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EXISTENCE OF INVARIANT WEAK UNITS IN BANACH LATTICES: COUNTABLY GENERATED LEFT AMENABLE SEMIGROUP OF OPERATORS

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ABSTRACT Let Σ be a countably generated left amenable semigroup and $\{T_\sigma \mid \sigma \in \Sigma\}$ be a representation of Σ as a semigroup of positive linear operators on a weakly sequentially complete Banach lattice E with a weak unit e It is assumed T_σ are uniformly bounded It is shown that a necessary and sufficient condition for the existence of a weak unit invariant under $\{T_\sigma \mid \sigma \in \Sigma\}$ is that $\inf_{\sigma\in\Sigma} H(T_\sigma e) > 0$ for all nonzero H in the positive dual cone of E

1. **Introduction.** The first asymptotic conditions for the existence of an equivalent invariant measure were given for a nonsingular point-transformation τ on a probability space (X, F, p). The following result was proved, independently, by Y. N. Dowker [8] and A. Calderon [4].

THEOREM 1.1. *Let (X, F, p) be a probability space. Let τ be a nonsingular measurable mapping from X into X. Then the following conditions are equivalent:*
(i) There exists an equivalent finite invariant measure.
(ii) $\liminf_n p(\tau^{-n}A) > 0$ *if* $p(A) > 0$.

E. Granirer [10] extended the above result to the case of (left) amenable semigroup of point-transformations.

If τ is a nonsingular transformation on (X, F, p) then τ generates a positive contraction T on $L_1(X, F, p)$; with each $f = d\phi/dp$ where ϕ is a finite measure, one associates $Tf = g$ given by $g = d(\phi o \tau^{-1})/dp$. Thus the problem of existence of equivalent invariant measures in (X, F, p) generalizes to the problem of existence of strictly positive fixed points in $L_1(X, F, p)$.

U. Sachdeva [16] obtained the following result in the case of a left amenable semigroup of positive linear contractions on $L_1(X, F, p)$.

THEOREM 1.2. *Let Σ be a left amenable semigroup and $\{T_\sigma \mid \sigma \in \Sigma\}$ be positive linear contractions on $L_1(X, F, p)$. Then the following conditions are equivalent:*
(i) There exists a strictly positive $f \in L_1$ such that $T_\sigma f = f$, $\forall \sigma \in \Sigma$.
(ii) $\inf_{\sigma\in\Sigma} \int_A T_\sigma 1\, dp > 0$, $\forall p(A) > 0$.

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Now we consider the generalization of the above result in a different direction. Consider a weakly sequentially complete Banach lattice E with a weak unit e, *i.e.*, an element $e \in E_+$ such that $e \wedge |f| = 0$ for $f \in E$ implies $f = 0$.

P. C. Shields [17] proved the result:

THEOREM 1.3. *Let T be a bounded positive linear operator on E. The following conditions are equivalent:*

(i) There exists an invariant weak unit v in E, i.e., weak unit v such that $Tv = v$.
(ii) $\inf_n H(T^n e) > 0$, $\forall H \in E^*_{++}$.

The usual argument that uses construction of a countably additive measure in proving similar results in the case of a point transformation on a probability space or a contraction on $L_1(X, F, p)$ does not work in the above case. Shields used the concept of countably additive functionals instead.

For all our results we consider a countably generated left amenable semigroup Σ and for the operators on E, we consider a positive linear operator representation $\{T_\sigma : \sigma \in \Sigma\}$ of Σ with $\sup\{\|T_\sigma\| : \sigma \in \Sigma\} = K < \infty$. We obtain a result similar to the above results in our general setting. More precisely, we have the following:

THEOREM 1.4. *Let E be a weakly sequentially complete Banach lattice with a weak unit e. Let $\{T_\sigma : \sigma \in \Sigma\}$ be a representation of a countably generated left amenable semigroup as a semigroup of uniformly bounded positive linear operators on E. Then the following conditions are equivalent:*

(i) There exists an invariant weak unit v in E, i.e., $\exists$ a weak unit $v \in E_+$ such that $T_\sigma v = v$, $\forall \sigma \in \Sigma$.
(ii) $\inf_{\sigma \in \Sigma} H(T_\sigma e) > 0$, $\forall H \in E^*_{++}$.

In our proof we use the concept of countably additive functionals as P. C. Shields [17] did. But, instead of the Theorem 4 of [17] we use a weaker result from our main reference [14]. Once a sub-invariant weak unit is found, our proof differs entirely as the approach in [17] will not work in our case.

