

A NECESSARY AND SUFFICIENT CONDITION FOR CERTAIN MARTINGALE INEQUALITIES IN BANACH FUNCTION SPACES

MASATO KIKUCHI*

Department of Mathematics, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan
e-mail: kikuchi@sci.u-toyama.ac.jp

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Abstract. Let X be a Banach function space over a nonatomic probability space. We investigate certain martingale inequalities in X that generalize those studied by A. M. Garsia. We give necessary and sufficient conditions on X for the inequalities to be valid.

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1. Introduction. It is well known that, for each $p \in [1, \infty)$, the Hardy space \mathcal{H}_p of martingales consists of those $f = (f_n)_{n \in \mathbb{Z}_+}$ for which $Sf \in L_p$, where Sf denotes the square function of f . It is also known to many researchers of martingale theory that, for each $q \in [2, \infty]$, the space \mathcal{K}_q consists of those $f = (f_n)_{n \in \mathbb{Z}_+}$ for which there exists a random variable $\gamma \in L_q$ satisfying

$$\mathbb{E}[|f_\infty - f_{n-1}|^2 | \mathcal{F}_n] \leq \mathbb{E}[\gamma^2 | \mathcal{F}_n]$$

almost surely (a.s.) for all $n \in \mathbb{Z}_+$, where $f_{-1} \equiv 0$. The norm of $f \in \mathcal{K}_q$ is defined to be the infimum of $\|\gamma\|_q$ over all $\gamma \in L_q$ satisfying the inequality above.

The space \mathcal{K}_q plays a crucial role in studying the dual space of \mathcal{H}_p . In fact, Garsia [5] proved that if $1 \leq p \leq 2$ and q is the conjugate exponent of p , then the dual space of \mathcal{H}_p is isomorphic to \mathcal{K}_q . Since \mathcal{K}_∞ coincides with BMO (the space of martingales of bounded mean oscillation), Garsia's result includes Fefferman's duality theorem which asserts that the dual space of \mathcal{H}_1 is isomorphic to BMO . On the other hand, Garsia also proved that if $2 \leq q < \infty$, then \mathcal{H}_q and \mathcal{K}_q coincide, and for all $f \in \mathcal{K}_q$,

$$\sqrt{2/q} \|Sf\|_q \leq \|f\|_{\mathcal{K}_q} \leq \|Sf\|_q. \quad (1.1)$$

Moreover, combining (1.1) with the Burkholder square function inequality ([2, Theorem 9]), we see that if $2 \leq q < \infty$, then there exists a constant $C_q > 0$ such that for any $f = (f_n) \in \mathcal{K}_q$,

$$C_q^{-1} \|f_\infty\|_q \leq \|f\|_{\mathcal{K}_q} \leq C_q \|f_\infty\|_q, \quad (1.2)$$

where $f_\infty := \lim_n f_n$ a.s.

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In this paper, we consider more general inequalities similar to those in (1.2). Given a Banach function space X (see Definition 1 below) and a filtration $\mathcal{F} = (\mathcal{F}_n)$, we introduce a Banach space of martingales, which we denote by $\mathcal{K}(X, \mathcal{F})$, and give necessary and sufficient conditions on X for the inequalities

$$C^{-1} \|f_\infty\|_X \leq \|f\|_{\mathcal{K}(X, \mathcal{F})} \leq C \|f_\infty\|_X$$

and

$$C^{-1} \underline{\lim}_{n \rightarrow \infty} \|f_n\|_X \leq \|f\|_{\mathcal{K}(X, \mathcal{F})} \leq C \overline{\lim}_{n \rightarrow \infty} \|f_n\|_X$$

to be valid. For a fixed filtration $\mathcal{F} = (\mathcal{F}_n)$, the definition of $\mathcal{K}(L_q, \mathcal{F})$ is slightly different from that of \mathcal{K}_q (cf. Definition 3 in Section 3). However, $\mathcal{K}(L_q, \mathcal{F})$ and \mathcal{K}_q in fact coincide for all $q \in [2, \infty]$.

2. Preliminaries. We deal with martingales on a *nonatomic* probability space $(\Omega, \Sigma, \mathbb{P})$. The assumption that Ω is nonatomic is essential. In addition, we have to deal with another probability space; let I be the interval $(0, 1]$ and let μ be Lebesgue measure on the σ -algebra \mathfrak{M} consisting of all Lebesgue measurable subsets of I . The reader may assume that these two probability spaces are the same. However, our argument will not be very simple by doing so.

Let X and Y be normed linear spaces. We write $X \hookrightarrow Y$ if X is continuously embedded in Y , that is, if $X \subset Y$ and the inclusion map is continuous.

DEFINITION 1. Let $(X, \|\cdot\|_X)$ be a Banach space of (equivalence classes of) random variables on Ω , or measurable functions on I . We call $(X, \|\cdot\|_X)$ a *Banach function space* if it satisfies the following conditions:

- (B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$;
- (B2) if $|x| \leq |y|$ a.s. and $y \in X$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$;
- (B3) if $0 \leq x_n \uparrow x$ a.s., $x_n \in X$ for all n , and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$.

If $x \notin X$, we let $\|x\|_X := \infty$.

Note that, in Definition 1, we may replace (B3) by the condition that

- (B3') if $0 \leq x_n \in X$ for all n and $\underline{\lim}_n \|x_n\|_X < \infty$, then $\underline{\lim}_n x_n \in X$ and $\|\underline{\lim}_n x_n\|_X \leq \underline{\lim}_n \|x_n\|_X$.

Let x and y be random variables on Ω , or measurable functions on I . We write $x \simeq_d y$ to mean that x and y have the same distribution.

DEFINITION 2. A Banach function space $(X, \|\cdot\|_X)$ is said to be *rearrangement-invariant* (r.i.) provided that

- (RI) if $x \simeq_d y$ and $y \in X$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

A rearrangement-invariant Banach function space will be simply called a *rearrangement-invariant space* or an *r.i. space*.

Typical examples of r.i. spaces are Lebesgue spaces L_p , Orlicz spaces L_Φ , Lorentz spaces $L_{p,q}$, and so on. An example of a Banach function space that is not r.i. is a weighted Lebesgue space. Let w be a strictly positive random variable such that $\mathbb{E}[w] = 1$, and let $1 < p < \infty$. If $w^{-1/(p-1)}$ is integrable, then the Lebesgue space L_p^w with respect to the measure $w d\mathbb{P}$ satisfies (B1)–(B3), and thus it is a Banach function space

(with respect to \mathbb{P}). It is known that L_p^w can be renormed so as to be r.i. if and only if $0 < \text{ess inf } w \leq \text{ess sup } w < \infty$ (cf. [6, Section 4]).

Let x be a random variable on Ω . The *nonincreasing rearrangement* of x , which we denote by x^* , is the nonincreasing right-continuous function on $I = (0, 1]$ defined by

$$x^*(t) := \inf\{\lambda > 0 \mid \mathbb{P}(|x| > \lambda) \leq t\} \quad \text{for all } t \in I,$$

with the convention that $\inf \emptyset = \infty$. Note that x^* is characterized as the nonincreasing right-continuous function that has the same distribution (with respect to μ) as $|x|$.

If ϕ is a measurable function on I , then the nonincreasing rearrangement ϕ^* is defined by regarding ϕ as a random variable on the probability space (I, \mathfrak{M}, μ) .

Let x and y be integrable random variables on Ω , or measurable functions on I . We write $x < y$ if

$$\int_0^t x^*(s) \, ds \leq \int_0^t y^*(s) \, ds \quad \text{for all } t \in I.$$

Then it is obvious that $x \simeq_d y$ if and only if $x < y < x$.

A Banach function space $(X, \|\cdot\|_X)$ is said to be *universally rearrangement-invariant* (u.r.i.) provided that

(URI) if $x < y$ and $y \in X$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$.

