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STRUCTURE RESULTS FOR FUNCTION LATTICES

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1. Introduction. For partially ordered sets X and Y let Y^X denote the set of all order-preserving maps of X to Y partially ordered by $f \leqq g$ if and only if $f(x) \leqq g(x)$ for each $x \in X$ [**1**; **4**; **6**]. If X is unordered then Y^X is the usual direct product of partially ordered sets, while if both X and Y are finite unordered sets then Y^X is the commonplace exponent of cardinal numbers. This generalized exponentiation has an important vindication especially for those partially ordered sets that are lattices.

Let L be a lattice and let P be a partially ordered set; then L^P is a lattice (see Figure). We call L^P a *function lattice*. This concept is a powerful tool in codifying lattices, especially finite distributive lattices. As is well-known, if D is any finite distributive lattice then there is a finite partially ordered set P such that $D \cong \mathbf{2}^P$, where $\mathbf{2}$ denotes the two-element chain [**2**]. Moreover, the dual of P is isomorphic to the partially ordered set of all join irreducible elements of D. Function lattices have also provided fresh insight and suggested new results in lattice theory. Particularly noteworthy is the duality theory between the category of finite distributive lattices with $\{0, 1\}$-preserving homomorphisms and the category of finite partially ordered sets with order preserving maps [**2**] (cf. [**5**; **7**; **8**]). The purpose of this paper is to establish certain natural structure results for arbitrary function lattices.

For a partially ordered set P, let $c(P)$ denote the number of connected components of P. For a lattice L, let Con (L) denote the lattice of all congruence relations of L, let L^σ denote the lattice of all *nonempty* ideals of L and let Cen (L) denote the centre of L. As is customary the positive integer n shall denote the unordered n-element set. The following theorem summarizes our main results.

THEOREM. *Let L be a lattice and let P be a finite partially ordered set with $c(P) = m$ and $|P| = n$. Then*

(i) Con $(L^P) \cong (\text{Con }(L))^n$

and

(ii) $(L^P)^\sigma \cong (L^\sigma)^P$;

moreover, if L is bounded then

(iii) Cen $(L^P) \cong (\text{Cen }(L))^m$.

2. Congruence relations. A well-known result, implicit in G. Birkhoff's early work on distributive lattices [**2**], asserts that Con $(D) \cong \mathbf{2}^n$ for a finite

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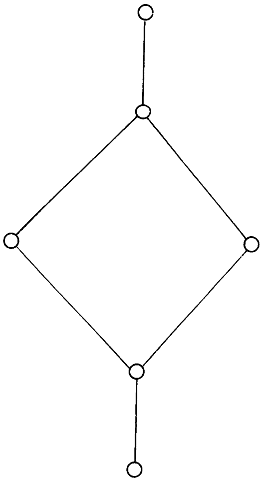
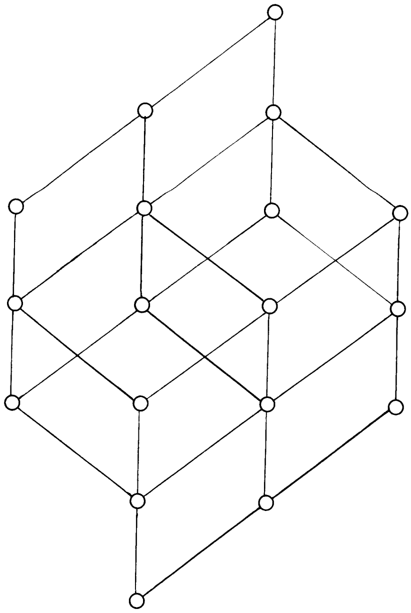


