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EXPLICIT SOLUTIONS OF PYRAMIDAL DIOPHANTINE EQUATIONS

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1. **Introduction.** Let $P_{m,k}$ denote the set of pyramidal numbers

$$P_{m,k} = \left\{\binom{m}{k} \,\middle|\, m, k \in N;\, k \text{ fixed} \geq 2\right\}. \tag{1.1}$$

The question has been asked whether there exist elements p, q, r in $P_{m,k}$ such that $p+q=r$ or, as the problem is usually posed,

$$k!\,p+k!\,q = k!\,r. \tag{1.2}$$

The case $k=2$ has been studied by Sierpinski [6] and Khatri [3]; the case $k=3$ by Oppenheim [4] and Segal [5]; recently Fraenkel [2] has generalized (1.1) to the larger set

$$\begin{aligned} P_{m,k,d} = \Bigg\{\binom{m,d}{k} \,\Bigg|\, & m, k, d \in N; \\ & k \text{ fixed} \geq 2;\, k!\binom{m,d}{k} = \prod_{i=0}^{k-1}(m+id)\Bigg\} \end{aligned} \tag{1.3}$$

and has also investigated the cases $k=2, 3$. But these authors succeeded in finding only one infinite class of tuples $(p, q, r) \in P_{m,k}$ or $\in P_{m,k,d}$ satisfying (1.2). In this paper infinitely many classes of solutions of (1.2) each containing infinitely many tuples (p, q, r) are stated explicitly. In addition related Diophantine equations are studied. The following results are obtained:

(i) solutions of the Diophantine equation

$$x(x+d)+y(y+d) = z(z+d) \tag{1.4}$$

are stated explicitly;

(ii) solutions of infinitely many classes (each containing infinitely many elements) of the Diophantine equation

$$x(x+d)(x+2d)+y(y+d)(y+2d) = z(z+d)(z+2d) \tag{1.5}$$

are stated explicitly;

(iii) many infinite classes of solutions of the Diophantine equation

$$x(x+1)(x+2)(x+3)+y(y+1)(y+2)(y+3) = z(z+2) \tag{1.6}$$

are stated explicitly.

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2. **The equation** $x(x+d)+y(y+d)=z(z+d)$, d **arbitrary**. The methods used in this paper are mainly based on solving a Pellian equation. Concerning notations and formulas the author's papers [1] should be consulted. We shall use the following theorems:

(I) The solutions of the Pellian equation

$$\text{(2.1)} \qquad \begin{cases} x^2-my^2 = 1 \ (m \text{ not a perfect square}); & [\sqrt{m}] = D \geq 1; \\ \sqrt{m} = [D, \overline{b_1, b_2, \ldots, b_{n-1}, 2D}]; & n \geq 1; \quad b_i = b_{n-1-i}, \end{cases}$$

are given by the formulas

$$\text{(2.2)} \qquad \begin{aligned} &x_k = A^{(nk)}+DA^{(nk+1)}, \quad y_k = A^{(nk+1)}, \quad nk = 2u \quad (k = 1, 2, \ldots \text{ or } 2, 4, \ldots) \\ &A^{(0)} = 1, \quad A^{(1)} = 0; \quad A^{(v+2)} = A^{(v)}+b_vA^{(v+1)} \quad (v = 0, 1, \ldots) \end{aligned}$$

All solutions of (2.1) are obtainable from

$$\text{(2.3)} \qquad x_k+\sqrt{m}\, y_k = (x_s+\sqrt{m}\, y_s)^k; \qquad s = \min(nk) \quad (k = 1, 2, \ldots)$$

(II) The Diophantine equation

$$\text{(2.4)} \qquad x^2-my^2 = -1, \qquad \sqrt{m} = [D, \overline{b_1, b_2, \ldots, b_{n-1}, 2D}] \quad (m, D \text{ as in (I)})$$

is solvable iff $n-1=2u$. The solutions are given by the formulas

$$\text{(2.5)} \qquad \begin{aligned} x_{2k-1} &= A^{((2k-1)n)}+DA^{((2k-1)n+1)}; \\ y_{2k-1} &= A^{((2k-1)n+1)} \quad (k = 1, 2, \ldots). \end{aligned}$$

