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A RELATION BETWEEN POSITIVE DEPENDENCE OF SIGNAL AND THE VARIABILITY OF CONDITIONAL EXPECTATION GIVEN SIGNAL

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Abstract

Let S_1 and S_2 be two signals of a random variable X, where $G_1(s_1 \mid x)$ and $G_2(s_2 \mid x)$ are their conditional distributions given $X = x$. If, for all s_1 and s_2, $G_1(s_1 \mid x) - G_2(s_2 \mid x)$ changes sign at most once from negative to positive as x increases, then the conditional expectation of X given S_1 is greater than the conditional expectation of X given S_2 in the convex order, provided that both conditional expectations are increasing. The stochastic order of the sufficient condition is equivalent to the more stochastically increasing order when S_1 and S_2 have the same marginal distribution and, when S_1 and S_2 are sums of X and independent noises, it is equivalent to the dispersive order of the noises.

Keywords: Conditional expectation; convex order; stochastically increasing order; dispersive order

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1. Introduction

A simple Bayesian signal extraction model assumes that $S = X + U$, where X is a real random variable that is unobservable, U is a random variable that represents noise, and S is a noisy signal of X. Suppose that X and U are independent, X has a normal distribution with mean μ_X and variance σ_X^2, and U has a normal distribution with variance σ_U^2. Then the conditional expectation of X given $S = s$ is

$$\frac{\sigma_X^2}{\sigma_X^2 + \sigma_U^2}(s - \mu_X) + \mu_X,$$

which is strictly increasing in s. Its variance, $\sigma_X^4/(\sigma_X^2 + \sigma_U^2)$, is decreasing in the variance of the noise, σ_U^2. When the variance of the noise is smaller, S becomes more dependent on X. Hence, the variability of the conditional expectation of X given S increases as S becomes more dependent on X.

When S is less dependent on X, it is a bad signal, so the conditional expectation should be more concentrated on the *ex ante* mean. We extend this relation and show that if a signal of a random variable becomes more dependent on the random variable in a dependence stochastic order, then the dispersion of the conditional expectation of the random variable given the signal increases in a variability stochastic order. This result can be used to analyze how changes in the reliability of a signal affect the behaviour of risk-neutral Bayesian decision makers.

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For variability stochastic order, we use the convex order, which is a weak variability stochastic order between random variables that have the same mean. For dependence order, we use the condition that $G_1(s_1 \mid x) - G_2(s_2 \mid x)$ changes sign at most once from negative to positive as x increases for all s_1 and s_2, where S_1 and S_2 are two signals of X and $G_1(s_1 \mid x)$ and $G_2(s_2 \mid x)$ are the respective conditional cumulative distributions of S_1 and S_2 given $X = x$. We show that the latter dependence order is sufficient for the conditional expectation of X given S_1 to be greater than the conditional expectation of X given S_2 in the convex order, provided that both conditional expectations are increasing.

When S_1 and S_2 have the same marginal distributions, the dependence order of the sufficient condition coincides with the more stochastically increasing order, and when signals are sums of the random variable and independent noises, the order is equivalent to the dispersive order of the noises.

We summarize the definitions and properties of variability stochastic orders and dependence orders in Section 2. Section 3 includes our main proposition and its proof.

2. Variability and dependence orders

This section summarizes the definitions and properties of univariate variability orders and bivariate dependence orders that we use in Section 3. Standard references are Shaked and Shanthikumar (1994), Müller and Stoyan (2002), and Joe (1997).

Let X_1 and X_2 be two real random variables. We say that X_2 is larger than X_1 *in the convex order* (i.e. $X_1 \leq_{\text{cx}} X_2$) if $\mathrm{E}[\phi(X_1)] \leq \mathrm{E}[\phi(X_2)]$ for all convex functions ϕ provided that the expectations exist (see Shaked and Shanthikumar (1994, p. 55) and Müller and Stoyan (2002, p. 15)). Since $\phi(x) = x$ and $\phi(x) = -x$ are both convex, $X_1 \leq_{\text{cx}} X_2$ implies that $\mathrm{E}[X_1] = \mathrm{E}[X_2]$ if both X_1 and X_2 have expectations. When $\mathrm{E}[X_1] = \mathrm{E}[X_2]$, $X_1 \leq_{\text{cx}} X_2$ if and only if $\mathrm{E}[\max(X_2 - y, 0)] \geq \mathrm{E}[\max(X_1 - y, 0)]$ for all y (see Shaked and Shanthikumar (1994, p. 57)). We use this property in the proof of Proposition 1, below.

We say that X_2 is larger than X_1 *in the dispersive order* if

$$F_1^{-1}(\beta) - F_1^{-1}(\alpha) \leq F_2^{-1}(\beta) - F_2^{-1}(\alpha),$$

for all α, β, $0 < \alpha \leq \beta \leq 1$, where F_1^{-1} and F_2^{-1} are the right-continuous inverses of F_1 and F_2, respectively, which are the cumulative distributions of X_1 and X_2, respectively. This definition is equivalent to the condition that $F_1(x + c) - F_2(x)$ changes sign from negative to positive at most once for all c (i.e. if $F_1(x' + c) \geq F_2(x')$ then $F_1(x'' + c) \geq F_2(x'')$ for all $x'' > x'$, for all c); see Shaked and Shanthikumar (1994, p. 69) and Müller and Stoyan (2002, p. 40).