Clearly condition (URI) implies condition (RI), while the converse is not true in general. However, if the underlying measure space is nonatomic, then condition (RI) implies condition (URI) (cf. [1, Theorem 4.6, p. 61]). Thus, in our argument, we need not distinguish u.r.i. spaces from r.i. spaces.

Now let us recall Luxemburg’s representation theorem. If X is an r.i. space over Ω , then there exists a unique Banach function space \widehat{X} over I such that:

- $x \in X$ if and only if $x^* \in \widehat{X}$;
- $\|x\|_X = \|x^*\|_{\widehat{X}}$ for all $x \in X$.

In fact \widehat{X} consists of those functions ϕ for which

$$\|\phi\|_{\widehat{X}} := \sup \left\{ \int_0^1 \phi^*(s) y^*(s) \, ds \mid \|y\|_{X'} \leq 1 \right\} < \infty,$$

where

$$\|y\|_{X'} := \sup\{\mathbb{E}[|xy|] \mid x \in X, \|x\|_X \leq 1\}. \tag{2.1}$$

We call $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ the *Luxemburg representation* of $(X, \|\cdot\|_X)$. For example, the Luxemburg representation of $L_p(\Omega)$ is $L_p(I)$. For more details, see [1, pp. 62–64].

Now let Z_1 and Z_2 be r.i. spaces over I , and let T be a linear operator whose domain contains Z_1 . We write $T \in B(Z_1, Z_2)$ to mean that the restriction of T to Z_1 is a bounded operator on Z_1 into Z_2 . If $Z_1 = Z_2 = Z$, we also write $T \in B(Z)$ for $T \in B(Z, Z)$.

In order to state our results, we need the notion of Boyd indices, which are defined as follows. Given a measurable function ϕ on I , we define a function $D_s\phi$ on I by setting

$$(D_s\phi)(t) := \begin{cases} \phi(st) & \text{if } st \in I, \\ 0 & \text{otherwise.} \end{cases}$$

If Z is an r.i. space over I , then $D_s \in B(Z)$ and $\|D_s\|_{B(Z)} \leq (1/s) \vee 1$ for all $s > 0$, where $\|D_s\|_{B(Z)}$ denotes the operator norm of D_s (restricted to Z). The *lower* and *upper Boyd indices* of an r.i. space Z are defined by

$$\alpha_Z := \sup_{0 < s < 1} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s} \quad \text{and} \quad \beta_Z := \inf_{1 < s < \infty} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s},$$

respectively. Then we have

$$\alpha_Z = \lim_{s \downarrow 0} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s}, \quad \beta_Z = \lim_{s \uparrow \infty} \frac{\log \|D_{s^{-1}}\|_{B(Z)}}{\log s}$$

and

$$0 \leq \alpha_Z \leq \beta_Z \leq 1.$$

If X is an r.i. space over Ω , we define the Boyd indices of X by $\alpha_X := \alpha_{\widehat{X}}$ and $\beta_X := \beta_{\widehat{X}}$, where \widehat{X} is the Luxemburg representation of X . For instance, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1, \infty]$. See [1, pp. 148–149] for details.

We conclude this section by introducing operators \mathcal{P} , \mathcal{Q} and \mathcal{R} . For a measurable function ϕ on I , we define

$$\begin{aligned} (\mathcal{P}\phi)(t) &:= \frac{1}{t} \int_0^t \phi(s) \, ds, \quad t \in I, \\ (\mathcal{Q}\phi)(t) &:= \int_t^1 \frac{\phi(s)}{s} \, ds, \quad t \in I, \end{aligned}$$

and

$$(\mathcal{R}\phi)(t) := \int_0^1 \frac{\phi(s)}{s+t} \, ds, \quad t \in I,$$

provided that these integrals exist for all $t \in I$. It is easy to verify that if ϕ is nonnegative and integrable, then

$$\frac{1}{2}(\mathcal{P}\phi + \mathcal{Q}\phi) \leq \mathcal{R}\phi \leq \mathcal{P}\phi + \mathcal{Q}\phi \quad \text{on } I, \tag{2.2}$$

$$\mathcal{P}(\mathcal{Q}\phi) = \mathcal{P}\phi + \mathcal{Q}\phi \quad \text{on } I, \tag{2.3}$$

and

$$\mathcal{Q}(\mathcal{P}\phi) = \mathcal{P}\phi + \mathcal{Q}\phi - \int_0^1 \phi(s) \, ds \quad \text{on } I. \tag{2.4}$$

Note that each of the operators \mathcal{P} and \mathcal{Q} is the (formal) adjoint of the other. It is known that $\mathcal{P} \in B(Z)$ (resp. $\mathcal{Q} \in B(Z)$) if and only if $\beta_Z < 1$ (resp. $\alpha_Z > 0$). Furthermore, by (2.2) we have that $\mathcal{R} \in B(Z)$ if and only if $\alpha_Z > 0$ and $\beta_Z < 1$. See [1, p. 150] for details (cf. [10]).

3. Results. Let \mathbb{F} denote the collection of all filtrations of $(\Omega, \Sigma, \mathbb{P})$, where by *filtration* of $(\Omega, \Sigma, \mathbb{P})$ we mean a nondecreasing sequence of sub- σ -algebras of Σ . Given $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$, we denote by $\mathcal{M}(\mathcal{F})$ the space of all martingales with respect

to \mathcal{F} and \mathbb{P} , and we denote by $\mathcal{M}_u(\mathcal{F})$ the linear subspace of $\mathcal{M}(\mathcal{F})$ consisting of all uniformly integrable martingales. Recall that every $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$ converges a.s.; we let $f_\infty := \lim_n f_n$ a.s. for each $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$.

Henceforth we adopt the convention that $f_{-1} \equiv 0$ for any $f = (f_n) \in \mathcal{M}(\mathcal{F})$.

DEFINITION 3. Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω . We denote by $\Gamma_f(X, \mathcal{F})$ the set of all nonnegative, $\sigma(\bigcup_{n=0}^\infty \mathcal{F}_n)$ -measurable random variables $\gamma \in X$ satisfying

$$\sup_{m \geq n} \mathbb{E}[|f_m - f_{n-1}| | \mathcal{F}_n] \leq \mathbb{E}[\gamma | \mathcal{F}_n] \quad \text{a.s., } n \in \mathbb{Z}_+. \tag{3.1}$$

The space $\mathcal{K}(X, \mathcal{F})$ is defined to be the set of $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$ for which $\Gamma_f(X, \mathcal{F}) \neq \emptyset$. The norm of $f \in \mathcal{K}(X, \mathcal{F})$ is given by

$$\|f\|_{\mathcal{K}(X, \mathcal{F})} := \inf\{\|\gamma\|_X \mid \gamma \in \Gamma_f(X, \mathcal{F})\}.$$

For martingales $f \in \mathcal{M}(\mathcal{F})$ that are not in $\mathcal{K}(X, \mathcal{F})$, we let $\|f\|_{\mathcal{K}(X, \mathcal{F})} := \infty$.

Note that if $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$, then (3.1) can be rewritten as

$$\mathbb{E}[|f_\infty - f_{n-1}| | \mathcal{F}_n] \leq \mathbb{E}[\gamma | \mathcal{F}_n] \quad \text{a.s., } n \in \mathbb{Z}_+.$$

Note also that $\mathcal{K}(X, \mathcal{F})$ is a Banach space. Indeed, it is not hard to show that $\mathcal{K}(X, \mathcal{F})$ has the Riesz-Fischer property, that is, that if $\{f^{(k)}\}$ is a sequence in $\mathcal{K}(X, \mathcal{F})$ such that $\sum_{k=1}^\infty \|f^{(k)}\|_{\mathcal{K}(X, \mathcal{F})} < \infty$, then the series $\sum_{k=1}^\infty f^{(k)}$ converges in $\mathcal{K}(X, \mathcal{F})$. As is well known, a normed linear space that has the Riesz-Fischer property is complete. Thus $\mathcal{K}(X, \mathcal{F})$ is a Banach space.