It shall be noted that in the case of cyclic semigroup of operators generated by a positive linear contraction T, there are equivalent conditions for the existence of invariant weak units derived in terms of the Cesàro Averages of the operator T ([9], [4], [18] and [3]). As there is no straight forward generalization of such averages to our case, we do not deal with similar conditions in this paper.

In Section 2, we have some preliminary results.

In Section 3, we define operator averages that would generalize Cesàro Averages to the case of countably generated left amenable semigroup of operators. Then we prove a crucial asymptotic invariant property of the averages.

In Section 4, we give the proof of Theorem 1.4. The operator averages defined in Section 3 are used in the construction of an invariant weak unit in Section 4.

2. **Preliminary results.**

2.1. *On Banach lattices.* Let E be a Banach lattice. We denote the set all elements $f \in E$ such that $f \geq 0$ by E_+ and we define $E_{++} = E_+ \backslash \{0\}$.

The dual of the Banach lattice E is denoted by E^*.

We consider a Banach lattice E satisfying the following conditions:

(A) E has a weak unit e, *i.e.*, there exists $e \in E_+$ such that for any $f \in E$, $e \wedge |f| = 0 \Rightarrow f = 0$.

(B) Any norm bounded increasing sequence in E has a strong limit.

The condition (B) is equivalent to weak sequential completeness ([14], p. 34). The condition (B) implies order continuity of the lattice norm (OCN):

(OCN) For every downwards directed net $(f_\iota, i \in I)$ with $\wedge_{\iota \in I} f_\iota = 0$, one has $\lim_\iota \|f_\iota\| = 0$

Another condition that is equivalent to (OCN) is that every order interval $[f, g] = \{h : f \leq h \leq g\}$ in E is weakly compact. ([14], p. 28).

We denote strong convergence in E by $\longrightarrow$ and weak convergence in E by $\xrightarrow{w}$. We denote order convergence of monotone nets by $\uparrow$ or $\downarrow$.

DEFINITION 2.1. An ideal in a Banach lattice E is a linear subspace D for which $y \in D$ whenever $|y| \leq |x|$ for some $x \in D$.

We have the following result from Lindenstrauss ([14], p. 28).

THEOREM 2.2. *A Banach lattice E is order continuous if and only if the canonical image of E into its second dual E^{**} is an ideal of E^{**}.*

Now consider the following theorem which is due to H. Nakano [15]. This result assumes only the condition (B).

THEOREM 2.3. *If $\{a_\alpha\}_{\alpha \in \Lambda}$ is an increasing net of bounded norm in E_+, then $\vee a_\alpha$ exists in E. Furthermore there exists a sequence $\{\alpha_n\} \in \Lambda$ such that*

$$\vee a_\alpha = \vee_n a_{\alpha_n}.$$

2.2. *On countably additive functionals.* Now we consider some results on countably additive functionals. These results are based on the work of P. C. Shields [17] and these results generalize the corresponding results on countably additive measures by L. Sucheston [18].

Let L be a Banach lattice satisfying the following condition (σ-completeness in order):

(SC) Suppose that $\{x_n\}$ is an increasing sequence in L and for some $x \in L$ one has $x_n \leq x$, $n = 1, 2, \ldots$. Then $\vee x_n$ exists.

Note that the condition (SC) follows from condition (B).

Let F and G be in L^*. Then $F \wedge G$ and $F \vee G$ are given by

$$(F \wedge G)(b) = \inf_{b=b_1+b_2. b_1. b_2 \geq 0} [F(b_1) + G(b_2)], \quad \forall b \in L, b \geq 0$$

and

$$(F \vee G)(b) = \sup_{b=b_1+b_2. b_1. b_2 \geq 0} [F(b_1) + G(b_2)], \quad \forall b \in L, b \geq 0.$$

and also it follows that for any Banach lattice L, L^* is a Banach lattice satisfying the condition (SC) ([14], p. 3).

DEFINITION 2.4. We say that a positive linear functional G on L is countably additive if any sequence $\{y_n\}$ in L is such that $y_n \downarrow 0$ then $G(y_n) \downarrow 0$.

One has the following result. (see [17] for the proof.)

LEMMA 2.5. *Let F, G be positive linear functionals on L, and let G be countably additive.*

If $b \geq 0$ and $(F \wedge G)(b) = 0$, then we can find $a_k \geq 0$, $k = 0, 1, 2, \ldots$ such that

$$b = \sum_{k=1}^{\infty} a_k \text{ and}$$
$$0 = G(a_0) = F(a_1) = F(a_2) = \cdots .$$

2.3. *On amenable semigroups.* Let Σ be a semigroup and $l_\infty(\Sigma)$ denote the Banach space of bounded real-valued functions on Σ, with the supremum norm.