(III) Let u_0, v_0 be a solution of the Diophantine equation

$$\text{(2.6)} \qquad u^2-mv^2 = N$$

(N an integer not a perfect square; m as before). Let x_k, y_k be the solutions of $x^2-my^2=1$ ($x^2-my^2=-1$). Then infinitely many solutions of $u^2-mv^2=N$ ($u^2-mv^2=-N$) are given by

$$\text{(2.7)} \qquad u_k+\sqrt{m}\, v_k = (u_0+\sqrt{m}\, v_0)(x_k+\sqrt{m}\, y_k).$$

We shall now solve equation (1.4). Introducing the parameter t by means of

$$\text{(2.8)} \qquad y = x+t$$

(1.4) takes the form $x(x+d)+(x+t)(x+t+d)=z(z+d)$. Transforming this equation into a quadratic form in x and z, we obtain

$$\text{(2.9)} \qquad 2(2x+t+d)^2+2t^2-d^2 = (2z+d)^2.$$

Introducing the notation

$$\text{(2.10)} \qquad 2z+d = u, \qquad 2x+t+d = v, \qquad d^2-2t^2 = N,$$

we obtain from (2.9)

$$\text{(2.11)} \qquad u^2-2v^2 = -N.$$

We face the problem of finding representatives of the classes of solutions of $u^2-2v^2=-N$. By Theorem (III), it suffices to solve $u^2-2v^2=N$, since $u^2-2v^2=-1$ is solvable. But d and t are arbitrarily chosen parameters, and since $N=d^2-2t^2$, we obtain from $u^2-2v^2=d^2-2t^2$

$$u_0 = d_0; \qquad v_0 = t_0, \tag{2.12}$$

where the subscripts denote special values for u, v, d, and t. The smallest solution of $u^2-2v^2=-1$ is the vector (1, 1) which can be verified directly or from the expansion $\sqrt{2}=[1, \bar{2}]$. From these considerations and formulas (2.6), (2.7) we obtain

$$\begin{cases} u_k+v_k\sqrt{2} = (a_{2k+1}+b_{2k+1}\sqrt{2})(d_0+t_0\sqrt{2}), \\ a_{2k+1}^2-2b_{2k+1}^2 = -1, \quad (k = 1, 2, \ldots) \end{cases} \tag{2.13}$$

Calculating the values of u, v from (2.13), and substituting them in (2.10), infinitely many solutions of (1.4) are thus given by

$$\begin{cases} x = \frac{1}{2}[(a_{2k+1}-1)t_0+(b_{2k+1}-1)\,d_0], \\ y = \frac{1}{2}[(a_{2k+1}+1)t_0+(b_{2k+1}-1)\,d_0], \\ z = \frac{1}{2}[(a_{2k+1}-1)d_0+2t_0b_{2k+1}]. \end{cases} \tag{2.14}$$

It is easy to prove that a_{2k+1}, b_{2k+1} are both odd, so that x, y, z from (2.14) are integral. Likewise it is verified without difficulty that (x, y, z) from (2.14) is a primitive solution in the sense that $(x, y, z)=1$ if and only if $(d_0, t_0)=1$.

For $k=1, 2$, we obtain from (2.14) the special cases

$$\begin{cases} x = 3t_0+2d_0; \quad y = 4t_0+2d_0; \quad z = 5t_0+3d_0. \\ x = 20t_0+14d_0; \quad y = 21t_0+14d_0; \quad z = 29t_0+20d_0. \end{cases} \tag{2.15}$$

The following solutions of

$$x(x+1)+y(y+1) = z(z+1) \tag{2.16}$$

where found by the author independently of (2.14):

$$\begin{cases} x = 2nk+n; \\ y = (n^2-1)k+\frac{1}{2}n(n+1)-1; \\ z = (n^2+1)k+\frac{1}{2}n(n+1) \quad (k, n \text{ any integers}). \end{cases} \tag{2.17}$$

From (2.17) one obtains Sierpinski's solution for (2.16) with $k=0$, and the three solutions of Khatri for $n=2, -2, 4$.