We can now state the main result of this paper.

THEOREM 1. *Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω . Then the following are equivalent:*

- (i) *there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$,*

$$C^{-1} \underline{\lim}_{n \rightarrow \infty} \|f_n\|_X \leq \|f\|_{\mathcal{K}(X, \mathcal{F})} \leq C \overline{\lim}_{n \rightarrow \infty} \|f_n\|_X; \tag{3.2}$$

- (ii) *there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$,*

$$C^{-1} \|f_\infty\|_X \leq \|f\|_{\mathcal{K}(X, \mathcal{F})} \leq C \|f_\infty\|_X; \tag{3.3}$$

- (iii) *there exists a norm $\|\cdot\|_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is a rearrangement-invariant space such that $\alpha_X > 0$ and $\beta_X < 1$.*

REMARK 1. Suppose that (iii) of Theorem 1 holds. Then (3.2) can be rewritten as

$$K^{-1} \sup_{n \in \mathbb{Z}_+} \|\|f_n\|\|_X \leq \|f\|_{\mathcal{K}(X, \mathcal{F})} \leq K \sup_{n \in \mathbb{Z}_+} \|\|f_n\|\|_X, \tag{3.4}$$

where K is a positive constant, independent of f . To see this, let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}(\mathcal{F})$. Then $f_n < f_{n+1}$ for all n (see [7, Remark 4.3]), and hence (URI) with $\|\cdot\|_X$ replaced by $\|\cdot\|_X$ implies that $\|\|f_n\|\|_X \leq \|\|f_{n+1}\|\|_X$ for all n . Thus we may replace

both $\overline{\lim} \|f_n\|_X$ and $\underline{\lim} \|f_n\|_X$ in (3.2) with a constant multiple of $\sup \|f_n\|_X$ to obtain (3.4).

As we shall see in the last section, Theorem 1 is a consequence of Propositions 1, 2, and 3 below.

PROPOSITION 1. *Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω . Suppose that one of the following four conditions holds:*

- (i) *the first inequality of (3.2) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$;*
- (ii) *the second inequality of (3.2) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$;*
- (iii) *the first inequality of (3.3) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$;*
- (iv) *the second inequality of (3.3) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$.*

Then there exists a norm $\|\cdot\|_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is a rearrangement-invariant space.

PROPOSITION 2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be rearrangement-invariant spaces over Ω , and let $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ and $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ be their Luxemburg representations. Then the following are equivalent:*

- (i) *there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$,*

$$\|f\|_{\mathcal{K}(X, \mathcal{F})} \leq C \sup_{n \in \mathbb{Z}_+} \|f_n\|_Y; \tag{3.5}$$

- (ii) *there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$,*

$$\|f\|_{\mathcal{K}(X, \mathcal{F})} \leq C \|f\|_Y; \tag{3.6}$$

- (iii) $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$.

REMARK 2. As mentioned before, $\mathcal{P} \in B(\widehat{X})$ if and only if $\beta_{\widehat{X}} < 1$. Hence by Propositions 1 and 2, the following are equivalent:

- the second inequality of (3.2) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$;
- the second inequality of (3.3) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$;
- X can be renormed so that it is an r.i. space with $\beta_X < 1$.

PROPOSITION 3. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(\widehat{X}, \|\cdot\|_{\widehat{X}})$, and $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ be as in Proposition 2.*

- (i) *Suppose that $\mathcal{R} \in B(\widehat{Y}, \widehat{X})$. Then there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$,*

$$\sup_{n \in \mathbb{Z}_+} \|f_n\|_X \leq \|Mf\|_X \leq C \|f\|_{\mathcal{K}(Y, \mathcal{F})}. \tag{3.7}$$

Here Mf denotes the maximal function of f , that is, $Mf := \sup_{n \in \mathbb{Z}_+} |f_n|$.

(ii) Suppose that there exists a positive constant C such that the inequality

$$\sup_{n \in \mathbb{Z}_+} \|f_n\|_X \leq C \|f\|_{\mathcal{K}(Y, \mathcal{F})} \tag{3.8}$$

holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$. Then $\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$.

REMARK 3. From (2.2), (2.3), and (2.4), we see that the hypothesis $\mathcal{R} \in B(\widehat{Y}, \widehat{X})$ in (i) of Proposition 3 is equivalent to each of the following:

- (a) $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$ and $\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$;
- (b) $\mathcal{P}\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$;
- (c) $\mathcal{Q}\mathcal{P} \in B(\widehat{Y}, \widehat{X})$.

Incidentally, in order to prove that (c) implies (a), we have to use the hypotheses $L_\infty(I) \hookrightarrow \widehat{X}$ and $\widehat{Y} \hookrightarrow L_1(I)$ (cf. (B1)).

REMARK 4. Let $(X, \|\cdot\|_X)$ be an r.i. space over Ω . There is a characterization of those r.i. spaces Y for which $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$. Define $H(X)$ to be the set of all $x \in L_1(\Omega)$ such that $\|x\|_{H(X)} := \|\mathcal{P}x^*\|_{\widehat{X}} < \infty$. Then $(H(X), \|\cdot\|_{H(X)})$ is an r.i. space and $\mathcal{P} \in B(\widehat{H(X)}, \widehat{X})$. Moreover $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$ if and only if $Y \hookrightarrow H(X)$.

There is a similar result concerning the boundedness of \mathcal{Q} . Define $K(X)$ to be the set of all $x \in L_1(\Omega)$ such that $\|x\|_{K(X)} := \|\mathcal{Q}x^*\|_{\widehat{X}} < \infty$. If the function $t \mapsto -\log t$ belongs to \widehat{X} , then $K(X)$ is an r.i. space and $\mathcal{Q} \in B(\widehat{K(X)}, \widehat{X})$. Moreover $\mathcal{Q} \in B(\widehat{Y}, \widehat{X})$ if and only if $Y \hookrightarrow K(X)$, provided that $-\log t \in \widehat{X}$. See [7] for details.

4. Proof of Proposition 1. We begin with a lemma.

LEMMA 1. Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω , and let S_+ be the set of all nonnegative simple random variables on Ω . Then the following are equivalent:

- (i) there is a constant $c > 0$ such that if $x, y \in S_+$, $x \simeq_d y$, and $x \wedge y \equiv 0$, then $\|y\|_X \leq c \|x\|_X$;
- (ii) there is a constant $c > 0$ such that if $x, y \in X$ and $x \simeq_d y$, then $\|y\|_X \leq c \|x\|_X$;
- (iii) there is a norm $\|\cdot\|_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is an r.i. space.

A complete proof of this lemma can be found in [8]. For convenience, we sketch the proof here (cf. [9]).

Proof. (iii) \Rightarrow (i). Obvious.

