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MANIFOLDS THAT FAIL TO BE CO-DIMENSION 2 FIBRATORS NECESSARILY COVER THEMSELVES

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Abstract

Let N be a closed s-Hopfian n-manifold with residually finite, torsion free $\pi_1(N)$ and finite $H_1(N)$. Suppose that either $\pi_k(N)$ is finitely generated for all $k \geq 2$, or $\pi_k(N) \cong 0$ for $1 < k < n-1$, or $n \leq 4$. We show that if N fails to be a co-dimension 2 fibrator, then N cyclically covers itself, up to homotopy type.

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1. Introduction

The advantage of approximate fibration is that on one hand there exists an exact homotopy sequence but on the other hand there are more such approximate fibrations available. (See [3–5] for the definition and usefulness of approximate fibrations.)

To detect approximate fibrations, Daverman introduced the concept of co-dimension 2 fibrator as follows [7].

A closed n-manifold N^n is a *co-dimension* 2 *fibrator* (respectively, a *co-dimension* 2 *orientable fibrator*) if, whenever $p : M \to B$ is a proper map from an arbitrary (respectively, orientable) $(n+2)$-manifold M to a 2-manifold B such that each $p^{-1}(b)$ is shape equivalent to N, then $p : M \to B$ is an approximate fibration.

All closed s-Hopfian manifolds with either trivial fundamental group or Hopfian fundamental group and nonzero Euler characteristic or hyper-Hopfian fundamental group are known to be co-dimension 2 fibrators [6, 7, 16, 18, 19].

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For the sake of simplicity, we say that a closed manifold N satisfies **(CP)** if N cyclically covers itself nontrivially, up to homotopy type, and say that a closed manifold N is **(F)** if N fails to be a co-dimension 2 fibrator. Not only the torus and the Klein bottle but also S^1-bundles satisfy **(CP)**.

It is well known [7, Theorem 4.2] that if a closed manifold N satisfies **(CP)**, then N is **(F)**. What can we say about the converse? Recently, Daverman [9] proves that the converse is not true in general, by showing that $S^3 \times L(p, q)$ fails to be a co-dimension 2 fibrator but it cannot cover itself cyclically, where $L(p, q)$ is a Lens space.

It is natural to ask *when the converse is true*. A continuation of earlier investigations launched on [17], this paper adds evidence for a claim that the converse is true for many interesting manifolds. More precisely, we have the following

THEOREM. *Suppose that a closed s-Hopfian n-manifold N with residually finite, torsion free $\pi_1(N)$ and finite $H_1(N)$ is* **(F)**. *Then, N satisfies* **(CP)**, *provided either*

(1) *$\pi_k(N)$ is finitely generated for all $k \geq 2$, or*
(2) *$\pi_k(N) \cong 0$ for $1 < k < n - 1$, in particular, aspherical manifold, or*
(3) *$n \leq 4$.*

2. Definitions and preliminaries

Throughout this paper, the symbols $\simeq$ and $\cong$ denote a homotopy equivalence and an isomorphism, respectively. All manifolds are understood to be finite dimensional, connected and metric.

For a closed manifold N, a proper map $p : M \to B$ is *N-like* if each fiber $p^{-1}(b)$ is shape equivalent to N. For simplicity, we shall assume that each fiber $p^{-1}(b)$ in an N-like map to be an ANR having the homotopy type of N.

Let N and N' be (not necessarily closed) n-manifolds and $f : N \to N'$ be a map. Denote the kth cohomology group of N with $\mathbb{G}$-coefficients and compact supports by $H^k_C(N; \mathbb{G})$. If both N and N' are orientable, then the *degree of f* is the nonnegative integer d such that the induced endomorphism $f^* : H^n_C(N; \mathbb{Z}) \cong \mathbb{Z} \to H^n_C(N'; \mathbb{Z}) \cong \mathbb{Z}$ amounts to multiplication by d, up to sign. In general, the *degree* mod 2 *of f* is the integer $d \in \{0, 1\}$ such that the induced endomorphism $f^* : H^n_C(N; \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H^n_C(N'; \mathbb{Z}_2) \cong \mathbb{Z}_2$ amounts to multiplication by d.

