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SEPARATION OF FUNCTIONS

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1. **Introduction.** Eames [2], and Jeffery [5], consider separation of sets in a measure space and show that, if A is separated from B, then

$$m^*(A \cup B) = m^*A + m^*B,$$

where m^* denotes outer measure.

In this paper we consider the class, $\mathscr{F}^+$, of nonnegative bounded real-valued functions of a real variable. We define an outer integral of f over E, $I^*(f; E)$, which satisfies

$$I^*(f+g; E) \leq I^*(f; E) + I^*(g; E),$$

for all $f, g \in \mathscr{F}^+$. A definition of separation of functions is introduced, so that, if f is separated from g, then

$$I^*(f+g; E) = I^*(f; E) + I^*(g; E),$$

for all $f, g \in \mathscr{F}^+$ and $E \subseteq R$.

2. **Preliminaries.** The Lebesgue outer and inner linear measure of the linear set A is denoted by m^*A and m_*A respectively. We also denote the outer and inner planar Lebesgue measure of the planar set A by m^*A and m_*A, respectively. It is understood from the context whether m^* denotes planar or linear outer Lebesgue measure.

The set M is a measurable cover of the set A if M is a measurable superset of A which satisfies $m_*(M-A)=0$. It is shown in [1] that if M is a measurable cover of A and E is measurable then $M \cap E$ is a measurable cover of $A \cap E$.

Let $f \in \mathscr{F}^+$. We denote by $\bar{f}$ the measurable cover function introduced in [3]. It is shown in [3] that $\bar{f}$ satisfies the following:

(i) $\bar{f}$ is measurable and $\bar{f}(x) \geq f(x)$, for all x;

(ii) if $h(x)$ is measurable and $h(x) \geq f(x)$, for all x, then $h(x) \geq \bar{f}(x)$, a.e. (thus if f is measurable, then $f(x) = \bar{f}(x)$, a.e.).

Let $\bar{D}(x, A)$ denote either the strong or symmetric Lebesgue upper density of the linear set A at the point x. The bounded function f is called measurable at the point x if

$$\bar{D}(x, \{y : f(y) \neq \bar{f}(y)\}) = 0.$$

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We then write $f \in C_x$. If f is measurable then it is measurable at every point, i.e. $C_x = \emptyset$.

It is shown in [3] that, for any bounded function f, the set $F = \{x : f \in C_x\}$ is measurable.

THEOREM 1. *If f is bounded, then the restriction of f to $F = \{x : f \in C_x\}$ is measurable.*

Proof. We show that $f(x) = \bar{f}(x)$, for almost all $x \in F$, so that since F is a measurable set and $\bar{f}$ is a measurable function, the result follows.

Suppose otherwise, then there is a subset A of F such that $m^*A > 0$ and, if $y \in A$, then $f(y) \neq \bar{f}(y)$. By the Lebesgue density theorem, there exists $x \in A$ such that $\bar{D}(x, A) > 0$. Thus $\bar{D}(x, \{y : f(y) \neq \bar{f}(y)\}) > 0$, so that $f \notin C_x$. Since $x \in F$, this is a contradiction.

THEOREM 2. *If f is measurable and g is any bounded function, then $\overline{f+g} = \bar{f} + \bar{g}$,* a.e.

Proof. By (i), $f+g \leq \bar{f} + \bar{g}$. Therefore, by (ii), $\overline{f+g} \leq \bar{f} + \bar{g}$, a.e. By (i) and the measurability of f, $f+g = \bar{f} + g \leq \overline{f+g}$. Thus $g \leq \overline{f+g} - \bar{f}$ and therefore, by (ii), $\bar{g} \leq \overline{f+g} - \bar{f}$. Thus $\overline{f+g} = \bar{f} + \bar{g}$, a.e.

Let $f \in \mathscr{F}^+$, and let E be a linear set. The inner ordinate set of f relative to E is defined to be

$$\{(x, y) : x \in E \text{ and } 0 < y < f(x)\},$$

and is denoted by $0(f; E)$.

The outer integral of f over E is defined to be

$$m^*\{0(f; E)\},$$

and is denoted by $I^*(f; E)$. In case the set E and the function f are measurable, the above definition agrees with the usual definition of the Lebesgue integral of a nonnegative function (see, for example, [6]) and in this case we write $I^*(f; E) = I(f; E)$. Thus, if D and E are disjoint measurable sets and f and g are measurable functions, then

$$I(f+g; E) = I(f; E) + I(g; E),$$

and

$$I(f; D \cup E) = I(f; D) + I(f; E).$$

The following theorem is essential to the work of this paper. A proof may be found in [6].