A linear functional ϕ on $l_\infty(\Sigma)$ is called a *mean* if

$$\inf_\sigma h(\sigma) \leq \phi(h) \leq \sup_\sigma h(\sigma)$$

for any h $\in l_\infty(\Sigma)$.

Let 1_σ, $\sigma \in \Sigma$ be the evaluation functional given by $1_\sigma h = h(\sigma)$.

With each linear functional ϕ on $l_\infty(\Sigma)$ of the form $\phi = \sum_{k=1}^m \beta_k 1_{\sigma_k}$, where $\sum_{k=1}^m |\beta_k| < \infty$, we associate the l_1-norm $\|\phi\|_{l_1} = \sum_{k=1}^m |\beta_k|$.

A mean ϕ is called a *finite mean* if

$$\phi = \sum_{k=1}^m \beta_k 1_{\sigma_k}$$

for some $\beta_k \geq 0$ such that $\sum_{k=1}^m \beta_k = 1$ and $\sigma_k \in \Sigma$.

Let L_σ, $\sigma \in \Sigma$ be the left shift defined on $l_\infty(\Sigma)$ by

$$L_\sigma h(\rho) = h(\sigma\rho), \quad \forall h \in l_\infty(\Sigma).$$

For $\psi = \sum_{k=1}^n \beta_k 1_{\sigma_k} \in l_\infty^*(\Sigma)$, we define

$$L_\psi h = \sum_{k=1}^m \beta_k L_{\sigma_k} h.$$

Furthermore, if we choose ψ to be a finite mean then L_ψ will be a contraction on l_∞. Indeed, if $\psi = \sum_{k=1}^n \beta_k 1_{\sigma_k}$ with $\sum_{k=1}^n \beta_k = 1$ and $\beta_k \geq 0$ then for any $h \in l_\infty(\Sigma)$ we have

$$\|L_\psi h\|_\infty = \left\| \sum_{k=1}^m \beta_k L_{\sigma_k} h \right\|_\infty \leq \sum_{k=1}^m \beta_k \cdot \|L_{\sigma_k} h\|_\infty \leq \sum_{k=1}^m \beta_k \cdot \|h\|_\infty = \|h\|_\infty.$$

A mean ϕ is said to be a *left invariant mean* if

$$\phi(L_\sigma h) = \phi(h), \quad \forall h \in l_\infty(\Sigma), \quad \forall \sigma \in \Sigma.$$

Note that for every left invariant mean ϕ and for every finite mean ψ, we have $\phi(L_\psi h) = \phi(h)$.

Similarly we define right invariant means. A mean which is left invariant as well as right invariant is called an *invariant mean*.

A semigroup Σ is called *left amenable* if there exists a left invariant mean. A semigroup Σ is called *right amenable* if there exists a right invariant mean. A semigroup Σ is called *amenable* if there exists an invariant mean.

Obviously under the abelian assumption on the semigroup Σ, the concepts of left amenability, right amenability and amenability all coincide. It is also known that all the abelian semigroups are amenable ([6]).

Day [6] proved the following result on a left amenable semigroup:

THEOREM 2.6 (DAY). *Let Σ be a left amenable semigroup. Then there exists a net ψ_α of finite means such that ψ_α converges in norm to left invariance, i.e.,* $\lim_\alpha \|\psi_\alpha L_\sigma - \psi_\alpha\|_{l_1} = 0$, $\forall \sigma \in \Sigma$

In the case of countably generated left amenable semigroups, by a diagonal argument we have the following consequence, which is due to Sachdeva [16].

THEOREM 2.7. *Let Σ be a countably generated left amenable semigroup. Then there exists a sequence ψ_n of finite means such that ψ_n converges in norm to left invariance, i.e.,* $\lim_n \|\psi_n L_\sigma - \psi_n\|_{l_1} = 0$, $\forall \sigma \in \Sigma$

3. **Existence of operator averages converging in norm to left invariance.** Consider any finite mean of the form

$$\psi = \sum_{k=1}^{m} \beta_k 1_{\sigma_k}, \text{ where } \sum_{k=1}^{m} \beta_k = 1, \ \beta_k \geq 0 \text{ and } \sigma_k \in \Sigma. \tag{1}$$

With this finite mean we associate the operator averages A_ψ given by

$$A_\psi = \sum_{k=1}^{m} \beta_k T_{\sigma_k}. \tag{2}$$

Also, it follows that for any $f \in E$ and for any $F \in E^*$, if we define $h \in l_\infty(\Sigma)$ by

$$h(\sigma) = F(T_\sigma f)$$

then we have

$$F[A_\psi(f)] = \psi(h). \tag{3}$$

Indeed, both expressions above are equal to $\sum_{k=1}^{m} \beta_k F(T_{\sigma_k} f)$.

