC(r) + (x, y) : x, y, r \in \mathbb{R}, r > 0\}$ is called the Maltese cross topology. Since $\widetilde{\tau_{MC}}$ consists of sets open in the natural topology, $\mathbb{R}^2$ equipped with τ_{MC} is not a topological group, but, of course, it is semitopological. However, since τ_{MC} and τ_{nat} are mutually coinitial, the space satisfies condition (2) of Theorem 4.1, so '+' is quasicontinuous. The authors express thanks to T. Banakh for suggesting this example.

5. Classical examples

EXAMPLE 5.1. Let $X = \mathbb{R}$, let $\mathcal{L}$ be the family of Lebesgue measurable sets and let $\mathcal{N}$ be the σ-ideal of Lebesgue null sets. Moreover, let $\mathbb{D}$ denote the family of dense sets with respect to the Euclidean topology on $\mathbb{R}$. Then

$$DS_{\mathcal{N}}(\mathcal{L}\backslash\mathcal{N}) = \mathbb{D}.$$

Indeed, the inclusion '$\supset$' is simply the Smital lemma. Now let $P \notin \mathbb{D}$ and let (a, b) be an interval such that $(a, b) \cap P = \emptyset$. Then $L = (0, \frac{1}{2}(b - a)) \in \mathcal{L}\backslash\mathcal{N}$ and $(\frac{1}{2}(b + a), b) \subset (L + P)'$. Hence $P \notin DS_{\mathcal{N}}(\mathcal{L}\backslash\mathcal{N})$.

An analogous argument works in the next example.

EXAMPLE 5.2. Let $X = \mathbb{R}$, let $\mathcal{B}$ be the family of sets with the Baire property and let $\mathcal{K}$ be the σ-ideal of meagre sets. Then

$$DS_{\mathcal{K}}(\mathcal{B}\backslash\mathcal{K}) = \mathbb{D}.$$

From Proposition 3.1, it follows that

$$(\mathcal{L}\backslash\mathcal{N})\Uparrow \subset DS^{(2)}_{\mathcal{N}}(\mathcal{L}\backslash\mathcal{N}) = DS_{\mathcal{N}}(\mathbb{D}).$$

The question arises whether it is true that $(\mathcal{L}\backslash\mathcal{N})\Uparrow = DS^{(2)}_{\mathcal{N}}(\mathcal{L}\backslash\mathcal{N})$. Similarly, is it true that $DS^{(2)}_{\mathcal{K}}(\mathcal{B}\backslash\mathcal{K}) = (\mathcal{B}\backslash\mathcal{K})\Uparrow$? The negative answer for both questions was supplied recently by Filipczak *et al.* in [3].

EXAMPLE 5.3. Let H be the Hamel basis on $\mathbb{R}$ such that $1 \in H$. Let A be the linear span (over $\mathbb{Q}$) of the set $H\backslash\{1\}$. Let $C = A \triangle (-\infty, 0]$ (where $\triangle$ denotes the symmetric difference). Then C is not Lebesgue measurable and does not have the Baire property, but, for every dense set $D \subset \mathbb{R}$, the set $C + D$ covers the whole real line apart from at most one point.

QUESTION 5.4. *How can we characterise the family* $DS^{(2)}_{\mathcal{N}}(\mathcal{L}\backslash\mathcal{N})$?

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