Let (S_1, X_1) and (S_2, X_2) be two pairs of random variables that have the same marginal distributions, and let $G_i(s \mid x)$ be the conditional distribution of S_i given $X_i = x$, $i = 1, 2$. Then $G_2(s \mid x)$ is *more stochastically increasing* than $G_1(s \mid x)$ if $G_2^{-1}(G_1(s \mid x) \mid x)$ is increasing in x (see Joe (1997, p. 40)). This is the case if and only if $G_1(s_1 \mid x) - G_2(s_2 \mid x)$ changes sign at most once from negative to positive as x increases, for all s_1 and s_2 (i.e. if $G_1(s_1 \mid x') \geq G_2(s_2 \mid x')$ then $G_1(s_1 \mid x'') \geq G_2(s_2 \mid x'')$ for all $x'' \geq x'$, for all s_1 and s_2); see Joe (1997, p. 41).

3. Our model and proposition

Let (X, S_1, S_2) be a random variable in $\mathbb{R}^3$, where S_1 and S_2 can be interpreted as signals of X. Let the marginal distribution function of X be $F(x)$ and let $G_i(s \mid x)$ be the conditional distribution of S_i given $X = x$, for $i = 1, 2$. We assume that X has finite expectation. Let $m_i(s)$ be the conditional expectation of X given $S = s$, i.e.

$$\iint_{s \in A} m_i(s)\,\mathrm{d}G_i(s \mid x)\,\mathrm{d}F(x) = \iint_{s \in A} x\,\mathrm{d}G_i(s \mid x)\,\mathrm{d}F(x),$$

for all real Borel sets A. If the conditional distributions are absolutely continuous and $g_i(s \mid x)$ is the conditional density, then

$$m_i(s) = \frac{\int x g_i(s \mid x)\,\mathrm{d}F(x)}{\int g_i(s \mid x)\,\mathrm{d}F(x)},$$

provided that the denominator is strictly positive.

If $m_i(s)$ is increasing then the distribution function of S_i is

$$H_i(t) = \int G_i(m_i^{-1}(t) \mid x)\,\mathrm{d}F(x), \qquad i = 1, 2,$$

where m_i^{-1} is the inverse of m_i. We now state our main proposition.

Proposition 1. *Suppose that X has an expected value and $m_1(s)$ and $m_2(s)$ are increasing. If $G_1(s_1 \mid x) - G_2(s_2 \mid x)$ changes sign at most once from negative to positive as x increases, then $m_1(S_1) \leq_{\mathrm{cx}} m_2(S_2)$.*

Proof. Since $\mathrm{E}[m_1(S_1)] = \mathrm{E}[X] = \mathrm{E}[m_2(S_2)]$, it is sufficient to show that

$$\varphi(y) := \mathrm{E}[\max(m_2(S_2) - y, 0)] - \mathrm{E}[\max(m_1(S_1) - y, 0)] \geq 0, \quad \text{for all } y.$$

Since

$$\varphi(y) = \iint_{m_2(s) \geq y} (m_2(s) - y)\,\mathrm{d}G_2(s \mid x)\,\mathrm{d}F(x) - \iint_{m_1(s) \geq y} (m_1(s) - y)\,\mathrm{d}G_1(s \mid x)\,\mathrm{d}F(x),$$

we have $\lim_{y \to \infty} \varphi(y) = 0$, and since $\mathrm{E}[m_1(S_1)] = \mathrm{E}[m_2(S_2)]$, we have $\lim_{y \to -\infty} \varphi(y) = 0$. Hence, if $\varphi'(y) = 0$ implies that $\varphi(y) \geq 0$, we have $\varphi(y) \geq 0$ for all y.

Since another expression for $\varphi(y)$ is

$$\begin{aligned}\varphi(y) &= \int_y^\infty (t - y)\,\mathrm{d}H_2(t) - \int_y^\infty (t - y)\,\mathrm{d}H_1(t) \\ &= -\int_y^\infty \{(1 - H_2(t)) - (1 - H_1(t))\}\,\mathrm{d}t,\end{aligned}$$

we obtain

$$\begin{aligned}\varphi'(y) &= H_1(y) - H_2(y) \\ &= \int \{G_1(m_1^{-1}(y) \mid x) - G_2(m_2^{-1}(y) \mid x)\}\,\mathrm{d}F(x).\end{aligned}$$