$\mathbf{2}^{\mathbf{2}\times\mathbf{2}}$



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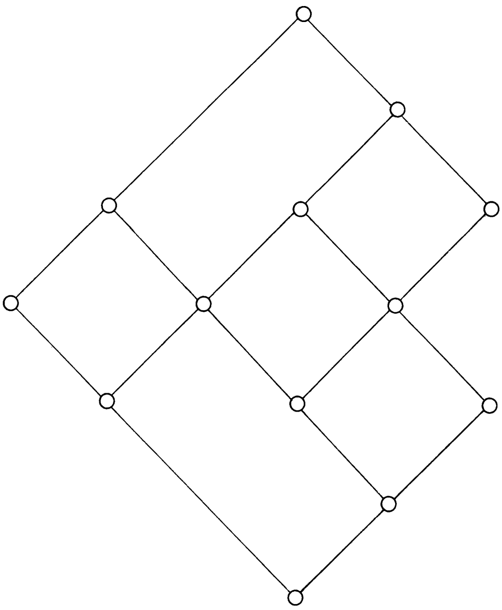

$\mathbf{N}_5{}^{\mathbf{2}}$

FIGURE

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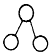

N_5

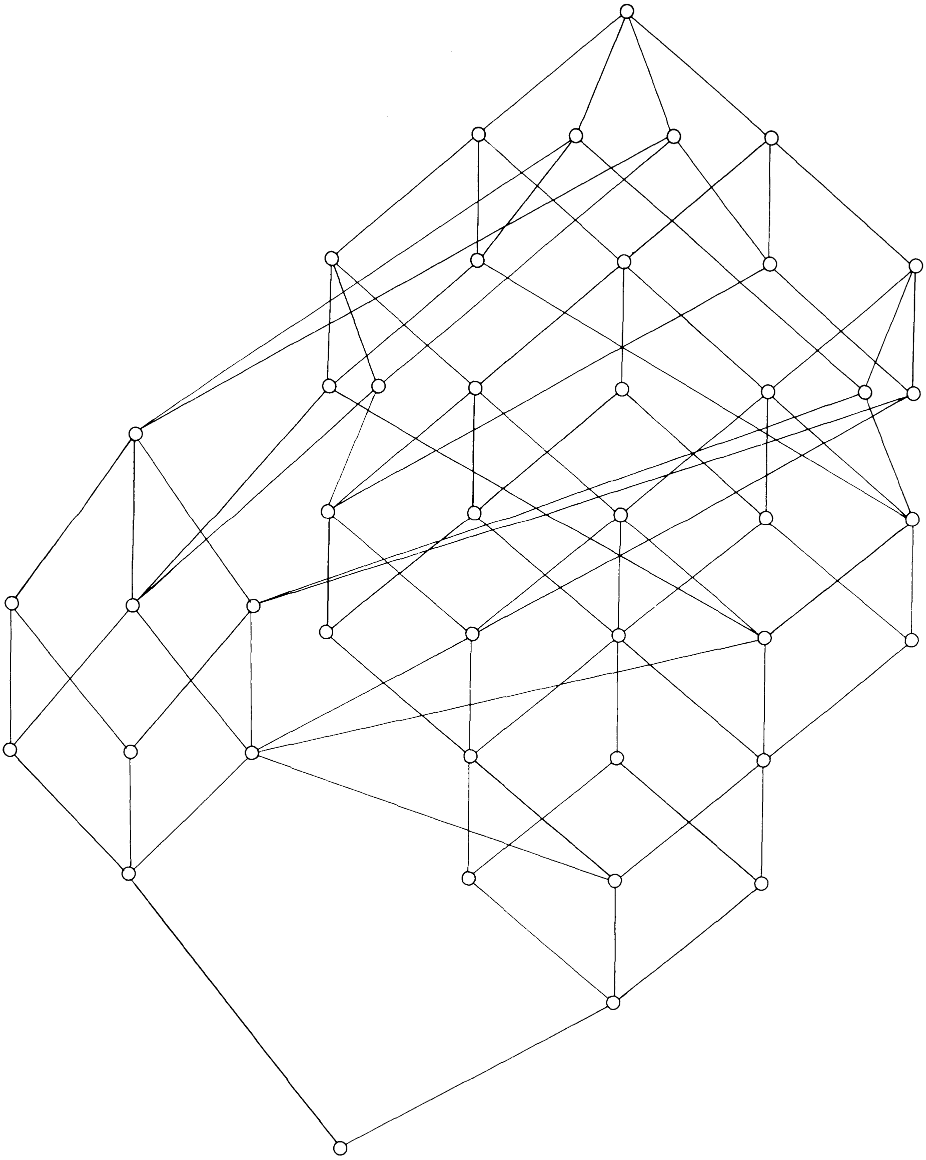

FIGURE (*Cont'd.*)