Suppose that N is a closed n-manifold and a proper map $p : M \to B$ is N-like. Let G be the set of all fibers, that is, $G = \{p^{-1}(b) : b \in B\}$. Put $C = \{p(g) \in B : g \in G$ and there exist a neighbourhood U_g of g in M and a retraction $R_g : U_g \to g$ such that $R_g \mid g' : g' \to g$ is a degree one map for all $g' \in G$ in $U_g\}$, and $C' = \{p(g) \in B : g \in G$ and there exist a neighbourhood U_g of g in M and a retraction $R_g : U_g \to g$ such

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that $R_g \mid g' : g' \to g$ is a degree one mod 2 map for all $g' \in G$ in $U_g\}$. Call C the *continuity set of* p and C' the mod 2 *continuity set of* p. Coram and Duvall showed [5] that C and C' are dense, open subsets of B.

Call a closed manifold N *Hopfian* if it is orientable and every degree one map $N \to N$ which induces a π_1-isomorphism is a homotopy equivalence. A closed manifold N is *s-Hopfian* if N is Hopfian when N is orientable and N_H is Hopfian when N is non-orientable, where N_H is the covering space of N corresponding to $H = \bigcap_{i\in I}\{H_i : [\pi_1(N) : H_i] = 2\}$. By Hall's Theorem (for any finitely generated group G, the number of subgroups of G having any fixed finite index is finite), the index set I is finite, and so H has a finite index in $\pi_1(N)$. All closed manifolds with virtually nilpotent or finite fundamental group, all closed aspherical manifolds, and all closed n-manifolds ($n \le 4$) are examples of s-Hopfian manifolds. Whether all closed manifolds are s-Hopfian is related to the famous old problem of Hopf [14].

A group Γ is said to be *Hopfian* if every epimorphism $f : \Gamma \to \Gamma$ is necessarily an isomorphism. A finitely presented group Γ is said to be *hyper-Hopfian* if every homomorphism $f : \Gamma \to \Gamma$ with $f(\Gamma)$ normal and $\Gamma/f(\Gamma)$ cyclic is an isomorphism (onto). A group Γ is said to be *residually finite* if for any non-trivial element x of Γ there is a homomorphism f from Γ onto a finite group K such that $f(x) \neq 1_K$. It is well known that every finitely generated residually finite group is Hopfian.

Given a group Γ, we use Γ' to denote its commutator subgroup.

PROPOSITION 2.1 ([15] or [10]). *Let $\psi : \Gamma \to \Gamma$ be an endomorphism of a finitely generated, residually finite group Γ with $\Gamma' \subset \psi(\Gamma)$. Then there exists an integer $k \ge 0$ for which ψ restricts to a monomorphism on $\psi^k(\Gamma)$. Moreover, if Γ/Γ' is finite, then* ker ψ *is finite.*

The next proposition is an easy consequence of the work of Epstein [13].

PROPOSITION 2.2 ([2, Lemma 3.2]). *Let M and N be manifolds and $f : M \to N$ a proper map such that $f_\# : \pi_1(M) \to \pi_1(N)$ is an isomorphism. Let $q' : N' \to N$ and $q'' : M'' \to M$ be coverings such that $q''_\#(\pi_1(M'')) = f^{-1}(q_\#(\pi_1(N')))$. Suppose that $f' : M'' \to N'$ is a lifting of $f \circ q''$ with $f \circ q'' = f' \circ q'$. Then $\deg f = \deg f' \in \mathbb{Z}_2$. Moreover, if M and N are orientable, then $\deg f = \deg f' \in \mathbb{Z}$.*

The following is basic for investigating co-dimension 2 fibrators.