Hence, it is sufficient to show that $\int\{G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x)\}\,\mathrm{d}F(x)=0$ implies that $\varphi(y)\ge 0$. But, since $m_i(s)$ is a conditional expectation, we obtain

$$\begin{aligned}
\varphi(y) &= \iint_{m_2(s)\ge y}(m_2(s)-y)\,\mathrm{d}G_2(s\mid x)\,\mathrm{d}F(x)-\iint_{m_1(s)\ge y}(m_1(s)-y)\,\mathrm{d}G_1(s\mid x)\,\mathrm{d}F(x)\\
&= \iint_{m_2(s)\ge y}(x-y)\,\mathrm{d}G_2(s\mid x)\,\mathrm{d}F(x)-\iint_{m_1(s)\ge y}(x-y)\,\mathrm{d}G_1(s\mid x)\,\mathrm{d}F(x)\\
&= \int(x-y)\{\{1-G_2(m_2^{-1}(y)\mid x)\}-\{1-G_1(m_1^{-1}(y)\mid x)\}\}\,\mathrm{d}F(x)\\
&= \int(x-y)\{G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x)\}\,\mathrm{d}F(x)\\
&= \int(x-y)\,\mathrm{sgn}(G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x))\\
&\qquad\times|G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x)|\,\mathrm{d}F(x), \qquad (1)
\end{aligned}$$

where

$$\mathrm{sgn}(x)=\begin{cases}1 & \text{if } x\ge 0,\\ -1 & \text{if } x<0.\end{cases}$$

Expression (1) is the covariance between $\mathrm{sgn}(G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x))$ and $x-y$, evaluated with the probability measure that has a density function proportional to $|G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x)|$ with respect to the measure generated by F, because

$$\begin{aligned}
&\int \mathrm{sgn}(G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x))|G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x)|\,\mathrm{d}F(x)\\
&\quad=\int(G_1(m_1^{-1}(y)\mid x)-G_2(m_2^{-1}(y)\mid x))\,\mathrm{d}F(x)\\
&\quad=0.
\end{aligned}$$

Since $G_1(s_1\mid x)-G_2(s_2\mid x)$ changes sign at most once from negative to positive as x increases, $\mathrm{sgn}(G_1(s_1\mid x)-G_2(s_2\mid x))$ changes from -1 to 1 at most once as x increases; so it is increasing. Since $x-y$ is also increasing, their covariance is nonnegative.

We can obtain several corollaries by combining Proposition 1 with the following remarks.

Remark 1. It is easy to show that a tight sufficient condition for $m_i(s)$ to be nondecreasing is that the conditional density function $g_i(s\mid x)$ is TP_2, i.e. $s'\ge s$ and $x'\ge x$ imply that $g_i(s\mid x)g_i(s'\mid x')\ge g_i(s'\mid x)g_i(s\mid x')$. Here, 'tight' means that $m_i(s)$ is nondecreasing for any distribution of X. When the signal is the sum of a random variable and independent noise, $g_i(s\mid x)$ is TP_2 if the noise has log-concave density.

Remark 2. When S_1 and S_2 have the same margin, $G_1(s_1\mid x)-G_2(s_2\mid x)$ changes sign at most once from negative to positive as x increases if and only if $G_2(s\mid x)$ is more stochastically increasing than $G_1(s\mid x)$.

Remark 3. When $S_i=X+U_i$, where X and U_i are independent, $G_1(s_1\mid x)-G_2(s_2\mid x)$ changes sign at most once from negative to positive as x increases if and only if U_1 is larger than U_2 in the dispersive order. When the distribution function of U_i is Ψ_i, we have

$$G_1(s_1\mid x)-G_2(s_2\mid x)=\Psi_1(s_1-x)-\Psi_2(s_2-x),$$

so that $G_1(s_1 \mid x) - G_2(s_2 \mid x)$ changes sign at most once from negative to positive as x increases for all s_1 and s_2 if and only if $\Psi_2(c + x) - \Psi_1(x)$ changes sign at most once from negative to positive as x increases for all c, which is equivalent to $U_2 \leq_{\text{disp}} U_1$.

If two random variables have the same expectation, then the dispersive order is a stronger variability order than the convex order (see Shaked and Shanthikumar (1994, p. 74)). When U_1 and U_2 have normal distributions with the same mean, both orders are equivalent to the comparison using variance. The convex order is not always weaker than the dispersive order. For example, if U_1 and U_2 do not have expectations, as is the case when both have Cauchy distributions, then they cannot be compared by convex order, but they may be ordered by dispersive order, and then Proposition 1 holds.

When both random variables have the same expectation, the dispersive order is stronger than the convex order, and the more stochastically increasing order is stronger than the concordant order, where (X, S_2) is more concordant than (X, S_1) if

$$\mathrm{P}[(X, S_2) \geq (x, s)] \geq \mathrm{P}[(X, S_1) \geq (x, s)], \quad \text{for all } (x, s).$$

In this sense, the proof of Proposition 1 uses a fairly strong assumption. We may be able to obtain weaker sufficient conditions, especially when both signals have the same marginal distribution, and therefore obtain other results by simultaneously strengthening the stochastic variability order of the conditional expectation and the dependence order between the random variable and the signal.

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