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distributive lattice D with n join irreducible elements. As $D \cong \mathbf{2}^P$, where P^d is isomorphic to the partially ordered set of all join irreducible members of D, it follows that Con $(\mathbf{2}^P) \cong (\text{Con } (\mathbf{2}))^n$. Our aim in this section is to prove the analogue for arbitrary function lattices.

THEOREM 2.1. *Let L be a lattice and let P be a finite partially ordered set with n elements. Then*

$$\text{Con } (L^P) \cong (\text{Con } (L))^n.$$

The lemma upon which our proof is based is due to S. Comer.

LEMMA 2.2. *Let A be an algebra and let* Con (A) *be distributive. If A is a subdirect product of finitely many algebras $B_i (i \in I)$ then, for $\theta \in$* Con (A), *there exist $\varphi_i \in$* Con (B_i) $(i \in I)$ *such that, for all $x, y \in A$,*

$$x\, \theta\, y \quad \textit{if and only if} \quad x_i\, \varphi_i\, y_i$$

for all $i \in I$.

Proof. Let χ_i be the kernel of the projection map of A onto B_i; that is,

$$x\, \chi_i\, y \quad \text{if and only if} \quad x_i = y_i$$

for all $x, y \in A$. The meet of the congruence relations χ_i is the trivial congruence relation on A. By the Second Isomorphism Theorem, $\theta \vee \chi_i$ induces a congruence relation φ_i on B_i such that

$$x(\theta \vee \chi_i)y \quad \text{if and only if} \quad x_i\, \varphi_i\, y_i$$

for all $x, y \in A$. Since Con (A) is distributive,

$$\bigwedge_{i \in I} (\theta \vee \chi_i) = \theta.$$

Therefore, $x\, \theta\, y$ if and only if $x_i\, \varphi_i\, y_i$ for all $i \in I$.

Proof of Theorem 2.1. Let p be a minimal element of P and let $Q = P - \{p\}$. We may identify L^P with the sublattice K of $L \times L^Q$ where

$$K = \{(a, f) | a \in L, f \in L^Q \text{ and } a \leqq f(x), \text{ for all } p \leqq x \text{ and } x \in Q\}.$$

Then K is a subdirect product of L and L^Q.

We define a map F of Con $(L) \times$ Con (L^Q) to Con (K) by

$$(a, f)F(\varphi, \chi)(b, g) \quad \text{if and only if} \quad a\, \varphi\, b \text{ and } f\, \chi\, g$$

for $\varphi \in$ Con (L), $\chi \in$ Con (L^Q) and $(a, f), (b, g) \in K$. F is obviously order-preserving; that F maps Con $(L) \times$ Con (L^Q) onto Con (K) follows from Lemma 2.1. We show that F is inverse order-preserving. We suppose that

$$F(\varphi, \chi) \leqq F(\varphi', \chi')$$

for $\varphi, \varphi' \in$ Con (L) and $\chi, \chi' \in$ Con (L^Q). Let $a, b \in L$ with $a\, \varphi\, b$. Define a map

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f of Q to L by

$$f(x) = a \vee b$$

for all $x \in Q$. Then $(a, f), (b, f) \in K$ and $(a, f)F(\varphi, \chi)(b, f)$; hence, $(a, f)F(\varphi', \chi')(b, f)$ so $a \varphi' b$. Also, given $f, g \in L^Q$ with $f \chi g$, let

$$a = \bigwedge_{x \in Q} (f(x) \wedge g(x)).$$

Then $(a, f), (a, g) \in K$ and $(a, f)F(\varphi, \chi)(a, g)$. It follows that $f \chi' g$.