LEMMA 2.3. *Let N be a closed s-Hopfian n-manifold with Hopfian fundamental group. If N is* (**F**)*, then at least one of the following two cases occurs*:

Case 1: *There is an N-like proper map $p : M^{n+2} \to \mathbb{R}^2$ defined on an $(n+2)$-manifold M which is an approximate fibration over $\mathbb{R}^2 \setminus \mathbf{0}$, but not an an approximate fibration over $\mathbb{R}^2$, such that $p^{-1}(\mathbf{0})$ is a strong deformation retract of $p^{-1}(\mathbb{R}^2) \equiv$ (say) L under a retraction $R : L \to p^{-1}(\mathbf{0})$.*

Case 2: *There is an N-like proper map $p : M^{n+2} \to \overline{\mathbb{H}}$ defined on an $(n+2)$-manifold M which is an approximate fibration over $\mathbb{H}$, but not an an approximate fibration over $\overline{\mathbb{H}}$, such that $p^{-1}(\mathbf{0})$ is a strong deformation retract of $p^{-1}(\overline{\mathbb{H}}) \equiv$ (say) L under a retraction $R : L \to p^{-1}(\mathbf{0})$ and for all $\mathbf{a} \in \partial\mathbb{H}$, $R|p^{-1}(\mathbf{a}) : p^{-1}(\mathbf{a}) \to p^{-1}(\mathbf{0})$ is a homotopy equivalence, where $\overline{\mathbb{H}} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and $\partial\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.*

In either case, $(R|p^{-1}(x))_\#(\pi_1(p^{-1}(x))) \neq \pi_1(p^{-1}(\mathbf{0}))$ for some $x (\neq \mathbf{0}) \in \mathbb{R}$.

PROOF. If a closed s-Hopfian n-manifold N with Hopfian fundamental group is **(F)**, there is an N-like proper map $p : M^{n+2} \to B$ defined on an $(n+2)$-manifold M which is not an approximate fibration. Hence $p : M^{n+2} \to B$ is not an approximate fibration at x for some $x \in B$. Here $x \in C'$ or $x \in \partial B$.

For the case of $x \in C'$, applying [16, Theorem 3.1] and [7, Proposition 2.8], we can localize the situation into Case 1. Applying [7, Proposition 2.8] for the case of $x \in \partial B$, we can localize the situation into Case 2. In either case, the Hopfian hypotheses on N and $\pi_1(N)$ gives $(R|p^{-1}(x))_\#(\pi_1(p^{-1}(x))) \neq \pi_1(p^{-1}(\mathbf{0}))$ for some $x (\neq \mathbf{0}) \in \mathbb{R}$. □

3. Proof of Main Theorem

Suppose that a closed s-Hopfian n-manifold N with residually finite, torsion free $\pi_1(N)$ and finite $H_1(N)$ is **(F)**. By Lemma 2.3, at least one of the two cases occurs. Since the method of the proof of Case 2 is basically same as Case 1, we only prove Case 1.

Put $g = p^{-1}(x)$ and $g_0 = p^{-1}(\mathbf{0})$. Take the covering $\alpha : L_I \to L \equiv p^{-1}(\mathbb{R}^2)$ corresponding to $\mathrm{incl}_\#(\pi_1(g))$. Take the covering $\beta : L_{IH} \to L_I$ corresponding to $H = \cap_t\{H_t \leq \pi_1(L_I) : [\pi_1(L_I) : H_t] = 2\}$ and then take the universal covering $\gamma : \tilde{L} \to L_{IH}$. Consider the following commutative diagram.

$$\begin{array}{ccccc}
\tilde{g} & \xrightarrow{\widetilde{\mathrm{incl}}} & \tilde{L} & \xrightarrow{\tilde{R}} & \tilde{g}_0 \\
\downarrow{\scriptstyle \gamma|} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma|} \\
\beta^{-1}(g) & \xrightarrow{\widetilde{\mathrm{incl}}_I} & L_{IH} & \xrightarrow{\tilde{R}_I} & g_{IH} \\
\downarrow{\scriptstyle \beta|} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta|} \\
g & \xrightarrow{\mathrm{incl}_I} & L_I & \xrightarrow{R_I} & \alpha^{-1}(g_0) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha|} \\
g & \xrightarrow{\mathrm{incl}} & L & \xrightarrow{R} & g_0 .
\end{array}$$

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Here, incl_I and R_I are liftings of the inclusion map incl and $R \circ q$, respectively. $\tilde{g}$ and $\tilde{g}_0$ are the universal covering of N.