3. **The equation** $x(x+d)(x+2d)+y(y+d)(y+2d)=z(z+d)(z+2d)$. For convenience we write the title equation in the form

$$(x-d)x(x+d)+(y-d)y(y+d) = (z-d)z(z+d). \tag{3.1}$$

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This is easily rearranged in the form

$$y(y^2-d^2) = (z-x)(z^2+zx+x^2-d^2). \tag{3.2}$$

We introduce a new parameter a by putting

$$y = a(z-x). \tag{3.3}$$

From (3.2), (3.3) we obtain, after easy rearrangements,

$$[2(a^3-1)z-(2a^3+1)x]^2-(12a^3-3)x^2 = 4d^2(a-1)(a^3-1), \tag{3.4}$$

and, denoting

$$2(a^3-1)z-(2a^3+1)x = du; \qquad x = dv, \tag{3.5}$$

$$u^2-(12a^3-3)v^2 = 4(a^3-1)(a-1). \tag{3.6}$$

Surprisingly, the Pellian equation (3.6) has the special solution

$$v_0 = 1; \qquad u_0 = 2a(a+1)-1. \tag{3.7}$$

Indeed, we have

$$\begin{aligned}[2a(a+1)-1]^2-(12a^3-3)\cdot 1 &= 4a^4+8a^3+4a^2-4a^2-4a+1-12a^3+3\\ &= 4a^4-4a^3-4a+4 = 4(a^3-1)(a-1).\end{aligned}$$

If s_k, t_k are solutions of

$$s^2-(12a^3-3)t^2 = 1, \tag{3.8}$$

then infinitely many solutions of (3.6) are given by

$$u_k+\sqrt{12a^3-3}\,v_k = [2a(a+1)-1+\sqrt{12a^3-3}][s_k+\sqrt{12a^3-3}\,t_k]. \tag{3.9}$$

Thus

$$\begin{cases} u_k = [2a(a+1)-1]s_k+(12a^3-3)t_k, \\ v_k = [2a(a+1)-1]t_k+s_k \quad (k = 1, 2, \ldots). \end{cases} \tag{3.10}$$

From (3.5), (3.10) we now obtain

$$\begin{aligned} x_k &= dt_k[2a(a+1)-1]+ds_k; \\ 2(a^3-1)z_k &= (2a^3+1)x_k+du_k \\ &= d[(2a^3+1)(2a^2+2a-1)t_k+(2a^3+1)s_k \\ &\qquad +(2a^2+2a-1)s_k+(12a^3-3)t_k] \\ &= d[(4a^5+4a^4+10a^3+2a^2+2a-4)t_k+(2a^3+2a^2+2a)s_k] \\ &= 2d[(a^2+a+1)(2a^3+3a-2)t_k+a(a^2+a+1)s_k]. \end{aligned}$$

The reader can easily verify this interesting factorization in virtue of which we obtain

$$2(a^3-1)z_k = 2d[(2a^3+3a^2-2)t_k+as_k](a^2+a+1)$$

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and, after cancelling by $2(a^2+a+1)\neq 0$,

$$(a-1)z_k = d[(2a^3+3a-2)t_k+as_k].$$

We have thus obtained for x, y, z the expressions

$$\begin{cases} x_k = d[(2a^2+2a-1)t_k+s_k]; \\ y_k = da(a-1)^{-1}[3(2a-1)t_k+s_k]; \\ z_k = d(a-1)^{-1}[(2a^3+3a-2)t_k+as_k]. \end{cases} \tag{3.11}$$

The value of y_k was calculated from those of x_k, z_k and (3.3). Formulas (3.11) are most remarkable. We shall first investigate the case $a=2$. Then $d \mid (x, y, z)$ and we have to put $d=1$ in order to obtain primitive solutions. This gives

THEOREM 1. *An infinity of solutions of the Diophantine equation*

$$(x-1)x(x+1)+(y-1)y(y+1) = (z-1)z(z+1) \tag{3.12}$$

is given by

$$x = 11t_k+s_k; \qquad y = 18t_k+2s_k; \qquad z = 20t_k+2s_k \tag{3.13}$$

where s_k, t_k are all solutions of

$$s^2-93t^2 = 1. \tag{3.14}$$

We shall illustrate Theorem 1 by an example. We obtain

$$\begin{aligned} \sqrt{93} &= [9,\overline{1,1,1,4,6,4,1,1,1,18}]; \\ s_k &= A^{(10k)}+9A^{(10k+1)}; \qquad t_k = A^{(10k+1)}. \end{aligned} \tag{3.15}$$