(i) \Rightarrow (ii). Assume that (i) holds. We first show that if $x, y \in X$, $x \simeq_d y$, and $|x| \wedge |y| \equiv 0$, then $\|y\|_X \leq c \|x\|_X$. For such x and y , there are sequences $\{x_n\}$ and $\{y_n\}$ in S_+ such that $x_n \simeq_d y_n$ for all $n \in \mathbb{Z}_+$, and such that $0 \leq x_n \uparrow |x|$ and $0 \leq y_n \uparrow |y|$. Since by assumption $\|y_n\|_X \leq c \|x_n\|_X$ for all n , we can apply (B3) to obtain $\|y\|_X \leq c \|x\|_X$. Next we show that (ii) holds, or equivalently, that

$$\sup\{\|y\|_X \mid x, y \in X, x \simeq_d y, \|x\|_X \leq 1\} < \infty. \tag{4.1}$$

Suppose $x, y \in X$, $x \simeq_d y$, and $\|x\|_X \leq 1$. We choose a positive number λ so that $\mathbb{P}(|x| > \lambda) = \mathbb{P}(|y| > \lambda) \leq 1/3$, and let $x' := |x|1_{\{|x| > \lambda\}}$ and $y' := |y|1_{\{|y| > \lambda\}}$. Here, and in what follows, 1_A denotes the indicator function of A . Then, since $\mathbb{P}(x' = 0, y' = 0) \geq 1/3$, there exists a random variable z such that $\{z \neq 0\} \subset \{x' = 0, y' = 0\}$ and $z \simeq_d x'$ (cf. [4, (5.6), p. 44]). Since $x' \wedge z \equiv 0$ and $y' \wedge z \equiv 0$, we have that

$\|y'\|_X \leq c \|z\|_X \leq c^2 \|x'\|_X \leq c^2$. Hence, letting $\mathbf{1}$ denote the constant function with value one, we obtain

$$\|y\|_X \leq \|y'\|_X + \lambda \|\mathbf{1}\|_X \leq c^2 + \lambda \|\mathbf{1}\|_X,$$

which proves (4.1).

(ii) \Rightarrow (iii). For each $x \in L_1(\Omega)$, we define

$$\| \|x\| \|_X := \sup \left\{ \int_0^1 x^*(s) y^*(s) ds \mid \|y\|_{X'} \leq 1 \right\},$$

where $\|y\|_{X'}$ is defined as in (2.1). Then the set of all $x \in L_1(\Omega)$ such that $\| \|x\| \|_X < \infty$ forms an r.i. space. Moreover, under the assumption that (ii) holds, one can show that $\| \|x\| \|_X < \infty$ if and only if $x \in X$, and in this case $\|x\|_X \leq \| \|x\| \|_X \leq c \|x\|_X$ for all $x \in X$. □

Proof of Proposition 1. Suppose first that (ii) of Proposition 1 holds. We show that (i) of Lemma 1 holds. Let x and y be nonnegative simple random variables such that $x \simeq_d y$ and $x \wedge y \equiv 0$. Then we can write

$$x = \sum_{j=1}^{\ell} \alpha_j 1_{A_j} \quad \text{and} \quad y = \sum_{j=1}^{\ell} \alpha_j 1_{B_j},$$

where $\alpha_j > 0$ for each $j \in \{1, 2, \dots, \ell\}$, and where $\{A_j\}_{j=1}^{\ell}$ and $\{B_j\}_{j=1}^{\ell}$ are sequences of sets in Σ such that:

- $\mathbb{P}(A_j) = \mathbb{P}(B_j)$ for each $j \in \{1, 2, \dots, \ell\}$;
- $A_j \cap A_k = B_j \cap B_k = \emptyset$ whenever $j \neq k$;
- $(\bigcup_{j=1}^{\ell} A_j) \cap (\bigcup_{j=1}^{\ell} B_j) = \emptyset$.

Let $\Lambda_j := A_j \cup B_j$ for each $j \in \{1, 2, \dots, \ell\}$. We define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$ by

$$\mathcal{F}_n := \begin{cases} \sigma\{\Lambda_j \mid j = 1, 2, \dots, \ell\} & \text{if } n = 0, \\ \Sigma & \text{if } n \geq 1. \end{cases} \quad \text{and} \quad f := \mathbb{E}[x \mid \mathcal{F}_n], \quad n \in \mathbb{Z}_+.$$

Suppose that $\gamma \in \Gamma_f(X, \mathcal{F})$. Then since $f_0 = 2^{-1}(x + y)$ and $f_n = x$ for $n \geq 1$,

$$\frac{y}{2} \leq f_0 = \mathbb{E}[|f_1 - f_0| \mid \mathcal{F}_1] \leq \mathbb{E}[\gamma \mid \mathcal{F}_1] = \gamma \quad \text{a.s.}$$

Hence $\|y\|_X \leq 2 \| \gamma \|_X$, which implies $\|y\|_X \leq 2 \|f\|_{\mathcal{K}(X, \mathcal{F})}$. Combining this with the second inequality of (3.2), we obtain $\|y\|_X \leq 2C \|x\|_X$. Thus (i) of Lemma 1 holds.

If (iv) of Proposition 1 holds, we can use exactly the same argument as above to show that (i) of Lemma 1 holds.

Suppose next that (i) of Proposition 1 holds. Let x and y be as above, and let C be the constant appearing in (3.2). Of course, we may assume that $C \geq 1$. (In fact, one can deduce that $C \geq 1$.) For each $j \in \{1, 2, \dots, \ell\}$, we choose $B'_j \in \Sigma$ so that $B'_j \subset B_j$ and $\mathbb{P}(B'_j) = C^{-1} \mathbb{P}(B_j)$. This is possible, since $(\Omega, \Sigma, \mathbb{P})$ is nonatomic. Now let $\Lambda_j := A_j \cup B'_j$ for each $j \in \{1, 2, \dots, \ell\}$, and define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$

as above. Then, letting $y' := \sum_{j=1}^{\ell} \alpha_j 1_{B_j}$, we have

$$f_n = \begin{cases} C(C+1)^{-1}(x+y') & \text{if } n = 0, \\ x & \text{if } n \geq 1. \end{cases}$$

It is easy to see that $(C+1)^{-1}(x+C^2y') \in \Gamma_f(X, \mathcal{F})$. Since $y' \leq y$, we have

$$\|f\|_{\mathcal{K}(X, \mathcal{F})} \leq \frac{1}{C+1} \|x\|_X + \frac{C^2}{C+1} \|y\|_X.$$

On the other hand, the first inequality of (3.2) implies $\|x\|_X \leq C \|f\|_{\mathcal{K}(X, \mathcal{F})}$. Therefore

$$\|x\|_X \leq \frac{C}{C+1} \|x\|_X + \frac{C^3}{C+1} \|y\|_X,$$

which implies $\|x\|_X \leq C^3 \|y\|_X$. Thus (i) of Lemma 1 holds. Exactly the same argument applies if (iii) holds, and Proposition 1 is proved. □

5. Proof of Proposition 2. In order to prove Proposition 2, we need four lemmas, which will also be used in the proof of Proposition 3.

LEMMA 2. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(\widehat{X}, \|\cdot\|_{\widehat{X}})$, and $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ be as in Proposition 2, and let $f = (f_n)_{n \in \mathbb{Z}_+}$ be a martingale.

(i) If $f = (f_n)_{n \in \mathbb{Z}_+}$ is uniformly integrable, then

$$\|f_\infty\|_X = \lim_{n \rightarrow \infty} \|f_n\|_X = \sup_{n \in \mathbb{Z}} \|f_n\|_X.$$

(ii) If $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$ and $\sup_{n \in \mathbb{Z}_+} \|f_n\|_Y < \infty$, then $Mf = \sup_{n \in \mathbb{Z}_+} |f_n| \in X$ and

$$\|Mf\|_X \leq \|\mathcal{P}\|_{B(\widehat{Y}, \widehat{X})} \cdot \sup_{n \in \mathbb{Z}_+} \|f_n\|_Y,$$

where $\|\mathcal{P}\|_{B(\widehat{Y}, \widehat{X})}$ stands for the operator norm of $\mathcal{P}: \widehat{Y} \rightarrow \widehat{X}$.

Proof. (i) Assume that $f = (f_n)$ is uniformly integrable. Then $f_n < f_{n+1} < f_\infty$ for all $n \in \mathbb{Z}_+$ (see [7, Remark 4.3]). Hence, by (URI) and (B3'),

$$\sup_{n \in \mathbb{Z}_+} \|f_n\|_X \leq \|f_\infty\|_X \leq \lim_{n \rightarrow \infty} \|f_n\|_X = \sup_{n \in \mathbb{Z}_+} \|f_n\|_X,$$

as desired. Of course, if $f_\infty \notin X$, then $\|f_\infty\|_X = \sup_n \|f_n\|_X = \infty$.