First, we claim that $R_I \circ \mathrm{incl}_I$ induces a π_1-isomorphism.

Since p is an approximate fibration over $\mathbb{R}^2 \setminus \mathbf{0}$, there is a homotopy exact sequence

$$\pi_1(g) \to \pi_1(p^{-1}(\mathbb{R}^2 \setminus \mathbf{0})) \to \pi_1(\mathbb{R}^2 \setminus \mathbf{0}) \cong \mathbb{Z} \to 1$$

showing $\pi_1(p^{-1}(\mathbb{R}^2 \setminus \mathbf{0}))/\mathrm{incl}_\#(\pi_1(g)) \cong \mathbb{Z}$. Because g has the homotopy type of a co-dimension 2 compactum from L, the inclusion $p^{-1}(\mathbb{R}^2 \setminus \mathbf{0}) \to L$ induces an epimorphism φ of fundamental groups. It follows directly that $R_\# \varphi \mathrm{incl}(\pi_1(g))$ is a normal subgroup of $\pi_1(g_0)$ having cyclic cokernel. Hence, $\mathrm{incl}_\#(\pi_1(g))$ contains the commutator subgroup $\pi_1(L)'$ of $\pi_1(L)$. Since $\pi_1(L) \cong \pi_1(g_0) \cong \pi_1(g) \cong \pi_1(N)$ is residually finite and $\pi_1(N)/\pi_1(N)' \cong H_1(N)$ is finite, by Proposition 2.1, $\ker(\mathrm{incl}_\#)$ is finite. But since $\pi_1(g)$ is torsion free, $\ker(\mathrm{incl}_\#)$ should be trivial, that is, $\mathrm{incl}_\# : \pi_1(g) \to \pi_1(L)$ is a monomorphism. Consequently, $(\mathrm{incl}_I)_\# : \pi_1(g) \to \pi_1(L_I)$ is an isomorphism, for $q_\# \circ (\mathrm{incl}_I)_\# = \mathrm{incl}_\#$ and $q_\#$ is a monomorphism.

Since there is no upper semicontinuous decomposition of an orientable $(n+2)$-manifold consisting entirely of nonorientable n-manifolds [7, Proposition 2.9], the orientability of L_{IH} implies the orientability of $\beta^{-1}(g)$. So by [11, Lemma 5.5], the index $[\pi_1(g_{IH}) : (\tilde{R}_I \circ \widetilde{\mathrm{incl}}_I)_\#(\pi_1(\beta^{-1}(g)))]$ equals to the degree of the map $\tilde{R}_I \circ \widetilde{\mathrm{incl}}_I$. Applying the fact [18, Lemma 3.2] that $R_I \circ \mathrm{incl}_I$ induces a π_1-isomorphism if and only if $\tilde{R}_I \circ \widetilde{\mathrm{incl}}_I$ induces a π_1-isomorphism, we see that the degree of the map $\tilde{R}_I \circ \widetilde{\mathrm{incl}}_I$ must be one.

(1) First assume that $\pi_k(N)$ is finitely generated for all $k \geq 2$.
By Proposition 2.2, we have that the degree of the map $\tilde{R} \circ \widetilde{\mathrm{incl}}$ is one. Hence $\tilde{R} \circ \widetilde{\mathrm{incl}}$ induces H^k_C-monomorphisms for all k. By the fact of $H^k_C(X; \mathbb{G}) \cong H_{n-k}(X; \mathbb{G})$ [21, page 388], $\tilde{R} \circ \widetilde{\mathrm{incl}}$ induces H_k-epimorphisms for all k. But since $\pi_k(N)$ is finitely generated for all $k \geq 2$, $H_k(\tilde{g}) \cong H_k(\tilde{g}_0)$ is a finitely generated Abelian group [22, page 509] (and so it is Hopfian) so that $(\tilde{R} \circ \widetilde{\mathrm{incl}})_k : H_k(\tilde{g}) \to H_k(\tilde{g}_0)$ is an isomorphism. Appealing to the Hurewicz Theorem, we see that $\pi_k(\tilde{g}) \to \pi_k(\tilde{L})$ is an isomorphism for all $k \geq 2$. Whitehead's Theorem ensures that the composition $g \to L_I \to \alpha^{-1}(g_0)$ is a homotopy equivalence. But since $\alpha| : \alpha^{-1}(g_0) \to g_0$ is a covering map, $g_0 \simeq N$ satisfies **(CP)**.