We construct our sequence of operator averages $\{A_{\psi_n}\}$ from the sequence of finite means $\{\psi_n\}$, which converges in mean to left invariance. Note that the Theorem 2.7 guarantees the existence of such sequences of finite means.

First we shall show that the averages A_{ψ_n} asymptotically satisfy the left invariant property.

DEFINITION 3.1. Consider the operator averages A_{ψ_n}, $n = 1, 2, \ldots$, of the form $A_{\psi_n} = \sum_{k=1}^{m_n} \beta_{n_k} T_{\sigma_{n_k}}$, where $\sum_{k=1}^{m_n} \beta_{n_k} = 1$. We say that the operator averages converge in norm to the left invariance if

$$\|T_\sigma A_{\psi_n} - A_{\psi_n}\| \longrightarrow 0, \text{ as } n \longrightarrow \infty \text{ for each } \sigma \in \Sigma.$$

THEOREM 3.2. *Let Σ be a countably generated left amenable semigroup, with $\{\psi_n\}$ being a sequence of finite means which converges in norm to left invariance. Let A_{ψ_n} be the operator averages associated with ψ_n, as described above. Assume* $\sup\{\|T_\sigma\| : \sigma \in \Sigma\} = K < \infty$. *Then operator averages A_{ψ_n} converge in norm to the left invariance.*

PROOF. Let $\tau \in \Sigma$ be fixed.

Let $F \in E^*$ and $f \in E$. Define $h \in l_\infty(\Sigma)$ by

$$h(\sigma) = F(T_\sigma f), \quad \sigma \in \Sigma.$$

Let

$$\psi_n = \sum_{k=1}^{m_n} \beta_{n_k} 1_{\sigma_{n_k}} \text{ where } \sum_{k=1}^{m_n} \beta_{n_k} = 1,\ \beta_{n_k} \geq 0 \text{ and } \sigma_{n_k} \in \Sigma. \tag{4}$$

Then it follows that

$$\psi_n(h) = F(A_{\psi_n} f)$$

and

$$\begin{aligned}
\psi_n(L_\tau h) &= \sum_{k=1}^{m_n} \beta_{n_k} F(T_{\tau\sigma_{n_k}} f) \\
&= (T_\tau^* F)\Big(\sum_{k=1}^{m_n} \beta_{n_k} T_{\sigma_{n_k}} f\Big) \\
&= (T_\tau^* F)(A_{\psi_n} f).
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
|F(T_\tau A_{\psi_n} f - A_{\psi_n} f)| &= |(T_\tau^* F)(A_{\psi_n} f) - F(A_{\psi_n} f)| \\
&= |\psi_n L_\tau(h) - \psi_n(h)| \\
&\leq \|\psi_n L_\tau - \psi_n\|_{l_1} \cdot \|h\|_\infty \\
&\leq \|\psi_n L_\tau - \psi_n\|_{l_1} \cdot \sup_{\sigma \in \Sigma} |F(T_\sigma f)| \\
&\leq \|\psi_n L_\tau - \psi_n\|_{l_1} \cdot \|F\| \cdot K \cdot \|f\| \\
&\leq \|\psi_n L_\tau - \psi_n\|_{l_1} \cdot K \cdot \|f\|, \quad \text{if } \|F\| \leq 1.
\end{aligned}$$

Since $F \in E^*$ is arbitrary, we have

$$\sup_{\|F\| \leq 1} |F(T_\tau A_{\psi_n} f - A_{\psi_n} f)| \leq \|\psi_n L_\tau - \psi_n\|_{l_1} \cdot K \cdot \|f\|.$$

Therefore we have

$$||(T_\tau A_{\psi_n} - A_{\psi_n})f|| = ||T_\tau A_{\psi_n} f - A_{\psi_n} f|| \leq ||\psi_n L_\tau - \psi_n||_{l_1} \cdot K \cdot ||f||.$$

The above result being true for each $f \in E$, we obtain

$$||T_\tau A_{\psi_n} - A_{\psi_n}|| \leq ||\psi_n L_\tau - \psi_n||_{l_1} \cdot K.$$

But, by assumption

$$||\psi_n L_\tau - \psi_n||_{l_1} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore we have

$$||T_\tau A_{\psi_n} - A_{\psi_n}|| \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

This concludes the proof of Theorem 3.2.