(ii) As shown in the proof of [6, Proposition 3], for each $n \in \mathbb{Z}_+$,

$$(M_n f)^*(t) \leq (\mathcal{P} f_n^*)(t), \quad t \in I,$$

where $M_n f := \sup_{0 \leq m \leq n} |f_m|$. Therefore

$$\|M_n f\|_X = \|(M_n f)^*\|_{\widehat{X}} \leq \|\mathcal{P} f_n^*\|_{\widehat{X}} \leq \|\mathcal{P}\|_{B(\widehat{Y}, \widehat{X})} \cdot \sup_{n \in \mathbb{Z}_+} \|f_n\|_Y < \infty.$$

Since $M_n f \uparrow Mf$, it follows from (B3) that $Mf \in X$ and

$$\|Mf\|_X \leq \|P\|_{B(\widehat{Y}, \widehat{X})} \cdot \sup_{n \in \mathbb{Z}_+} \|f_n\|_Y,$$

as desired. □

LEMMA 3. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be r.i. spaces over Ω .*

- (i) *If (3.6) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$, then $\widehat{Y} \hookrightarrow \widehat{X}$, or equivalently $Y \hookrightarrow X$.*
- (ii) *If (3.8) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$, then $\widehat{Y} \hookrightarrow \widehat{X}$, or equivalently $Y \hookrightarrow X$.*

Proof. Let $x \in Y$. We define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$ by

$$\mathcal{F}_n := \begin{cases} \{\emptyset, \Omega\} & \text{if } n = 0, \\ \Sigma & \text{if } n \geq 1, \end{cases} \quad \text{and} \quad f_n := \mathbb{E}[x | \mathcal{F}_n], \quad n \in \mathbb{Z}_+.$$

Then, for any $\gamma \in \Gamma_f(X, \mathcal{F})$,

$$|x - \mathbb{E}[x]| = \mathbb{E}[|f_1 - f_0| | \mathcal{F}_1] \leq \mathbb{E}[\gamma | \mathcal{F}_1] = \gamma \quad \text{a.s.},$$

and hence $|x| \leq \gamma + \|x\|_1 \leq \gamma + d \|x\|_Y$ a.s., where d is a positive constant such that $\|\cdot\|_1 \leq d \|\cdot\|_Y$ on Y . Therefore $\|x\|_X \leq \|\gamma\|_X + d \|\mathbf{1}\|_X \|x\|_Y$, which implies

$$\|x\|_X \leq \|f\|_{\mathcal{K}(X, \mathcal{F})} + d \|\mathbf{1}\|_X \|x\|_Y.$$

Suppose that (3.6) holds for this $f = (f_n)$. Then

$$\|x\|_X \leq C \|f_\infty\|_Y + d \|\mathbf{1}\|_X \|x\|_Y = (C + d \|\mathbf{1}\|_X) \|x\|_Y,$$

which shows that $Y \hookrightarrow X$. Moreover, if $\phi \in \widehat{Y}$, then there exists $y \in Y$ such that $y^* = \phi^*$ on I (see [4, (5.6), p. 44]). Hence $\|\phi\|_{\widehat{X}} = \|y\|_X \leq C' \|y\|_Y = C' \|\phi\|_{\widehat{Y}}$, where $C' := C + d \|\mathbf{1}\|_X$. This shows that $\widehat{Y} \hookrightarrow \widehat{X}$ and (i) is proved.

To prove (ii), suppose that (3.8) holds for $f = (f_n)$ defined above. If we let $\eta := |x| + \|x\|_1$, then $\eta \in \Gamma_f(Y, \mathcal{F})$ and hence

$$\|f\|_{\mathcal{K}(Y, \mathcal{F})} \leq \|\eta\|_Y \leq \|x\|_Y + \|\mathbf{1}\|_Y \|x\|_1 \leq (1 + d \|\mathbf{1}\|_Y) \|x\|_Y.$$

Then by (i) of Lemma 2 and (3.8),

$$\|x\|_X = \sup_{n \in \mathbb{Z}_+} \|f_n\|_X \leq C(1 + d \|\mathbf{1}\|_Y) \|x\|_Y.$$

Thus $Y \hookrightarrow X$ and $\widehat{Y} \hookrightarrow \widehat{X}$. This completes the proof. □

Before stating the next lemma, we introduce the following notation: if Z is a Banach function space over I , then $\mathcal{D}(Z)$ denotes the set of all functions in Z that are *nonnegative, nonincreasing, and right-continuous*.

LEMMA 4. *Let $(Z_1, \|\cdot\|_{Z_1})$ and $(Z_2, \|\cdot\|_{Z_2})$ be r.i. spaces over I .*

- (i) *If there is a constant $c > 0$ such that $\|P\phi\|_{Z_2} \leq c \|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, then $P \in B(Z_1, Z_2)$ and $\|P\|_{B(Z_1, Z_2)} \leq c$.*

(ii) If there is a constant $c > 0$ such that $\|Q\phi\|_{Z_2} \leq c\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, then $Q \in B(Z_1, Z_2)$ and $\|Q\|_{B(Z_1, Z_2)} \leq c$.

Proof. (i) Suppose $\|\mathcal{P}\phi\|_{Z_2} \leq c\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, and let $\psi \in Z_1$ be arbitrary. According to [1, Lemma 2.1, p. 44], we have $|\mathcal{P}\psi| \leq \mathcal{P}\psi^*$. Since $\psi^* \in \mathcal{D}(Z_1)$, it follows that

$$\|\mathcal{P}\psi\|_{Z_2} \leq \|\mathcal{P}\psi^*\|_{Z_2} \leq c\|\psi^*\|_{Z_1} = c\|\psi\|_{Z_1},$$

as desired.

(ii) Suppose $\|Q\phi\|_{Z_2} \leq c\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, and let $\psi \in Z_1$ be arbitrary. As shown in the proof of [6, Lemma 3], $|Q\psi| \leq Q|\psi| \leq Q\psi^*$. Hence

$$\|Q\psi\|_{Z_2} \leq \|Q\psi^*\|_{Z_2} \leq c\|\psi^*\|_{Z_1} = c\|\psi\|_{Z_1},$$

as desired. □

Note that since $(\Omega, \Sigma, \mathbb{P})$ is nonatomic, there exists a random variable ξ such that

$$\xi^*(t) = 1 - t \quad \text{for all } t \in I. \tag{5.1}$$

It is easy to prove the following:

LEMMA 5. Let ξ be a random variable satisfying (5.1), and define a family of sets $\{A(t) \in \Sigma \mid t \in [0, 1]\}$ by setting

$$A(t) := \{\omega \in \Omega \mid \xi(\omega) > 1 - t\} \quad \text{for each } t \in [0, 1].$$

Let $\phi \in L_1(I)$ and let $x := \phi(1 - \xi)$. Then:

- (i) $x^*(t) = \phi^*(t)$ for all $t \in I$;
- (ii) $A(s) \subset A(t)$ whenever $0 \leq s \leq t \leq 1$;
- (iii) $\mathbb{P}(A(t)) = t$ for all $t \in [0, 1]$;
- (iv) $\int_{A(t)} x \, d\mathbb{P} = \int_0^t \phi(s) \, ds$ for all $t \in [0, 1]$.

We are now ready to prove Proposition 2.

Proof of Proposition 2. (iii) \Rightarrow (i). Assume that $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$. Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and let $f = (f_n) \in \mathcal{M}(\mathcal{F})$. To prove (3.5), we may assume $\sup_n \|f_n\|_Y < \infty$. Then by (ii) of Lemma 2, $Mf \in X$ and

$$\|Mf\|_X \leq \|\mathcal{P}\|_{B(\widehat{Y}, \widehat{X})} \cdot \sup_{n \in \mathbb{Z}_+} \|f_n\|_Y.$$

On the other hand, since $2Mf \in \Gamma_f(X, \mathcal{F})$, we have that $\|f\|_{\mathcal{K}(X, \mathcal{F})} \leq 2\|Mf\|_X$. Therefore

$$\|f\|_{\mathcal{K}(X, \mathcal{F})} \leq 2\|\mathcal{P}\|_{B(\widehat{Y}, \widehat{X})} \cdot \sup_{n \in \mathbb{Z}_+} \|f_n\|_Y,$$

as desired.