We conclude that F is an isomorphism of Con $(L) \times$ Con (L^Q) onto Con (K). The conclusion now follows by induction on $|P| = n$.

Of course, every homomorphic image of a finite distributive lattice is distributive. Formulated in the framework of function lattices this means that any homomorphic image K of $\mathbf{2}^P$, where P is a finite partially ordered set, is isomorphic to $\mathbf{2}^Q$ for some $Q \subseteq P$. Note that $\mathbf{2}$ is a simple lattice.

COROLLARY 2.3 (cf. [**5**]). *Let L be a simple lattice and let P be a finite partially ordered set. A lattice K is a homomorphic image of L^P if and only if $K \cong L^Q$ for some subset Q of P.*

Proof. Let $Q \subseteq P$ and let $K \cong L^Q$. The map ψ of L^P to L^Q defined by $\psi(f) = f|Q, f \in L^P$, is a homomorphism of L^P onto L^Q; whence, K is a homomorphic image of L^P.

Conversely, let $K \cong L^P/\theta$, $\theta \in$ Con (L^P). As L is simple, Con $(L^P) \cong \mathbf{2}^{|P|}$, and there exist $2^{|P|}$ distinct congruence relations on L^P. However, there are $2^{|P|}$ distinct subsets of P and for each $Q \subseteq P$, θ_Q, defined by $f \equiv g(\theta_Q)$ if $f|Q = g|Q$, is a congruence relation on L^P. Let $\theta_{Q_1} = \theta_{Q_2}$ and let $x \in Q_1 - Q_2$. It is an easy matter to define functions f and g in L^P that satisfy $f(x) > g(x)$ in L and $f(y) = g(y)$ for all $y \neq x$ in P. Then $f \equiv g(\theta_{Q_2})$ while $f \not\equiv g(\theta_{Q_1})$, which is impossible. We conclude that there exist $2^{|P|}$ distinct congruence relations on L^P each induced, as above, by the subsets of P. Hence, $K \cong L^Q$ for some $Q \subseteq P$.

3. Ideal lattices. The aim of this section is to uncover the relationship that the ideal lattice of a function lattice L^P bears to the ideal lattice of L.

THEOREM 3.1. *Let L be a lattice and let P be a finite partially ordered set. Then there is a unique isomorphism of $(L^P)^\sigma$ onto $(L^\sigma)^P$ which is the identity on L^P.*

Proof. While $(L^P)^\sigma$ need not be an algebraic lattice (unless, of course, L has a least element), the join of a nonempty subset of $(L^P)^\sigma$ exists and every element of $(L^P)^\sigma$ is the join of compact elements. Moreover, L^P is a sublattice of $(L^P)^\sigma$ and the elements of L^P are precisely the compact elements of $(L^P)^\sigma$. In fact, these properties characterize $(L^P)^\sigma$ up to L^P-isomorphism; therefore, it is enough to show that $(L^\sigma)^P$ has the same properties.

Certainly L^P is a sublattice of $(L^\sigma)^P$. (We identify L^P in $(L^\sigma)^P$ with the set of functions $f \in (L^\sigma)^P$ satisfying $f(P) \subseteq L$, where L is identified with the principal ideals of L^σ.) Given $f \in (L^\sigma)^P$, we show that

$$f = \bigvee\{g \in L^P | f \geqq g \text{ in } (L^\sigma)^P\}.$$

Since the operations in $(L^\sigma)^P$ are performed pointwise, it suffices to show that, for $p \in P$,

$$f(p) = \bigvee \{g(p) | f \geqq g \text{ in } (L^\sigma)^P \text{ and } g \in L^P\}.$$