4. **Existence of invariant weak units.** In this section, we give the proof of Theorem 1.4.

First we prove the following lemma.

LEMMA 4.1. *Let e be any weak unit in E and* $H \in E^*_{++}$. *Suppose we have uniformly bounded linear operators,* $T_{\rho_n}, n = 1, 2, \ldots,$ *such that* $\sup_n ||T_{\rho_n}|| \leq K < \infty$. *Assume* $\lim_n H(T_{\rho_n} e) = 0$. *Then* $\lim_n H(T_{\rho_n} u) = 0$ *for any u in* E_+.

PROOF. Let u be any element in E_+. Given any $\epsilon > 0$, there exists an integer k and $w \in E_+$ such that

$$u = u \wedge ke + w \text{ and } ||w|| < \epsilon/2K||H||.$$

Since $\lim_n H(T_{\rho_n} e) = 0$, we can choose a positive integer N such that for $n > N$

$$H(T_{\rho_n} e) < \epsilon/2k.$$

Thus, for $n > N$, we get

$$\begin{aligned} H(T_{\rho_n} u) &\leq k \cdot H(T_{\rho_n} e) + H(T_{\rho_n} w) \\ &\leq k \cdot H(T_{\rho_n} e) + ||H|| \cdot \sup_\sigma ||T_\sigma|| \cdot ||w|| \\ &\leq k \cdot H(T_{\rho_n} e) + ||H|| \cdot K \cdot ||w|| \\ &\leq k \cdot \epsilon/2k + \epsilon/2 = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the result follows.

PROOF OF THEOREM 1.4. Assume that there exists an invariant weak unit v in E_+. Suppose (ii) does not hold. Then there exists $H \in E^*_{++}$ such that

$$\inf_\sigma H(T_\sigma e) = 0.$$

By Lemma 4.1, it follows

$$\inf_\sigma H(T_\sigma v) = 0.$$

But $T_\sigma v = v$ for any $\sigma \in \Sigma$; this implies

$$0 = \inf_\sigma H(T_\sigma v) = \inf_\sigma H(v) = H(v).$$

Since v is a weak unit this leads to $H = 0$, a contradiction. Thus we have (i) $\Rightarrow$ (ii).

Now we prove that (i) $\Rightarrow$ (ii). Assume $\inf_\sigma H(T_\sigma e) > 0$, $\forall H \in E^*_{++}$. Let ϕ be a left invariant mean on $l_\infty(\Sigma)$. Define $\lambda \in E^{**}_{++}$ by

$$\lambda(H) = \phi[H(T_\sigma e)], \quad H \in E^*_{++}. \tag{5}$$

Then we have

$$\lambda(H) > 0, \quad \forall H \in E^*_{++}. \tag{6}$$

For any $H \in E^*$ and $\tau \in \Sigma$, we have

$$\begin{aligned}(T^{**}_\tau \lambda)H &= \lambda(T^*_\tau H)\\ &= \phi[T^*_\tau H(T_\sigma e)]\\ &= \phi[H(T_\tau T_\sigma e)]\\ &= \phi[H(T_\sigma e)]\\ &= \lambda(H).\end{aligned}$$

Therefore for any $\sigma \in \Sigma$

$$T^{**}_\sigma \lambda = \lambda. \tag{7}$$

Now, define

$$u = \sup\{w \in E \mid 0 \leq w \leq \lambda\}. \tag{8}$$

Theorem 2.3 guarantees that $u \in E$. Also we have

$$0 \leq T_\sigma u = T^{**}_\sigma u \leq T^{**}_\sigma \lambda = \lambda.$$

Therefore, by equation (8) it follows that

$$T_\sigma u \leq u \tag{9}$$

Now, we shall show that u is a weak unit. Consider $(\lambda - u) \wedge e$. By Theorem 2.2, we get

$$(\lambda - u) \wedge e \in E_+.$$

Thus, from equation (8) it follows that

$$(\lambda - u) \wedge e = 0. \tag{10}$$

Indeed, since $u + (\lambda - u) \wedge e \leq u + (\lambda - u) = \lambda$, the above result follows from the maximality of u as given by equation (8).