THEOREM 3. *Let A and E be linear sets and let χ_A be the characteristic function of A. Let a be any nonnegative real number. Then*

$$I^*(a\chi_A; E) = aI^*(\chi_A; E) = am^*(A \cap E).$$

Also, $0(\chi_A; E)$ *is a planar measurable set if and only if* $A \cap E$ *is a linear measurable set.*

3. Properties of I*.

THEOREM 4. *Let* $f \in \mathscr{F}^+$ *and let* E *be a measurable linear set. Then* $M=0(\bar{f}; E)$ *is a measurable cover of* $A=0(f; E)$ *in the plane and*

$$I^*(f; E) = I(\bar{f}; E).$$

Proof. It is clear that M is a superset of A. If M is not a measurable cover of A, then there is a measurable set $D \subseteq M-A$, with $mD>0$. For a planar set E we use the notation $E^x=\{x:(x, y) \in E$, for some $y\}$, and for a linear set F we write $c(F)=\{(x, y):x \in F \,\&\, y>0\}$.

Let K be a measurable kernel of D^x. Then $D=[D \cap c(K)] \cup [D \cap c(D^x-K)]$. We then have

$$mD = m^*[D \cap c(K)]+m_*[D \cap c(D^x-K)],$$

(see [4, p. 61]). Since $m_*(D^x-K)=0$, we have $m_*[D \cap c(D^x-K)]=0$. Also $D \cap c(K)$ is measurable and we therefore have

$$mD = m[D \cap c(K)].$$

Thus replacing D by $D \cap c(K)$, if necessary, we may assume that D^x is measurable.

Now let g_n be the measurable function defined by

$$g_n(x) = \max\left\{\bar{f}(x)-\frac{1}{n}\,; 0\right\}, \quad \text{if } x \in D^x;$$

$$g_n(x) = \bar{f}(x), \quad \text{if } x \in E-D^x.$$

Then $\{0(g_n, E)\}_n$ is an increasing sequence of sets with limit M. Hence

$$\lim_n m[0(g_n, E) \cap D] = mD,$$

and therefore, for some N, $m[0(g_N, E) \cap D]>0$.

Let H be a measurable kernel of $\{x:g_N(x)>y$ for some $(x, y) \in D\}$. Then $mH>0$, and we note that for $x \in H$, $g_N(x)=\bar{f}(x)-1/N$. Now we define

$$g(x) = \bar{f}(x), \quad \text{if } x \in E-H;$$

$$g(x) = g_N(x) = \bar{f}(x)-\frac{1}{N}, \quad \text{if } x \in H.$$

Clearly $g(x)\leq \bar{f}(x)$ on E and $\{x:g(x)<\bar{f}(x)\}=H$ has positive measure. Also $g\geq f$ on E, since on $E-H$, $g=\bar{f}\geq f$ and for $x \in H$, we have $g(x)=g_N(x)>y$, for some $(x, y) \in D$ and $y>f(x)$. Hence the properties of g contradicts (i) and (ii) and therefore M is a measurable cover of A.

By the definition of I and I^* it follows that $I^*(f, E)=I(\bar{f}, E)$.

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THEOREM 5. *Let $f \in \mathscr{F}^+$ and let $f(x)=0$ for $x \notin E$. Let M be a measurable cover of E. Then*

$$I^*(f; E) = I^*(f; M) = I(\bar{f}; M).$$

Hence $0(\bar{f}; M)$ *is a measurable cover of* $0(f, E)$.

Proof. Since $f(x)=0$, if $x \notin E$, we have $0(f; E)=0(f; M)$, so that $I^*(f; E)=I^*(f; M)$. By the previous theorem, the result follows.

THEOREM 6. *If E is measurable, then, for all $A \subseteq R$ and $f \in \mathscr{F}^+$,*

$$I^*(f; A) = I^*(f; A \cap E)+I^*(f; A-E).$$

Proof. The theorem follows easily from the measurability of $c(E)$.

THEOREM 7. *Let $f, g \in \mathscr{F}^+$ and let $E \subseteq R$. Then*

$$I^*(f+g; E) \leq I^*(f; E)+I^*(g; E).$$

Proof. Let $E_1=\{x:f(x)>0\}$ and $E_2=\{x:g(x)>0\}$, and let M_1 and M_2 be measurable covers of E_1 and E_2, respectively. Then $M_1 \cup M_2$ is a measurable cover of $E_1 \cup E_2 = \{x:f(x)+g(x)>0\}$.

Using Theorems 4 and 5 and the fact that $\overline{f+g} \leq \bar{f}+\bar{g}$, we have

$$\begin{aligned} I^*(f+g; E) &= I^*(f+g; E_1 \cup E_2) \\ &= I(\overline{f+g}; M_1 \cup M_2) \\ &\leq I(\bar{f}+\bar{g}; M_1 \cup M_2). \end{aligned}$$

Since $I(\bar{f}+\bar{g}; M_1 \cup M_2)$ is a Lebesgue integral it follows that $I^*(f+g; E) \leq I(\bar{f}; M_1)+I(\bar{g}; M_2)+I(\bar{f}; M_2-M_1)+I(\bar{g}; M_1-M_2)$. The function f is bounded and there is therefore a constant c such that $0 \leq f \leq c\chi_{m_1}$. Hence, by (ii), $0 \leq \bar{f} \leq c\chi_{m_1}$, a.e. Thus $\bar{f}(x)=0$ for almost all $x \notin M_1$. Thus $I(\bar{f}; M_2-M_1)=0$, and similarly $I(\bar{g}; M_1-M_2)=0$. Also, by Theorem 5, $I(\bar{f}; M_1)=I^*(f; E_1)$ and $I(\bar{g}; M_2)=I^*(g; E_2)$. It follows that

$$\begin{aligned} I^*(f+g; E) &\leq I^*(f; E_1)+I^*(g; E_2) \\ &= I^*(f; E)+I^*(g; E). \end{aligned}$$