Of course, for $p \in P$,

$$f(p) = \bigvee \{c \in L | f(p) \geqq c\};$$

therefore, it is enough to find, for each $c \in L$ with $f(p) \geqq c$, a function $g \in L^P$ such that $g \leqq f$ in $(L^\sigma)^P$ and $g(p) = c$. We define such a function g as follows: let $d \in L$ satisfy $d \leqq f(q)$, for all $q \in P$, and $d \leqq c$, and let

$$g(q) = \begin{cases} c & \text{if } q \geqq p \\ d & \text{if } q \ngeqq p. \end{cases}$$

Since L^P is closed under finite joins and every member of $(L^\sigma)^P$ is a join of elements of L^P, we see that all the compact elements of $(L^\sigma)^P$ must belong to L^P. Conversely, the compactness of elements $h \in L^P$ is assured by the fact that, for all $p \in P$, $h(p)$ is compact in L^σ, and that P is finite.

4. Factorization. The aim of this section is to examine the direct product factorizations of bounded function lattices. We shall, however, digress momentarily to review some further operations in the generalized arithmetic of partially ordered sets.

Let A and B be disjoint partially ordered sets. The *(disjoint) sum* $A + B$ is the set $A \cup B$ with partial ordering that induced by the partial orderings on A and B. A partially ordered set P is *connected* if $P = A + B$ implies $A = \emptyset$ or $B = \emptyset$. $P = P_1 + P_2 + \ldots + P_n$ is a *nondecomposable sum representation* of P if P_i is connected for each i; call P_i a *component* of P. A finite partially ordered set has a nondecomposable sum representation which is unique (up to the order of the components in the sum) [**1**]; let $c(P)$ denote the number of components of P. For partially ordered sets $A_i (i = 1, 2, \ldots, n)$ let $A_1A_2 \ldots A_n$ (or, $A_1 \times A_2 \times \ldots \times A_n$ or, $\prod_{i=1}^n A_i$) denote the usual direct product of partially ordered sets; call $A_i (i = 1, 2, \ldots, n)$ a *factor* of A and $A \cong \prod_{i=1}^n A_i$ a *factorization* of A.

It is easy to show that many "laws" valid for arithmetical operations hold for appropriate partially ordered set operations. We shall make use of the following (cf. [**1**]):

$$A^{B+C} \cong A^B A^C \quad \text{and} \quad (AB)^C \cong A^C B^C;$$

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moreover, if C is connected then also

$$(A + B)^C \cong A^C + B^C.$$

Let L be a lattice with least element 0_L and greatest element 1_L. An element a of L is a member of the *centre* Cen (L) of L if there is a factorization $L \cong_\varphi XY$ such that $\varphi(a) = (1_X, 0_Y)$. It is well-known that Cen (L) is a Boolean lattice [**3**].

Let $P = \{x_1, x_2, \ldots, x_k\}$ be a partially ordered set and let L be a bounded lattice. A function $f \in L^P$ may be identified with the k-tuple $(a_1, a_2, \ldots, a_k)$, where $f(x_i) = a_i$ for each $i = 1, 2, \ldots, k$. Let $f = (a_1, a_2, \ldots, a_k) \in$ Cen (L^P) and let $f' = (b_1, b_2, \ldots, b_k)$ be the unique complement of f in the Boolean lattice Cen (L^P); then $a_i \vee b_i = 1_L$ and $a_i \wedge b_i = 0_L$ for each $i = 1, 2, \ldots, k$. Also, the map ψ defined by $\psi(g) = (g \wedge f, g \wedge f')$ is an isomorphism of L^P onto $[0_{L^P}, f] \times [0_{L^P}, f']$.

Loosely speaking, the main result of this section will show that a factorization of a bounded function lattice L^P arises either from a factorization of L or a sum decomposition of P. To this end we first establish a few preliminary results.

LEMMA 4.1. *If* $f = (a_1, a_2, \ldots, a_k) \in$ Cen (L^P) *then* $a_i \in$ Cen (L) *for each* $i = 1, 2, \ldots, k$.