Now, suppose $H(u) = 0$ for some $H \in E_+^*$. Equation (10) implies that

$$\langle(\lambda - u) \wedge e, H\rangle = 0.$$

Therefore, by Lemma 2.5 applied with $L = E^*$, there exists a sequence $\{H_k\}$ in E_+^* such that

$$H = \sum_{k=0}^{\infty} H_n,$$
$$\langle(\lambda - u), H_k\rangle = 0, \quad \text{for } k = 0, 1, 2, \ldots \text{ and}$$
$$\langle e, H_0\rangle = 0.$$

But, $H(u) = 0$ and $0 \leq H_k \leq H$ implies

$$H_k(u) = 0, \quad \text{for } k = 0, 1, 2, \ldots;$$

this result, together with $\langle(\lambda - u), H_k\rangle = 0, k = 0, 1, 2, \ldots$ implies

$$\lambda(H_k) = 0, \quad \text{for } k = 1, 2, \ldots.$$

Therefore, by equation(6), we get

$$H_k = 0, \quad \text{for } k = 1, 2, \ldots.$$

Also, $\langle e, H_0\rangle = 0$ implies $H_0 = 0$. Therefore we have

$$H = \sum_{k=0}^{\infty} H_k = 0.$$

Thus we have shown for an arbitrary $H \in E_+^*$ that $H(u) = 0$ implies $H = 0$; hence u is a weak unit. Thus, we have a weak unit u such that $T_\sigma u \leq u$, for any $\sigma \in \Sigma$.

To complete the proof, it remains to find a weak unit v such that $T_\sigma v = v$ for all $\sigma \in \Sigma$. Consider a sequence $\{A_{\psi_n}\}$ of operator averages which converge in norm to left invariance. Since $0 \leq T_\sigma u \leq u$, we have

$$0 \leq A_{\psi_n} u \leq u.$$

Recall that order continuity (hence *a fortiori* condition (B)) implies that each order interval is weakly compact ([14], p. 28), and hence we have a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ such that $\{A_{\psi_{n_k}} u\}$ converges weakly to some element v in E_+.

CLAIM. v is a weak unit.

PROOF. Suppose the opposite. Then there exists an $H \in E^*_{++}$ such that $Hv = 0$. Thus we obtain

$$H(A_{\psi_{n_k}} u) \longrightarrow H(v) = 0.$$

This will imply

$$\inf_\sigma H(T_\sigma u) = 0.$$

Indeed, suppose $\inf_\sigma H(T_\sigma u) = \epsilon > 0$ then for any ψ_n given by

$$\psi_n = \sum_{k=1}^{m_n} \beta_{n_k} 1\sigma_{n_k}, \text{ where } \beta_{n_k} \geq 0 \text{ and } \sum_{k=1}^{m_n} \beta_{n_k} = 1,$$

we have

$$H(A_{\psi_n} u) = \sum_{k=1}^{m_n} \beta_{n_k} H(T\sigma_{n_k} u) \geq \sum_{k=1}^{m_n} \beta_{n_k} \epsilon = \epsilon.$$

Since $\inf_\sigma H(T_\sigma u) = 0$ and u is a weak unit, from Lemma 4.1 it follows that $\inf_\sigma H(T_\sigma e) = 0$. This contradicts our assumption.

CLAIM. v is such that $T_\sigma v = v$, $\forall \sigma \in \Sigma$.

PROOF. Take any $\sigma \in \Sigma$. Since $A_{\psi_{n_k}} u \xrightarrow{w} v$, as $n_k \longrightarrow \infty$, for any $H \in E^*_{++}$ we have,

$$\lim_{n_k \longrightarrow \infty} H(A_{\psi_{n_k}} u) = Hv. \tag{11}$$

Also for each $H \in E^*_{++}$ we have,

$$\lim_{n_k \longrightarrow \infty} (T^*_\sigma H)(A_{\psi_{n_k}} u) = (T^*_\sigma H)v$$

and hence,

$$\lim_{n_k \longrightarrow \infty} H(T_\sigma A_{\psi_{n_k}} u) = H(T_\sigma v). \tag{12}$$

But,

$$\lim_{n_k \longrightarrow \infty} |H(T_\sigma A_{\psi_{n_k}} u) - H(A_{\psi_{n_k}} u)| = 0. \tag{13}$$

Indeed, by Theorem (3.2), we have

$$|H(T_\sigma A_{\psi_{n_k}} u) - H(A_{\psi_{n_k}} u)| \leq ||H|| \cdot ||T_\sigma A_{\psi_{n_k}} - A_{\psi_{n_k}}||.||u|| \longrightarrow 0, \text{ as } n_k \longrightarrow \infty.$$

Equations (11), (12) and (13) imply

$$H(T_\sigma v) = Hv.$$

Therefore $H(T_\sigma v - v) = 0$; this result being true for any $H \in E^*_{++}$, we get

$$T_\sigma v = v.$$

This completes the proof of (i) $\Leftrightarrow$ (ii) of Theorem 1.4.