We calculate easily

$$A^{(10)} = 811; \qquad A^{(11)} = 1260; \qquad s_1 = 12151; \qquad t_1 = 1260.$$

We now obtain from (3.13)

$$x_1 = 26011; \qquad y_1 = 46982; \qquad z_1 = 49502. \tag{3.16}$$

Indeed

$$17{,}598{,}317{,}413{,}320+103{,}703{,}759{,}631{,}186 = 121{,}302{,}077{,}044{,}506.$$

This is the smallest solution of this infinite class of solutions of the title equation with $d=1$.

We shall now investigate the case $a>2$. In order that y_k, z_k, x_k be integers, and taking into account that $(s_k, t_k)=1$, $((a-1), a)=1$, one of the following two possibilities must hold

$$(a-1) \mid ([3(2a-1)t_k+s_k], [(2a^3+3a-2)t_k+s_k]) \tag{3.17}$$

in which case $d \mid (x_k, y_k, z_k)$, and we have to put $d=1$. If (3.17) does not hold, we put $d=a-1$ and have thus obtained

THEOREM 2. *Let* (s_k, t_k) *be a solution tuple of* $s^2-(12a^3-3)t^2=1$. *If, for a fixed* a, *(3.17) holds, then*

$$(3.18)\qquad \begin{cases} x_k = (2a^2+2a-1)t_k+s_k, \\ y_k = (a-1)^{-1}[3a(2a-1)t_k+as_k], \\ z_k = (a-1)^{-1}[(2a^3+3a-2)t_k+as_k] \end{cases}$$

is a solution of the equation $(x-1)x(x+1)+(y-1)y(y+1)=(z-1)z(z+1)$. *If (3.17) does not hold, then*

$$(3.19)\qquad \begin{cases} x_k = (a-1)[(2a^2+2a-1)t_k+s_k], \\ y_k = [3a(2a-1)t_k+as_k], \\ z_k = [(2a^3+3a-2)t_k+as_k] \end{cases}$$

is a solution of

$$(3.20)\qquad (x-d)x(x+d)+(y-d)y(y+d) = (z-d)z(z+d); \qquad d = a-1.$$

We shall illustrate Theorem 2 by numerical examples. Let

$$a = 3; \qquad s^2-321t^2 = 1; \qquad \sqrt{321} = [17, \overline{1, 10, 1, 34}],$$
$$s_k = A^{(4k)}+17A^{(4k+1)}; \qquad t_k = A^{(4k+1)}, \text{ by (2.2).}$$

We calculate easily: $A^{(4)}=11$, $A^{(5)}=12$, and from (3.21), for $k=1$,

$$s_1 = A^{(4)}+17A^{(5)} = 215; \qquad t_1 = A^{(5)} = 12.$$

We now obtain from (3.11), for $a=3$,

$$x_1 = 491d, \quad y_1 = \tfrac{1}{2}(1185d), \quad z_1 = \tfrac{1}{2}(1377d),$$

and have to put $d=2$; then

$$x_1 = 982; \qquad y_1 = 1185; \qquad z_1 = 1377$$

is a solution of $(x-2)x(x+2)+(y-2)y(y+2)=(z-2)z(z+2)$. Let

$$a = 4; \qquad s^2-765t^2 = 1; \qquad \sqrt{765} = [27,\overline{1,1,1,13,6,13,1,1,1,54}];$$
$$s_k = A^{(10k)}+27A^{(10k+1)}; \qquad t_k = A^{(10k+1)}.$$

We calculate, for $k=1$,

$$A^{(10)} = 6805; \qquad A^{(11)} = 10332; \qquad s_1 = 285769; \qquad t_1 = 10332;$$
$$x_1 = d(39t_1+s_1);$$
$$y_1 = (4d/3)(21t_1+s_1);$$
$$z_1 = (d/3)(138t_1+4s_1).$$

Since $3 \nmid s_1$, $3 \mid 21$; $3 \mid 138$, we have to put $d=3$ and obtain that

$$x_1 = 2066151; \qquad y_1 = 2010964; \qquad z_1 = 2568892$$

is a solution of $(x-3)x(x+3)+(y-3)y(y+3)=(z-3)z(z+3)$.