Proof. Let $f' = (b_1, b_2, \ldots, b_k)$ be the unique complement of f in Cen (L^P). Define the map φ_i of L to $[0_L, a_i] \times [0_L, b_i]$ by

$$\varphi_i(c) = (c \wedge a_i, c \wedge b_i)$$

$(i = 1, 2, \ldots, k)$. We observe that, for all $c \in L$, the function $f_c \in L^P$ defined by $f_c(y) = c$ for all $y \in P$ satisfies $f_c = (f_c \wedge f) \vee (f_c \wedge f')$. Therefore, $c = f_c(x_i) = (f_c(x_i) \wedge f(x_i)) \vee (f_c(x_i) \wedge f'(x_i)) = (c \wedge a_i) \vee (c \wedge b_i)$. It follows that φ_i is one-to-one and order-preserving $(i = 1, 2, \ldots, k)$.

Let $(c, d) \in [0_L, a_i] \times [0_L, b_i]$. Define $g \in L^P$ by

$$g(x_j) = \begin{cases} c & \text{if } x_j \geqq x_i \\ 0_L & \text{otherwise.} \end{cases}$$

Then $g \leqq f$. Similarly, the map h of P to L defined by

$$h(x_j) = \begin{cases} d & \text{if } x_j \geqq x_i \\ 0_L & \text{otherwise} \end{cases}$$

belongs to the interval $[0_{L^P}, f']$ in L^P. Since the map ψ of L^P to $[0_{L^P}, f] \times [0_{L^P}, f']$ defined by $\psi(g) = (g \wedge f, g \wedge f')$ is an isomorphism of L^P onto $[0_{L^P}, f] \times [0_{L^P}, f']$, there is $l = (e_1, e_2, \ldots, e_k)$ in L^P satisfying $(l \wedge f, l \wedge f') = (g, h)$. Therefore, $e_i \wedge a_i = c$ and $e_i \wedge b_i = d$ and φ_i maps L *onto* $[0_L, a_i] \times [0_L, b_i]$ for $i = 1, 2, \ldots, k$.

Finally, we show that φ_i^{-1} is order-preserving. Let $(c_1, d_1) \leqq (c_2, d_2)$ in $[0_L, a_i] \times [0_L, b_i]$. Again, choose $g_j, h_j (j = 1, 2)$ in L^P as above, satisfying

$0_{L^P} \leqq g_1 \leqq g_2 \leqq f, 0_{L^P} \leqq h_1 \leqq h_2 \leqq f'$ and $g_j(x_i) = c_j, h_j(x_i) = d_j (j = 1, 2)$. Let $r = (r_1, r_2, \ldots, r_k) = \psi^{-1}(g_1, h_1)$ and $s = (s_1, s_2, \ldots, s_k) = \psi^{-1}(g_2, h_2)$. Then $r \leqq s$ and $\varphi_i^{-1}(c_1, d_1) = r_i \leqq s_i = \varphi_i^{-1}(c_2, d_2)$ for $i = 1, 2, \ldots, k$.

Let $f = (a_1, a_2, \ldots, a_k) \in \text{Cen}(L^P)$ and let $f' = (b_1, b_2, \ldots, b_k)$ be the unique complement of f in Cen (L^P). In view of Lemma 4.1 $a_i, b_i \in \text{Cen}(L)$ are complementary $(i = 1, 2, \ldots, k)$. Let $x_j < x_k$ in P and let us suppose that $a_j < a_k$. Since $f' \in L^P$, $b_j = f'(x_j) \leqq f'(x_k) = b_k$; that is, $a_k \vee b_j \geqq a_j \vee b_j = 1_L$ and $a_k \wedge b_j \leqq a_k \wedge b_k = 0_L$. However, since Cen (L) is Boolean and a_j is the unique complement of b_j in Cen (L) this is impossible. Therefore, for any $f \in \text{Cen}(L^P)$, $f(x_i) = f(x_j)$ whenever x_i is comparable to x_j in P. This establishes