5. **Illustrations.** Let us consider a few special cases of our results.

EXAMPLE 5.1. Let $\Sigma = \{1, 2, 3, \ldots\}$ with addition. Then any $x \in l_\infty(\Sigma)$ is a bounded sequence of the form $(x_1, x_2, \ldots)$ with $x_n = x(n)$ and $\|x\| = \sup |x_n| < \infty$. Given $x = (x_1, x_2, x_3, \ldots)$, for any positive integer k define the sequence obtained by shifting the elements of the sequence x to the left by k places by $x^{(k)} = (x_{(k+1)}, x_{(k+2)}, \ldots)$. Then left shift operators L_k on $l_\infty(\Sigma)$ are given by $L_k(x) = x^{(k)}$.

Consider the finite means on $l_\infty(\Sigma)$ defined by

$$\psi_n = \frac{1}{n}\sum_{\iota=1}^{n} 1_\iota \in l_\infty^*(\Sigma).$$

We have

$$\psi_n(x) = \frac{1}{n}\sum_{\iota=1}^{n} 1_\iota(x) = \frac{1}{n}\sum_{\iota=1}^{n} x_\iota$$

$$(\psi_n L_k)(x) = \frac{1}{n}\sum_{\iota=1}^{n} 1_\iota(x^{(k)}) = \frac{1}{n}\sum_{\iota=1}^{n} x^{(k)}(i) = \frac{1}{n}\sum_{\iota=1}^{n} x_{\iota+k}$$

Therefore for any k, $1 \le k \le n$ we have

$$\begin{aligned} |(\psi_n - \psi_n L_k)(x)| &= \Big|\frac{1}{n}\sum_{\iota=1}^{n} x_\iota - \frac{1}{n}\sum_{\iota=1}^{n} x_{\iota+k}\Big| \\ &= \frac{1}{n}|x_1 + x_2 + \cdots + x_k - x_{n+1} - x_{n+2} - \cdots - x_{n+k}| \\ &\le \frac{1}{n}\cdot 2k \cdot \|x\| \end{aligned}$$

Thus we have

$$\|\psi_n - \psi_n L_k\|_{l_1} \le 2k/n \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Therefore $\{\psi_n\}$ is indeed a sequence of finite means on $l_\infty(\Sigma)$ converging in norm to left invariance.

By our definition, the operator averages associated with $\psi_n = \frac{1}{n}\sum_{\iota=1}^{\infty} 1_\iota$ are given by

$$A_n = A_{\psi_n} = \frac{1}{n}\sum_{\iota=1}^{n} T^\iota$$

which are the Cesàro Averages of the operator T. Thus in the case of cyclic group, the results of Theorem 1.4 reduces to the results in [17].

Furthermore by Theorem 3.2 we obtain that if T is power bounded then

$$\|T^n A_n - A_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty$$

This is a well-known result; in fact this result holds under a weaker condition, namely T is mean bounded and $\|T^n\|/n$ converges to zero.

EXAMPLE 5.2. Consider an abelian semigroup Σ generated by finitely many elements $\sigma_1, \sigma_2, \ldots, \sigma_d$.

Define $\psi_n^k \in l_\infty^*(\Sigma)$ for $k = 1, 2, \ldots, d$ by

$$\psi_n^k = \frac{1}{n} \sum_{i=1}^{n} 1_{\sigma_k^i}.$$

We have $\|\psi_n^k\|_{l_1} = 1$. Define the multiplication $\psi_n^k \cdot \psi_n^l$ by

$$\psi_n^k \cdot \psi_n^l = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1_{\sigma_k^i \sigma_l^j}.$$

Clearly the multiplication defined above is commutative. Let $\psi_n = \psi_n^1 \cdot \psi_n^2 \cdot \cdots \cdot \psi_n^d$. For any given σ_k in Σ we have

$$\begin{aligned}
&\|\psi_n - \psi_n L_{\sigma_k}\|_{l_1} \\
&\quad = \|\psi_n^1 \cdot \psi_n^2 \cdots \psi_n^{k-1} \cdot \psi_n^{k+1} \cdots \psi_n^d \cdot (\psi_n^k - \psi_n^k L_{\sigma_k})\|_{l_1} \\
&\quad \leq \|\psi_n^1\|_{l_1} \cdot \|\psi_n^2\|_{l_1} \cdots \|\psi_n^{k-1}\|_{l_1} \cdot \|\psi_n^{k+1}\|_{l_1} \cdot \cdots \cdot \|\psi_n^d\|_{l_1} \cdot \|(\psi_n^k - \psi_n^k L_{\sigma_k})\|_{l_1} \\
&\quad = \|(\psi_n^k - \psi_n^k L_{\sigma_k})\|_{l_1} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Note that the last step above follows by the same approach as in the previous example.

