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SEPARATION PRINCIPLES AND BOUNDED QUANTIFICATION

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This note is concerned with the implication $\mathrm{Sep}_{II}(Q) \to \mathrm{Sep}_I(Q)$ where Q is a class of subsets of some set S.

DEFINITION.

$$\mathrm{Sep}_I(Q) \equiv \forall X, Y\{X, Y \in Q \mathbin{\&} X \cap Y = \varnothing \\ \to \exists Z(Z, cZ \in Q \mathbin{\&} X \subset Z \mathbin{\&} Y \subset cZ)\}$$

$$\mathrm{Sep}_{II}(Q) \equiv \forall X, Y\{X, Y \in Q \to \exists U, V(cU, cV \in Q \mathbin{\&} U \cap V = \varnothing \\ \mathbin{\&} X - Y \subset U \mathbin{\&} Y - X \subset V)\}$$

where cZ denotes $S-Z$.

It is well-known that in general the above implication is false (e.g. let Q be the closed subsets of the reals). However in many cases of interest the class Q is separable$_{II}$ by a special mechanism dependent on the underlying structure of Q. These cases all fall into a common pattern (for a complete discussion of these examples and the analogies between them, see Addison's paper [1]). We have an index set I, a class Γ of relations on $I \times S$, and Q is the class of sets formed from Γ by universal quantification (intersection) over I. That is, for $X \subset S$,

$$X \in Q \text{ iff there is some } R \text{ in } \Gamma \text{ for which}$$

$$x \in X \equiv \forall i \in I\, R(i, x).$$

Furthermore, we have a binary relation $<$ on I, and Γ has properties:

(A) $R \in \Gamma \to \sim R \in \Gamma$
(B) $R, T \in \Gamma \to R \vee T \in \Gamma$
(C) $R \in \Gamma, i \in I \to \forall j < i R(j, x) \in \Gamma$

For example if Q is the class of G_δ sets of reals, Γ may be defined by

$R \in \Gamma$ iff $\forall i \in N(\{x\colon R(i, x)\}$ is in the Boolean algebra generated by the open sets)

where N is the natural numbers. Two sets X, Y in Q are normally separated$_{II}$ by reducing the sets cX, cY under the assumption that $<$ is a well-order. That is,

[1] The result in this note appeared in the author's Ph.D. thesis [2].

we have R, T in Γ for which

$$(1) \qquad x \in X \equiv \forall i R(i, x), \qquad x \in Y \equiv \forall i T(i, x)$$

and we define U, V by

$$x \in U \equiv \exists i({\sim}T(i, x) \;\&\; \forall j < i R(j, x))$$

$$x \in V \equiv \exists i({\sim}R(i, x) \;\&\; \forall j \leq i T(j, x)).$$

It follows from (A)–(C) that cU, cV are in Q, and obviously $X - Y \subset U$, $Y - X \subset V$. The only remaining requirement for U, V to separate$_{II}$ X, Y is disjointness:
(D) For any R, T in Γ,

$$\exists i({\sim}T(i, x) \;\&\; \forall j < i R(j, x)) \rightarrow {\sim}\exists i({\sim}R(i, x) \;\&\; \forall j \leq i T(j, x)).$$

We note (i) if $<$ is a linear order, then (D); (ii) if $<$ is a well-order, then cQ is reducible (as is well-known).

THEOREM. *If Q is formed from Γ as above, and Q is separable$_{II}$ in the standard fashion, i.e.* (A)–(D) *hold, then Q is separable$_I$.*

Proof. Let X, Y be as in (1) with $X \cap Y = \varnothing$. Let $W(i, x)$ be the relation ${\sim}T(i, x) \rightarrow \exists j < i {\sim}R(j, x)$. Then W is in Γ, and so the set Z defined by $\forall i W(i, x)$ is in Q. It is clear that $Y \subset Z$ and furthermore using $X \cap Y = \varnothing$ we can show $X \cap Z = \varnothing$. Using (A)–(D) we separate$_{II}$ X and Z by U, V where

$$x \in U \equiv \exists i({\sim}W(i, x) \;\&\; \forall j < i R(j, x))$$

$$x \in V \equiv \exists i({\sim}R(i, x) \;\&\; \forall j \leq i W(j, x)).$$

Expanding, we find that

$$x \in U \equiv \exists i({\sim}T(i, x) \;\&\; \forall j \leq i R(j, x))$$

i.e. $x \in U \equiv x \notin Z$. Since $X \cap Z = \varnothing$ we must have $X \subset U$, $Z \subset V$ and it follows that $V = Z$. This shows that $cZ \in Q$ and so Z is a separating$_I$ set for X and Y.

REFERENCES

1. J. W. Addison, *Separation principles in the hierarchies of classical and effective descriptive set theory*. Fundamenta Mathematicae XLVI (1958) pp. 123–135.

2. A. M. Dawes, *First-order hierarchies in general models and in models of Peano arithmetic*. Ph.D. thesis, University of Toronto, Toronto, 1972.

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