(i) \Rightarrow (ii). This is an immediate consequence of (i) of Lemma 2.

(ii) \Rightarrow (iii). Assume that (3.6) holds for any $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and any $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$. In view of Lemma 4, it suffices to show that for all $\phi \in \mathcal{D}(\widehat{Y})$,

$$\|\mathcal{P}\phi\|_{\widehat{X}} \leq k \|\phi\|_{\widehat{Y}} \tag{5.2}$$

with some constant $k > 0$, independent of ϕ . Let $\phi \in \mathcal{D}(\widehat{Y})$ and define $(\mathcal{P}\phi)(0) := \lim_{t \downarrow 0} (\mathcal{P}\phi)(t)$. Then $(\mathcal{P}\phi)(0)$ is finite if and only if $\phi \in L_\infty(I)$, and in this case $(\mathcal{P}\phi)(0) = \|\phi\|_\infty$. Bearing this in mind, we define a nonincreasing sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ in $[0, 1]$ by setting

$$t_0 := 1 \quad \text{and} \quad t_n := \inf\{s \in [0, 1] \mid (\mathcal{P}\phi)(s) \leq 2(\mathcal{P}\phi)(t_{n-1})\}, \quad n \geq 1.$$

Then $t_n \rightarrow 0$ and

$$(\mathcal{P}\phi)(t_n) \leq 2(\mathcal{P}\phi)(t_{n-1}) \quad \text{for all } n \geq 1. \tag{5.3}$$

(In fact, equality holds if and only if $2(\mathcal{P}\phi)(t_{n-1}) \leq (\mathcal{P}\phi)(0)$.)

Let x and $\{A(t) \mid t \in [0, 1]\}$ be as in Lemma 5. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$ by

$$\mathcal{F}_n := \sigma\{\Lambda \setminus A(t_n) \mid \Lambda \in \Sigma\}, \quad n \in \mathbb{Z}_+, \quad \text{and} \quad f_n := \mathbb{E}[x \mid \mathcal{F}_n], \quad n \in \mathbb{Z}_+. \tag{5.4}$$

Then by Lemma 5

$$f_n = \frac{1_{A(t_n)}}{\mathbb{P}(A(t_n))} \int_{A(t_n)} x d\mathbb{P} + x 1_{\Omega \setminus A(t_n)} = (\mathcal{P}\phi)(t_n) 1_{A(t_n)} + x 1_{\Omega \setminus A(t_n)} \quad (\text{a.s.}),$$

for each $n \in \mathbb{Z}_+$. Since $A(t_n) \downarrow \emptyset$ a.s., we have $f_\infty = x$ a.s. and hence for each $n \geq 1$,

$$\begin{aligned} \{(\mathcal{P}\phi)(t_{n-1}) - |x|\} 1_{A(t_{n-1}) \setminus A(t_n)} &\leq |f_\infty - f_{n-1}| 1_{A(t_{n-1}) \setminus A(t_n)} \\ &= \mathbb{E}[|f_\infty - f_{n-1}| \mid \mathcal{F}_n] 1_{A(t_{n-1}) \setminus A(t_n)} \quad (\text{a.s.}). \end{aligned} \tag{5.5}$$

Now let $\gamma \in \Gamma_f(X, \mathcal{F})$. Then for each $n \geq 1$,

$$\begin{aligned} \mathbb{E}[|f_\infty - f_{n-1}| \mid \mathcal{F}_n] 1_{A(t_{n-1}) \setminus A(t_n)} &\leq \mathbb{E}[\gamma \mid \mathcal{F}_n] 1_{A(t_{n-1}) \setminus A(t_n)} \\ &= \gamma 1_{A(t_{n-1}) \setminus A(t_n)} \quad (\text{a.s.}). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6), it follows that

$$\sum_{n=1}^\infty (\mathcal{P}\phi)(t_{n-1}) 1_{A(t_{n-1}) \setminus A(t_n)} \leq \gamma + |x| \quad (\text{a.s.}).$$

We write η for the sum on the left-hand side (which is a finite sum if $\phi \in L_\infty(I)$). Then by (5.3), we have for each $t \in I$,

$$\begin{aligned} (\mathcal{P}\phi)(t) &\leq \sum_{n=1}^\infty (\mathcal{P}\phi)(t_n) 1_{[t_n, t_{n-1})}(t) \\ &\leq 2 \sum_{n=1}^\infty (\mathcal{P}\phi)(t_{n-1}) 1_{[t_n, t_{n-1})}(t) = 2\eta^*(t) \leq 2(\gamma + |x|)^*(t). \end{aligned}$$

Therefore

$$\|\mathcal{P}\phi\|_{\widehat{X}} \leq 2\|(\gamma + |x|)^*\|_{\widehat{X}} = 2\|\gamma + |x|\|_X \leq 2(\|\gamma\|_X + \|x\|_X),$$

which implies

$$\|\mathcal{P}\phi\|_{\widehat{X}} \leq 2\|f\|_{\mathcal{K}(X, \mathcal{F})} + 2\|x\|_X.$$

By Lemma 3, we may replace $\|x\|_X$ with $d\|x\|_Y = d\|\phi\|_{\widehat{Y}}$, where d is a positive constant that is independent of x , and by (3.6) we may replace $\|f\|_{\mathcal{K}(X, \mathcal{F})}$ with $C\|f\|_{\infty} = C\|\phi\|_{\widehat{Y}}$. Thus (5.2) holds with $k=2(C+d)$. This completes the proof. \square

6. Proof of Proposition 3. In addition to lemmas in the previous section, we need one more lemma.

LEMMA 6. Let $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \in \mathbb{F}$ and $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$. If $\gamma \in \Gamma_f(L_1, \mathcal{F})$ and if $g = (g_n)_{n \in \mathbb{Z}_+}$ is the martingale defined by $g_n = \mathbb{E}[\gamma | \mathcal{F}_n]$, $n \in \mathbb{Z}_+$, then

$$\mathbb{E}[Mf] \leq 16\mathbb{E}[Mg].$$

Proof. Let $0 < \delta < 1 < b < \infty$ and let $0 < \lambda < \infty$. We define stopping times ρ , σ , and τ by

$$\rho := \min\{n \in \mathbb{Z}_+ \mid g_n > \delta\lambda\}, \quad \sigma := \min\{n \in \mathbb{Z}_+ \mid |f_n| > \lambda\},$$

and

$$\tau := \min\{n \in \mathbb{Z}_+ \mid |f_n| > b\lambda\}.$$

Here we follow the usual convention that $\min \emptyset = \infty$. Then, on the one hand,

$$\begin{aligned} \{Mf > b\lambda, Mg \leq \delta\lambda\} &= \{\tau < \infty, \rho = \infty\} \\ &\subset \{|f_\tau - f_{\sigma-1}| \geq (b-1)\lambda, \sigma < \rho\}. \end{aligned} \tag{6.1}$$

On the other hand, by assumption,

$$\mathbb{E}[|f_\tau - f_{\sigma-1}| 1_{\{\sigma < \rho\}} | \mathcal{F}_\sigma] \leq g_\sigma 1_{\{\sigma < \rho\}} \leq \delta\lambda 1_{\{\sigma < \infty\}} = \delta\lambda 1_{\{Mf > \lambda\}} \quad (\text{a.s.}) \tag{6.2}$$