Now consider two elements ρ_1, ρ_2 such that for $k = 1, 2$ one has

$$\lim_n \|(\psi_n - \psi_n L_{\rho_k})\|_{l_1} = 0.$$

Consider the product $\rho = \rho_1 \cdot \rho_2$. We have

$$\begin{aligned}
\|\psi_n - \psi_n L_\rho\|_{l_1} &= \|\psi_n - \psi_n L_{\rho_1} + \psi_n L_{\rho_1} - \psi_n L_{\rho_1 \rho_2}\|_{l_1} \\
&\leq \|\psi_n - \psi_n L_{\rho_1}\|_{l_1} + \|\psi_n L_{\rho_1} - \psi_n L_{\rho_1 \rho_2}\|_{l_1} \\
&\leq \|\psi_n - \psi_n L_{\rho_1}\|_{l_1} + \|\psi_n - \psi_n L_{\rho_2}\|_{l_1} \cdot \|L_{\rho_1}\|_{l_1} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Now consider any arbitrary element $\sigma \in \Sigma$. Then σ is given by a finite product of σ_k, $k = 1, 2, \ldots, d$. By repeating the above argument for the finite product of σ_k, we obtain

$$\lim_n \|(\psi_n - \psi_n L_\sigma)\|_{l_1} = 0.$$

Hence

$$\begin{aligned}
\psi_n &= \psi_n^1 \cdot \psi_n^2 \cdot \cdots \cdot \psi_n^d \\
&= \frac{1}{n^d} \sum_{i_1\, i_2\, \ldots i_k = 1}^{n} 1_{\sigma_1^{i_1}} \cdot 1_{\sigma_2^{i_2}} \cdot \cdots \cdot 1_{\sigma_d^{i_d}}.
\end{aligned}$$

are sequences of finite means converging in norm to left invariance.

By our definition, the corresponding operator averages are given by

$$A_{\psi_n} = \frac{1}{n^d} \sum_{i_1\, i_2\, \ldots\, i_k = 1}^{n} T_{\sigma_1}^{i_1} \cdot T_{\sigma_2}^{i_2} \cdot \cdots \cdot T_{\sigma_d}^{i_d}.$$

By Theorem 3 2, if the operators are uniformly bounded, then the above averages satisfy the condition

$$\lim_n \|A_{\psi_n} - T_{\sigma_k} A_{\psi_n}\| = 0, \quad k = 1, 2, \quad , d$$

For example, in the case of operators generated by two commutative operators S and T the operator averages are given by

$$A_{\psi_n} = \frac{1}{n^2} \sum_{i,j=1}^{n} S^i \ T^j$$

Also in this case Theorem 1 4 takes the following form

THEOREM 5 3 *Let E be a Banach lattice satisfying conditions (A) and (B) Let S, T be power bounded commutative operators Then the following conditions are equivalent*

(i) There exists an invariant weak unit v in E such that $Tv = v = Sv$

(ii) $\inf_{i,j} H(S^i T^j e) > 0$, $\forall H \in E^*_{++}$

Now let us consider an example of an amenable group which is not abelian

EXAMPLE 5 4 Consider a group Σ generated by two elements σ_1 and σ_2 such that $\sigma_i = \sigma_i^{-1}$, $i = 1, 2$ Dixmier [7] proved that such a group is amenable Therefore from Theorem 1 4, the following result follows

PROPOSITION 5 5 *Let E be a Banach lattice satisfying conditions (A) and (B) Let T_1 and T_2 be positive linear contractions on E such that $T_i^2 = I$, $i = 1, 2$ Then the following conditions are equivalent*

(i) There exists a weak unit v in E such that $T_1 v = v = T_2 v$

(ii) $\inf_T H(Te) > 0$, $\forall H \in E^*_{++}$

Here the infimum is taken over all T, where T denotes a finite product of the operators T_1 and T_2

In the case of point transformations acting on a probability space, a similar result was proved by Blum and Friedman [2]

Groups generated by more than two elements are not amenable (see Dixmier [7]) Therefore, the results of the above proposition do not readily extend to cases involving more than two elements

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