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4. **Solution of** $x(x+1)(x+2)(x+3)+y(y+1)(y+2)(y+3)=t(t+2)$. Rearranging the title equation, we obtain

$$\tfrac{1}{2}x(x+3)[\tfrac{1}{2}x(x+3)+1]+\tfrac{1}{2}y(y+3)[\tfrac{1}{2}y(y+3)+1] = z(z+1), \quad t = 2z. \tag{4.1}$$

If we substitute $-x$ for x in (2.16) and $-(n+1)$ for n in (2.17), we obtain that

$$\begin{cases} x = 2(n+1)k+(n+1), \quad (x-1 = 2(n+1)k+n), \\ y = n(n+2)k+\tfrac{1}{2}(n-1)(n+2), \\ z = (n^2+2n+2)k+\tfrac{1}{2}n(n+1), \end{cases} \tag{4.2}$$

is a solution of

$$x(x-1)+y(y+1) = z(z+1). \tag{4.3}$$

Now, (4.1) has the same structure as (4.3), with $x-1$ standing for $\frac{1}{2}x(x+3)$, and we thus obtain, in virtue of (4.2),

$$\begin{cases} \tfrac{1}{2}x(x+3) = 2(n+1)k+n, \\ \tfrac{1}{2}y(y+3) = n(n+2)k+\tfrac{1}{2}(n-1)(n+2), \\ \qquad\quad z = (n^2+2n+2)k+\tfrac{1}{2}n(n+1). \end{cases} \tag{4.4}$$

Eliminating k from the first two equations we obtain

$$2(n+1)y(y+3)-n(n+2)x(x+3) = -2(n+2),$$

or, writing the left side as quadratic form in x and y

$$2(n+1)(2y+3)^2-n(n+2)(2x+3)^2 = -(9n^2+8n-2). \tag{4.5}$$

(4.5) leads again to a Pell equation, but the author did not succeed to find a special solution of it for every n. For $n=2$, this is possible, and by the usual technique an infinity of solution vectors of (4.5) is found easily. The smallest solution is $x_1=31$; $y_1=36$; with these values and $n=2$, we obtain, from (4.4), $k=\frac{175}{2}$; substituting $n=2$, $k=\frac{175}{2}$ in (4.4), we obtain $z=878$. A solution of the title equation of this chapter is thus given by

$$(x, y, t) = (31, 36, 1756).$$

For other values of n many more infinite solution classes of the title equation can be constructed.

We shall outline still another method of solving the title equation. Rewriting it in the form

$$(x^2+3x)(x^2+3x+2)+(y^2+3y)(y^2+3y+2) = z(z+2) \tag{4.6}$$

and comparing (4.6) with $x(x+d)+y(y+d)=z(z+d)$, we make use of the solutions (2.14) of the latter, with $d=2$, to obtain

$$(4.7)\quad \begin{cases} x^2+3x = \frac{1}{2}t_0(a_{2k+1}-1)+b_{2k+1}-1, \\ y^2+3y = x^2+3x+t_0, \\ \qquad z = a_{2k+1}-1+t_0b_{2k+1} \quad (a_{2k+1}^2-2b_{2k+1}^2 = -1; \quad k = 1,2,\ldots) \end{cases}$$

We now substitute the value of t_0 from the second equation of (4.7) into the first and obtain

$$(4.8)\qquad (a_{2k+1}+1)x(x+3)-(a_{2k+1}-1)y(y+3) = 2(b_{2k+1}-1).$$

(4.8) leads again to a Pell equation. Choosing, for instance, $(a_{2k+1}, b_{2k+1})=(41, 29)$ one calculates easily: $(x, y)=(444, 455)$; then the value of t_0 is calculated which enables us to find z from the third equation of (4.7). We thus find that

$$(x, y, z) = (444, 455, 287778)$$

is a solution of the title equation.

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