Using (6.1) and (6.2), we have that

$$\begin{aligned} \mathbb{P}(Mf > b\lambda, Mg \leq \delta\lambda) &\leq \mathbb{P}(|f_\tau - f_{\sigma-1}| \geq (b-1)\lambda, \sigma < \rho) \\ &\leq \frac{1}{(b-1)\lambda} \mathbb{E}[|f_\tau - f_{\sigma-1}| 1_{\{\sigma < \rho\}}] \\ &\leq \frac{\delta}{b-1} \mathbb{P}(Mf > \lambda). \end{aligned}$$

Hence, by [3, Lemma 7.1],

$$\mathbb{E}[Mf] \leq \frac{b(b-1)}{\delta(b-b\delta-1)} \mathbb{E}[Mg],$$

provided $b - b\delta - 1 > 0$. Setting $b = 2$ and $\delta = 1/4$ gives the desired result. \square

Let $\mathcal{F} = (\mathcal{F}_n), f = (f_n)$, and $g = (g_n)$ be as in Lemma 6. Given $n \in \mathbb{Z}_+$ and $A \in \mathcal{F}_n$, we define $\mathcal{F}'_k := \mathcal{F}_{k+n}, f'_k := (f_{k+n} - f_{n-1}) 1_A$, and $g'_k := g_{k+n} 1_A = \mathbb{E}[\gamma 1_A | \mathcal{F}'_k]$ for each $k \in \mathbb{Z}_+$. Then $\mathcal{F}' = (\mathcal{F}'_k)_{k \in \mathbb{Z}_+} \in \mathbb{F}, f' = (f'_k)_{k \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F}'), g' = (g'_k)_{k \in \mathbb{Z}_+} \in \mathcal{M}(\mathcal{F}')$, and $\gamma 1_A \in \Gamma_{\mathcal{F}'}(L_1, \mathcal{F}')$. Hence by Lemma 6,

$$\mathbb{E}[(Mf - M_{n-1}f) 1_A] \leq \mathbb{E}[Mf'] \leq 16 \mathbb{E}[Mg'] \leq 16 \mathbb{E}[(Mg) 1_A].$$

Thus, under the same assumption as Lemma 6,

$$\mathbb{E}[Mf - M_{n-1}f | \mathcal{F}_n] \leq 16 \mathbb{E}[Mg | \mathcal{F}_n] \quad \text{a.s. for all } n \in \mathbb{Z}_+. \tag{6.3}$$

Proof of Proposition 3. (i) The first inequality of (3.7) is obvious. To prove the second inequality, suppose $\mathcal{R} \in B(\widehat{Y}, \widehat{X})$. Then $\mathcal{QP} \in B(\widehat{Y}, \widehat{X})$. Let $f = (f_n) \in \mathcal{K}(Y, \mathcal{F})$, let $\gamma \in \Gamma_f(Y, \mathcal{F})$, and let $g = (g_n)$ be the martingale defined as in Lemma 6. Then (6.3) holds. According to [7, Theorem 3.3] (or [6, Lemma 4]), we have that $(Mf)^* < 16 \mathcal{Q}(Mg)^*$. Furthermore we know that $(Mg)^* \leq \mathcal{P}g^*_{\infty} = \mathcal{P}\gamma^*$ on I (see the proof of [6, Proposition 3]). Therefore $(Mf)^* < 16 \mathcal{Q}(\mathcal{P}\gamma^*)$, which implies that

$$\begin{aligned} \|Mf\|_X &= \|(Mf)^*\|_{\widehat{X}} \leq 16 \|\mathcal{Q}(\mathcal{P}\gamma^*)\|_{\widehat{X}} \\ &\leq 16 \|\mathcal{QP}\|_{B(\widehat{Y}, \widehat{X})} \|\gamma^*\|_{\widehat{Y}} = 16 \|\mathcal{QP}\|_{B(\widehat{Y}, \widehat{X})} \|\gamma\|_Y. \end{aligned}$$

Thus the second inequality of (3.7) holds with $C = 16 \|\mathcal{QP}\|_{B(\widehat{Y}, \widehat{X})}$.

(ii) Suppose that (3.8) holds for any $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and any $f = (f_n) \in \mathcal{M}(\mathcal{F})$. In view of Lemma 4, it suffices to show that for all $\psi \in \mathcal{D}(\widehat{Y})$,

$$\|\mathcal{Q}\psi\|_{\widehat{X}} \leq k \|\psi\|_{\widehat{Y}} \tag{6.4}$$

with some constant $k > 0$, independent of ψ . To this end, we may assume that $\psi \not\equiv 0$; then $\psi > 0$ on some interval $(0, \delta]$, and hence $(\mathcal{Q}\psi)(t) \uparrow \infty$ as $t \downarrow 0$. Let $\varepsilon > 0$ be given. Since $(\mathcal{Q}\psi)(t)$ is continuous, we can find a sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ in I such that

$$t_0 = 1 \quad \text{and} \quad (\mathcal{Q}\psi)(t_n) = (\mathcal{Q}\psi)(t_{n-1}) + \varepsilon \quad n \geq 1. \tag{6.5}$$

It is obvious that $\{t_n\}$ is strictly decreasing and $t_n \rightarrow 0$. Let $\phi := (\mathcal{Q}\psi) - \psi$, and let x and $\{A(t) \mid t \in [0, 1]\}$ be as in Lemma 5. We again consider the martingale $f = (f_n)$ defined as in (5.4). Since $\mathcal{P}\phi = \mathcal{P}(\mathcal{Q}\psi) - \mathcal{P}\psi = \mathcal{Q}\psi$, we now have

$$f_n = (\mathcal{Q}\psi)(t_n) 1_{A(t_n)} + x 1_{\Omega \setminus A(t_n)} \quad \text{(a.s.) for all } n \in \mathbb{Z}_+.$$

It then follows that

$$\begin{aligned} \mathbb{E}[|f_{\infty} - f_{n-1}| | \mathcal{F}_n] &= \frac{1_{A(t_n)}}{t_n} \int_{A(t_n)} |x - (\mathcal{Q}\psi)(t_{n-1})| d\mathbb{P} \\ &\quad + |x - (\mathcal{Q}\psi)(t_{n-1})| 1_{A(t_{n-1}) \setminus A(t_n)} \quad \text{(a.s.) for all } n \in \mathbb{Z}_+. \end{aligned} \tag{6.6}$$

Since $(\mathcal{Q}\psi)(1 - \xi) 1_{A(t_n)} \geq (\mathcal{Q}\psi)(t_n) 1_{A(t_n)} > (\mathcal{Q}\psi)(t_{n-1}) 1_{A(t_n)}$, we have that

$$|x - (\mathcal{Q}\psi)(t_{n-1})| 1_{A(t_n)} \leq \{(\mathcal{Q}\psi)(1 - \xi) - (\mathcal{Q}\psi)(t_{n-1}) + \psi(1 - \xi)\} 1_{A(t_n)},$$

and hence

$$\begin{aligned} & \frac{1}{t_n} \int_{A(t_n)} |x - (\mathcal{Q}\psi)(t_{n-1})| d\mathbb{P} \\ & \leq \frac{1}{t_n} \int_{\{1-\xi < t_n\}} (\mathcal{Q}\psi)(1-\xi) d\mathbb{P} + \frac{1}{t_n} \int_{\{1-\xi < t_n\}} \psi(1-\xi) d\mathbb{P} - (\mathcal{Q}\psi)(t_{n-1}) \\ & = \frac{1}{t_n} \int_0^{t_n} (\mathcal{Q}\psi)(s) ds + \frac{1}{t_n} \int_0^{t_n} \psi(s) ds - (\mathcal{Q}\psi)(t_{n-1}). \end{aligned}$$

By (6.5) the right-hand side is equal to

$$\begin{aligned} & (\mathcal{P}(\mathcal{Q}\psi))(t_n) + (\mathcal{P}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1}) \\ & = 2(\mathcal{P}\psi)(t_n) + (\mathcal{Q}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1}) = 2(\mathcal{P}\psi)(t_n) + \varepsilon. \end{aligned}$$