(2) Next, assume that $\pi_k(N) \cong 0$ for $1 < k < n-1$.
Recall the work of Swarup [23]: *For a map $f : A \to B$ between closed oriented n-manifolds with π_1-isomorphism and $\pi_k(A) = \pi_k(B) = 0$ for $1 < k < n-1$, f is a homotopy equivalence if and only if* $\deg f = 1$.
Since the degree of the map $\tilde{R}_I \circ \widetilde{\mathrm{incl}}_I$ is one, by the work of Swarup, $\tilde{R}_I \circ \widetilde{\mathrm{incl}}_I$ is a homotopy equivalence.

(3) Finally, assume that $n \leq 4$.
The case of $n = 3$ is a special case of (2).

For the case of $n = 4$, apply the following consequence of the work of Hausmann [14]: *For any degree one map $f : A^4 \to B^4$ between closed 4-manifolds with π_1-isomorphism, f is a homotopy equivalence.*
Although Hausmann only proves the case $A = B$, just mimicking his proof and using the exact sequence of surgery with Poincaré duality, one may deduce the statement above.

4. Example and remarks

EXAMPLE ([12]). A closed n-manifold N, $n > 4$, which fails to be a co-dimension 2 fibrator but $H_1(N) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and every $\gamma \in \pi_1(N)$, $\gamma \neq 1$, has infinite order.

Apply Maunder's construction [20] to obtain a finite aspherical 2-complex K such that $H_1(K) \cong \mathbb{Z}_2$. Specify a PL embedding of K in an $(n+1)$-manifold M^{n+1}, and let S be the boundary of a regular neighbourhood of the image. Let Ω be the mapping cylinder of a 2-1 covering map $\Theta : \overline{S} \to S$; here Ω is a (non-orientable) twisted I-bundle over S. Form N by doubling Ω along $\overline{S}$, its boundary.

A routine computation involving a Mayer-Vietoris sequence confirms $H_1(N) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Note that $\pi_1(\Omega) \cong \pi_1(S) \cong \pi_1(K)$, from which it follows that $\pi_1(N) \cong \pi_1(\Omega) *_{\pi_1(\overline{S})} \pi_1(\Omega)$ is the fundamental group of an aspherical finite complex and, hence, no nontrivial element has finite order [1, Corollary VIII.2.5].

Such manifolds N fail to be co-dimension 2 fibrators, due to the existence of a 2-1 covering map $N \to N$ (see [7, Theorem 4.2]). For the most obvious 2-1 covering $\overline{N} \to N$, $\overline{N}$ will consist of two copies Ω_1, Ω_2 of Ω, arising as the preimage of one Ω in the target space, N, together with a 2-1 covering $\overline{\Omega}$ of the other copy of Ω used to form N. But here $\overline{\Omega}$ is simply $\overline{S} \times [0, 1]$, and $\overline{N} = \Omega_1 \cup (\overline{S} \times [0, 1]) \cup \Omega_2$ with attachments that reveal $\overline{N} \approx N$.

Note that for all $i \geq 2$, $\pi_i(N \times S^k)$ $(k \geq 2)$ is finitely generated.

REMARK. The condition of torsion free $\pi_1(N)$ cannot be omitted, since $S^3 \times L(p, q)$ fails to be a co-dimension 2 fibrator ([9]) but it cannot cover itself cyclically, where $L(p, q)$ is a Lens space.

On the other hand, the condition of finite $H_1(N)$ is also imperative, since N fails to be a co-dimension 2 fibrator but it cannot cover itself cyclically, where N is some Nil 3-manifold (See [8]).

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