4. **Separation of functions.**

DEFINITION. Let $f, g \in \mathscr{F}^+$. Let $F=\{x:f \notin C_x\}$ and let $G=\{x:g \notin C_x\}$. The function f *is separated from the function* g if $m(F \cap G)=0$.

THEOREM 8. *Let $f, g \in \mathscr{F}^+$ and $E \subseteq R$. Then, if f is separated from g,*

$$I^*(f+g; E) = I^*(f; E)+I^*(g; E).$$

Proof. We first prove the theorem under the assumption that g is measurable, (so that certainly f is separated from g, since $G=\emptyset$).

Let $f_E=f\chi_E$. Then, if M is a measurable cover of E, by Theorem 5, we have

$$\begin{aligned} I^*(f; E)+I^*(g; E) &= I^*(f_E; E)+I^*(g; E) \\ &= I(\bar{f}_E; M)+I(g; M). \end{aligned}$$

Now, using Theorem 2, and the fact that the terms on the right-hand side of the above equation are Lebesgue integrals, we have,

$$\begin{aligned} I^*(f; E)+I^*(g; E) &= I(\overline{f_E+g}; M) \\ &= I^*(f_E+g_E; E), \end{aligned}$$

where $g_E=g\chi_E$, and we have used Theorem 5. Since $0(f_E+g_E; E)=0(f+g; E)$, the result follows in the case when g is measurable.We note that, if the restriction of g to a measurable superset A of E is measurable, then the result also holds by replacing g by $g\chi_A$.

In the general case, by Theorem 1, f and g are measurable on the complements of the measurable sets F and G, respectively. Since $m(F \cap G)=0$, the restriction of g to F is measurable. Using the first part of this theorem and using Theorem 6, it therefore follows that

$$\begin{aligned} I^*(f; E)+I^*(g; E) &= I^*(f; E \cap F)+I^*(f; E-F)+I^*(g; E \cap F)+I^*(g; E-F) \\ &= I^*(f+g; E \cap F)+I^*(f+g; E-F) \\ &= I^*(f+g; E). \end{aligned}$$

THEOREM 9. *Let A and B be linear sets which satisfy $m(A \cap B)=0$. Let M and N be measurable covers of A and B respectively. Then the characteristic functions of A and B are separated if and only if $m(M \cap N)=0$.*

Proof. We note that $\chi_{A\cup B}=\chi_A+\chi_B$ a.e., since $m(A \cap B)=0$. We therefore have, by the previous theorem, that if χ_A is separated from χ_B, then

$$I^*(\chi_{A\cup B}; E) = I^*(\chi_A; E)+I^*(\chi_B; E),$$

for all $E \subseteq R$. Thus, by Theorem 3,

$$m^*((A \cup B) \cap E) = m^*(A \cap E)+m^*(B \cap E)$$

Now, if $m^*(A \cup B)$ is finite, we put $E=A \cup B$. We have

$$\begin{aligned} m^*(A \cup B) &= m^*A+m^*B \\ &= mM+mN \\ &= m(M \cup N)+m(M \cap N) \\ &\geq m^*(A \cup B), \end{aligned}$$

so that $m(M \cap N)=0$. If $m^*(A \cup B)$ is not finite, we put $E_n=(A \cup B) \cap (-n, n)$ and we conclude that $m(M \cap N \cap (-n, n))=0$, for all natural numbers n, so that $m(M \cap N)=0$.

Now suppose that $m(M \cap N)=0$, then we show that χ_A is separated from χ_B. By Theorems 1 and 2, $\chi_A=\bar{\chi}_A$, a.e. on the M complement of $\{x:\chi_A \notin C_x\}=F \subset M$. Similarly, $\{x:\chi_B \notin C_x\}=G \subset M$. Therefore $m(F \cap G)=0$, so that χ_A is separated. from χ_B.

The above theorem shows that the characteristic functions of sets are separated if and only if the sets are separated in the sense of the definition given in [2].

References

1. S. K. Berberian, *Measure and integration*, New York, 1965.
2. W. Eames, *A local property of measurable sets*, Canad. J. Math. **12** (1960), 632–640.
3. W. Eames and L. E. May, *Measurable cover functions*, Canad. Math. Bull. (4) **10** (1967), 519–523.
4. P. R. Halmos, *Measure theory*, Princeton, 1950.
5. R. L. Jeffery, *Sets of k-extent in n-dimensional space*, Trans. Amer. Math. Soc. **35** (1933), 629–647.
6. H. Kestelman, *Modern theories of integration*, Dover, New York, 1960.

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