Thus

$$\frac{1_{A(t_n)}}{t_n} \int_{A(t_n)} |x - (\mathcal{Q}\psi)(t_{n-1})| d\mathbb{P} \leq \{2(\mathcal{P}\psi)(t_n) + \varepsilon\} 1_{A(t_n)}. \tag{6.7}$$

As for the second term on the right-hand side of (6.6), we have

$$\begin{aligned} & |x - (\mathcal{Q}\psi)(t_{n-1})| 1_{A(t_{n-1}) \setminus A(t_n)} \\ & \leq \{(\mathcal{Q}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1}) + \psi(1-\xi)\} 1_{A(t_{n-1}) \setminus A(t_n)} \\ & = \{\psi(1-\xi) + \varepsilon\} 1_{A(t_{n-1}) \setminus A(t_n)}. \end{aligned} \tag{6.8}$$

From (6.6), (6.7), and (6.8), we see that for each $n \geq 1$,

$$\mathbb{E}[|f_\infty - f_{n-1}| | \mathcal{F}_n] \leq 2(\mathcal{P}\psi)(t_n) 1_{A(t_n)} + \psi(1-\xi) 1_{A(t_{n-1}) \setminus A(t_n)} + \varepsilon \quad (\text{a.s.}) \tag{6.9}$$

If we set $t_{-1} = 1$, then (6.9) remains valid for $n = 0$. Indeed,

$$\mathbb{E}[|f_\infty| | \mathcal{F}_0] = \|x\|_1 \leq \|\mathcal{Q}\psi\|_1 + \|\psi\|_1 = 2\|\psi\|_1 = 2(\mathcal{P}\psi)(t_0) \quad (\text{a.s.})$$

Let $\gamma_\varepsilon := 2\psi(1-\xi) + \varepsilon$. Then $\gamma_\varepsilon^* = 2\psi + \varepsilon \in \widehat{Y}$ and

$$\mathbb{E}[\gamma_\varepsilon | \mathcal{F}_n] = 2(\mathcal{P}\psi)(t_n) 1_{A(t_n)} + 2\psi(1-\xi) 1_{\Omega \setminus A(t_n)} + \varepsilon \quad (\text{a.s.})$$

Comparing this with (6.9), we see that $\gamma_\varepsilon \in \Gamma_f(Y, \mathcal{F})$. Thus

$$\|f\|_{\mathcal{K}(Y, \mathcal{F})} \leq \|\gamma_\varepsilon\|_Y \leq 2\|\psi\|_{\widehat{Y}} + \varepsilon \|\mathbf{1}\|_Y. \tag{6.10}$$

On the other hand, by (B3') we have that

$$\|\mathcal{Q}\psi\|_{\widehat{X}} - \|\psi\|_{\widehat{X}} \leq \|\phi\|_{\widehat{X}} = \|x\|_X \leq \liminf_{n \rightarrow \infty} \|f_n\|_X \leq \sup_{n \in \mathbb{Z}_+} \|f_n\|_X. \tag{6.11}$$

Using (3.8), (6.10) and (6.11), we obtain

$$\|\mathcal{Q}\psi\|_{\widehat{X}} \leq C(2\|\psi\|_{\widehat{Y}} + \varepsilon \|\mathbf{1}\|_Y) + \|\psi\|_{\widehat{X}}.$$

According to Lemma 3, there is a positive constant d such that $\|\cdot\|_{\widehat{X}} \leq d\|\cdot\|_{\widehat{Y}}$ on \widehat{Y} . Replacing $\|\psi\|_{\widehat{X}}$ by $d\|\psi\|_{\widehat{Y}}$ and letting $\varepsilon \downarrow 0$, we see that (6.4) holds with $k = 2C + d$. This completes the proof. □

7. Proof of Theorem 1. We conclude the paper with the proof of our main theorem.

Proof of Theorem 1. (i) \Rightarrow (iii). Suppose that (i) of Theorem 1 holds. Then by Proposition 1, there exists a norm $\|\cdot\|_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is an r.i. space. If $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}(\mathcal{F})$, then $\|f_n\|_X \leq \|f_{n+1}\|_X$ for all $n \in \mathbb{Z}_+$ (cf. Remark 1), and hence by (3.2),

$$K^{-1} \sup_{n \in \mathbb{Z}_+} \|f_n\|_X \leq \|f\|_{\mathcal{K}(X, \mathcal{F})} \leq K \sup_{n \in \mathbb{Z}_+} \|f_n\|_X. \tag{7.1}$$

Here $\|f\|_{\mathcal{K}(X, \mathcal{F})} = \inf\{\|\gamma\|_X \mid \gamma \in \Gamma_f(X, \mathcal{F})\}$ and K is a constant that is independent of f . It then follows from Propositions 2 and 3 that $\mathcal{P} \in B(\widehat{X})$ and $\mathcal{Q} \in B(\widehat{X})$, where \widehat{X} stands for the Luxemburg representation of $(X, \|\cdot\|_X)$. As mentioned at the end of Section 2, this means that $\alpha_X > 0$ and $\beta_X < 1$.

(iii) \Rightarrow (ii). Suppose that (iii) holds. This implies that $\mathcal{P} \in B(\widehat{X})$ and $\mathcal{R} \in B(\widehat{X})$. It then follows from Propositions 2 and 3 that (7.1) holds for any $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and any $f = (f_n) \in \mathcal{M}(\mathcal{F})$. If $f = (f_n)$ is uniformly integrable, then $\|f_\infty\|_X = \sup_n \|f_n\|_X$ by (i) of Lemma 2. Since the norms $\|\cdot\|_X$ and $\|\cdot\|_X$ are equivalent, we obtain (3.3).

(ii) \Rightarrow (i). Given a martingale $f = (f_n)$, we let $f^{(n)}$ denote the stopped martingale $(f_{n \wedge k})_{k \in \mathbb{Z}_+}$. Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}(\mathcal{F})$, and suppose that (ii) holds. Then, by the first inequality of (3.3),

$$\|f_n\|_X \leq C \|f^{(n)}\|_{\mathcal{K}(X, \mathcal{F})} \leq C \|f\|_{\mathcal{K}(X, \mathcal{F})} \quad \text{for all } n \in \mathbb{Z}_+,$$

where the second inequality follows from the fact that $\Gamma_f(X, \mathcal{F}) \subset \Gamma_{f^{(n)}}(X, \mathcal{F})$. From the inequality above, we easily obtain the first inequality of (3.2). Hence, by Proposition 1, there is a norm $\|\cdot\|_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is an r.i. space.

We now turn our attention to the second inequality of (3.2). Note that if $f = (f_n) \in \mathcal{M}(\mathcal{F})$ is uniformly integrable, then by the second inequality of (3.3) and (B3'),

$$\|f\|_{\mathcal{K}(X, \mathcal{F})} \leq C \|f_\infty\|_X \leq C \lim_{n \rightarrow \infty} \|f_n\|_X \leq \overline{\lim}_{n \rightarrow \infty} \|f_n\|_X. \tag{7.2}$$

Thus the required inequality holds for uniformly integrable martingales. Moreover, since the first inequality of (7.2) can be rewritten as $\|f\|_{\mathcal{K}(X, \mathcal{F})} \leq K \|f_\infty\|_X$, Proposition 2 implies that $\mathcal{P} \in B(\widehat{X})$.

Finally, let $f = (f_n) \in \mathcal{M}(\mathcal{F})$ be such that $\overline{\lim}_n \|f_n\|_X < \infty$. Then, since $\sup_n \|f_n\|_X < \infty$, Lemma 2 shows that $Mf \in X$. Therefore f is uniformly integrable, and satisfies (7.2). This completes the proof. \square

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