LEMMA 4.2. *If $P = P_1 + P_2 + \ldots + P_m$ is a nondecomposable sum representation of P and $f \in$ Cen (L^P) then the restriction of f to P_i is a constant function $(i = 1, 2, \ldots, m)$.*

Let $f \in \text{Cen}(L^P)$ and let $P = P_1 + P_2 + \ldots + P_m$ be a nondecomposable sum representation of P. Then, for each $i = 1, 2, \ldots, m$, there is $a_i \in L$ such that $f(x) = a_i$ for all $x \in P_i$. Indeed, we claim that

$$[0_{L^P}, f] \cong \prod_{i=1}^{m} [0_L, a_i]^{P_i}.$$

This is evident if we observe that $L^P = L^{P_1+P_2+\cdots+P_m} \cong_\varphi L^{P_1}L^{P_2} \ldots L^{P_m}$, where $\varphi(f) = (f_1, f_2, \ldots, f_m)$, and f_i denotes the restriction of f to P_i. It is easy to see that $[0_{L^P}, f] \cong \prod_{i=1}^m [0_{L^{P_i}}, f_i]$ and that $[0_{L^{P_i}}, f_i] \cong [0_L, a_i]^{P_i}$ since $f(x) = a_i$ for all $x \in P_i$.

THEOREM 4.3. *Let L be a bounded lattice and let P be a finite partially ordered set. If $L^P \cong XY$ then $X \cong L_1^{P_1}L_2^{P_2} \ldots L_m^{P_m}$ and $Y \cong K_1^{P_1}K_2^{P_2} \ldots K_m^{P_m}$ where $K_iL_i \cong L$ for each i and $P_1 + P_2 + \ldots + P_m$ is a nondecomposable sum representation of P.*

Proof. Let $f \in L^P$ be the element of Cen (L^P) satisfying $\varphi(f) = (1_X, 0_Y)$ where $L \cong_\varphi XY$. Also, let $P = P_1 + P_2 + \ldots + P_m$ be a nondecomposable sum representation of P. Then, by Lemma 4.2, there is $a_i \in L (i = 1, 2, \ldots, m)$ such that $f(x) = a_i$ for all $x \in P_i$. Hence,

$$X \cong [0_{L^P}, f] \cong \prod_{i=1}^{m} [0_L, a_i]^{P_i}$$

and

$$Y \cong [f, 1_{L^P}] \cong \prod_{i=1}^{m} [a_i, 1_L]^{P_i}.$$

By Lemma 4.1, $L \cong [0_L, a_i] \times [a_i, 1_L]$ $(i = 1, 2, \ldots, m)$.

COROLLARY 4.4. *Let L be a bounded lattice and let P be a finite partially ordered set. Then*

$$\text{Cen}\,(L^P) \cong (\text{Cen}\,(L))^{c(P)}.$$

Proof. Let $P = P_1 + P_2 + \ldots + P_m$ be a nondecomposable sum representation of P. Let $f \in \text{Cen}\,(L^P)$; then the restriction of f to P_i is a constant function, say $f(x) = a_i$ for all $x \in P_i$, $a_i \in L$ for $i = 1, 2, \ldots, m$. Define the map ψ of Cen (L^P) to $(\text{Cen}\,(L))^{c(P)}$ by $\psi(f) = (a_1, a_2, \ldots, a_m)$. ψ is evidently one-to-one and both ψ and ψ^{-1} are order-preserving. In order to see that ψ is onto, we need only observe that an m-tuple $(a_1, a_2, \ldots, a_m) \in (\text{Cen}\,(L))^{c(P)}$ induces a function $f \in L^P$ in the obvious manner; namely, choose $f \in L^P$ such that $f|P_i = a_i$; then, we have

$$L^P \cong \prod_{i=1}^{m} L^{P_i} \cong \left(\prod_{i=1}^{m} [0_L, a_i]^{P_i}\right)\left(\prod_{i=1}^{m} [a_i, 1_L]^{P_i}\right) \cong [0_{L^P}, f] \times [f, 1_{L^P}].$$

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