Seventh Meeting, 10*th June* 1904.

Mr CHARLES TWEEDIE, President, in the Chair.

The Turning-Values of a Cubic Function and the nature of the roots of a Cubic Equation.

By P. PINKERTON, M.A.

The first part of this paper depends on the theorem that if a, b, c are three positive quantities such that

$$a+b+c= \text{a constant},$$

then abc is a maximum when $a=b=c$; with the corollary that if a, b, c are three negative quantities such that

$$a+b+c= \text{a constant},$$

then abc is a minimum when $a=b=c$.

1. Consider the graph of

$$y=x(x-a)^2,$$

where a is positive.

The graph meets OX at the points (0, 0) (a, 0). The graph has a minimum point at (a, 0), for on shifting the origin to (a, 0) the equation becomes $$y=(\xi+a)\xi^2,$$
and a first approximation at the new origin is

$$y=a\xi^2,$$

so that the graph close to that point is of the form of a festoon. There is also a maximum point in the interval between $x=0$ and $x=a$. To determine the point we observe that $x(x-a)^2$ is a maximum when $2x(a-x)(a-x)$ is a maximum. Now each of these factors is positive in the interval considered and their sum is constant $(=2a)$; $\therefore$ a maximum value occurs when

$$2x=a-x \text{ i.e., when } x=\frac{a}{3}.$$

The maximum value is therefore $\dfrac{a}{3}\left(\dfrac{a}{3}-a\right)^2=\dfrac{4a^3}{27}$.

https://doi.org/10.1017/S0013091500002029 Published online by Cambridge University Press

2. Again considering the graph of

$$y = x(x+a)^2,$$

where a is positive, we observe that if the origin is shifted to the point $(-a, 0)$ a first approximation at the new origin is

$$y = -a\xi^2$$

which represents an inverted festoon. There is therefore a maximum point at $(-a, 0)$ and a minimum point in the interval $x = 0$ to $x = -a$. To find this point we observe that $x(x+a)^2$ has its minimum value when $2x(-a-x)(-a-x)$ is a minimum. Each of these factors is negative in the interval considered and their sum is constant $(= -2a)$;

$$\therefore \quad 2x = -a - x \text{ for the minimum point};$$

$$\therefore \quad x = -\frac{a}{3};$$

and the minimum value is $-\dfrac{4a^3}{27}$.

3. In general let

$$y = x^3 + px^2 + qx + r = (x+a)(x+b)^2 + c.$$

Equating coefficients, we have

$$p = a + 2b, \qquad (1)$$

$$q = 2ab + b^2, \qquad (2)$$

$$r = ab^2 + c. \qquad (3)$$

Eliminating a from (1) and (2),

$$3b^2 - 2bp + q = 0;$$

$\therefore$ real values of a, b, c can be found if

$$p^2 \gtreqless 3q.$$

If $p^2 = 3q$, it is clear that we can write

$$y = x^3 + px^2 + qx + r = \left(x + \frac{p}{3}\right)^3 + \left(r - \frac{p^3}{27}\right),$$

and by changing the origin to $\left(-\dfrac{p}{3},\ r - \dfrac{p^3}{27}\right)$ the equation takes the form $\eta = \xi^3$ which has a point of inflexion at the new origin and no turning-points.

If $p^2 > 3q$, $y = (x+a)(x+b)^2 + c$ where a, b, c are real. Shift the origin to $(-a, c)$ and the equation becomes

$$\eta = \xi(\xi - \overline{a-b})^2.$$

Hence, if $a-b$ is positive, there is a minimum turning-point at

$$\xi = a-b,\ \ \eta = 0\ \ i.e., \text{ at } x = -b,\ \ y = c.$$

Also there is a maximum turning-point at

$$\xi = \tfrac{1}{3}(a-b),\ \ \eta = \tfrac{4}{27}(a-b)^3$$

$$i.e., \text{ at } x = -a + \tfrac{1}{3}(a-b),\ \ y = c + \tfrac{4}{27}(a-b)^3.$$

If $(a-b)$ is negative, there is a maximum turning-point at

$$\xi = a-b,\ \eta = 0\ \ i.e., \text{ at } x = -b,\ y = c$$

and a minimum turning-point at

$$\xi = \tfrac{1}{3}(a-b),\ \ \eta = \tfrac{4}{27}(a-b)^3$$

$$i.e., \text{ at } x = -a + \tfrac{1}{3}(a-b),\ \ y = c + \tfrac{4}{27}(a-b)^3.$$

The graph whose equation is

$$y = Ax^3 + px^2 + qx + r$$

can clearly be reduced to the above case.

4. The nature of the roots of a cubic equation can be deduced from our knowledge of the above graphs.

Suppose the equation first brought to the form

$$x^3 + qx + r = 0.$$

Let $\qquad y = x^3 + qx + r = (x+a)(x+b)^2 + c.$

Here

$$a + 2b = 0, \qquad (1')$$

$$2ab + b^2 = q, \qquad (2')$$

$$ab^2 + c = r. \qquad (3')$$

If two roots are equal then clearly $c = 0$;

$\therefore$ by $(1')$ and $(2')$ $\qquad -3b^2 = q,$

and by $(1')$ and $(3')$ $\qquad -2b^3 = r;$

$$\therefore\ \ 4q^3 + 27r^2 = 0.$$

If the three roots are unequal $c \neq 0$.

In this case it is clear that if a_1, b_1, c_1 and a_2, b_2, c_2 are the two sets of solutions of $(1')$, $(2')$, $(3')$, then

$$a_1 + a_2 = 0,\ \ b_1 + b_2 = 0;$$

$$\therefore\ \ (a_1 - b_1) = -(a_2 - b_2).$$

https://doi.org/10.1017/S0013091500002029 Published online by Cambridge University Press

Suppose $a_1 - b_1$ to be positive and write

$$y = (x + a_1)(x + b_1)^2 + c_1.$$

It follows from the above that the minimum turning value is given by $y = c_1$.

Next, writing $\quad y = (x + a_2)(x + b_2)^2 + c_2,$

we observe that the maximum turning value is given by

$$y = c_2 ;$$

$\therefore$ OX will cut the graph of

$$y = x^3 + qx + r$$

in three different real points if $\frac{c_1}{c_2}$ is negative,

and in one real point and two imaginary points if $\frac{c_1}{c_2}$ is positive.

There are therefore three different real solutions of the equation, or one real and two imaginary,

according as	$\frac{c_1}{c_2}$	is negative or positive,	
$\therefore$,, ,,	$\frac{c_1{}^2}{c_1c_2}$	,, ,, ,, ,,	
$\therefore$,, ,,	c_1c_2	,, ,, ,, ,,	
$\therefore$,, ,,	$(r - a_1b^2)(r - a_2b^2)$	,, ,, ,, ,,	
$\therefore$,, ,,	$r^2 + a_1a_2b^4$	,, ,, ,, ,,	$(\because a_1 + a_2 = 0)$
$\therefore$,, ,,	$r^2 - a^2b^4$	,, ,, ,, ,,	
$\therefore$,, ,,	$r^2 - 4b^6$	,, ,, ,, ,,	by (1′)
$\therefore$,, ,,	$r^2 + \frac{4}{27}q^3$	,, ,, ,, ,,	by (1′) and (2′)
$\therefore$ according as	$4q^3 + 27r^2$	is negative or positive.	

Note on the Problem: To draw through a given point a transversal to (*a*) a given triangle (*b*) a given quadrilateral so that the intercepted segments may have (*a*) a given ratio (*b*) a given cross ratio.

By P. Pinkerton, M.A.