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NON-ISOMORPHIC BURNSIDE GROUPS OF EXPONENT p^2

K. K. HICKIN AND R. E. PHILLIPS

1. Introduction. 1.1. In the recent paper [**8**] Phillips has shown that for each prime p there are $2^{\aleph_0}$ non-isomorphic 2-generated p-groups. This same result was obtained independently by S. Jeanes and J. S. Wilson (unpublished) who show that the groups constructed in [**1**] have $2^{\aleph_0}$ non-isomorphic images. The groups in both of these proofs all have infinite exponent. In this paper we show that, for large enough primes p, there are $2^{\aleph_0}$ non-isomorphic 2-generated groups of exponent p^2. The primes p are restricted to the Novikov-Adjan primes, and the infinite Burnside groups $B(2, p)$ of Novikov-Adjan [**7**] play an essential role in our construction. Our precise result is

THEOREM 1. *For every prime p such that the Burnside group $B(2, p)$ is infinite, there is a set $\mathscr{H}$ of $2^{\aleph_0}$ 2-generated groups of exponent p^2 with the following properties. For any group G, let* $\mathrm{Fit}(G)$ = *the join of all the normal nilpotent subgroups of G. Then,*

(a) *for all $H \in \mathscr{H}$,* $\mathrm{Fit}(H)$ *has exponent p and is nilpotent of class $p - 1$, and* $\mathrm{Fit}(H)$ = *the locally finite radical of H; and*

(b) *for all $H \neq K \in \mathscr{H}$,* $\mathrm{Fit}(H) \not\cong \mathrm{Fit}(K)$.

1.2. Our method employs wreath products, the main idea coming from [**3**] and the later paper [**8**]. These techniques are all extensions of the method first put forth in the fundamental paper of Neumann and Neumann [**5**] where infinite subsets with special translational properties are used as the support of vectors in the base group to obtain embedding theorems.

Our 2-generated groups H occur as subgroups of $A \mathrm{Wr} B = G$ where A is a countable nilpotent group of exponent p and $B = \langle x, y\rangle$ is an infinite, 2-generated group of exponent p such that $\mathrm{Fit}(B) = 1$. The groups H have generators $\{xf, y\}$ where f is a certain element of the (unrestricted) base group Ω of G. These groups H have the following properties.

Let $N = \mathrm{Fit}(H)$.

(1) $N = H \cap \Omega$.

(2) For every $b \in B$, $A = N(b)$ = the projection of N on the coordinate b.

(3) $N' = (A')^B$ where A is identified with the 1-coordinate of Ω.

Conditions (2) and (3) guarantee that the structure of A is largely determined by the structure of H, and the proof of Theorem 1 then hinges on constructing a suitable class of groups A. In § 2 we will prove a proposition on wreath

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products which will yield the properties (1)–(3) above. The construction of the groups A and the proof of Theorem 1 are given in § 3.

To prove properties (2) and (3) it is necessary to choose the vector $f \in \Omega$ so that its support has a special translational property called "sparseness." This is defined as follows:

Definition. Let S and X be subsets of a group B. S is an *X-sparse subset of B* if for every pair of distinct $s, t \in S$,

$$SXs^{-1} \cap SXt^{-1} \subseteq \{1\}.$$

S is a *sparse subset of B* if S is $\{1\}$-sparse.

To prove (2) and (3) above we require that B possess an infinite subset S which is $\langle x \rangle$-sparse. We are able to prove the existence of infinite subsets with sparseness properties in a much wider context than required in the proof of Theorem 1. Because such subsets might have further utility, we devote §4 to the somewhat lengthy proof of

THEOREM 2. (Selection theorem for sparse sets). *Let G be a countable group, Φ = the FC-center of G, and $\Phi_2 = \{g \in G | g^2 \in \Phi\}$. Suppose that, for every finite subset $F \subseteq G$, $G \neq \Phi_2 F$. Then G has an infinite subset S with the following property: For every finite subset $X \subseteq G$, there is a subset $T \subseteq S$ such that*

(1) *$S - T$ is finite;*
(2) *$T \subseteq C_G(X \cap \Phi)$ (the centralizer);*
(3) *each $g \in TX$ is uniquely expressible as a product, $g = sx$ where $s \in S$ and $x \in X$; and*
(4) *for all $u \neq v \in T$, $SXu^{-1} \cap SXv^{-1} = X \cap \Phi$.*

COROLLARY. *Let B be an infinite group with trivial FC-center; thus $\Phi_2 = \{g \in B | g^2 = 1\}$. Suppose that, for every finite subset $F \subseteq B$, $B \neq F\Phi_2$. Then, for every finite subset $X \subseteq B$, B has an infinite subset which is X-sparse.*

Proof. Let $X \subseteq B$ be finite. A standard set-theoretic Lowenheim-Skolem argument shows that B has a countably infinite subgroup G such that $X \subseteq G$, G has trivial FC-center, and, for all finite $F \subseteq G$, $G \neq F\Phi_2$. Let $S \subseteq G$ be the subset obtained from Theorem 2. There is a subset $T \subseteq S$ such that T is infinite because of (1), and for all $u \neq v \in T$, $TXu^{-1} \cap TXv^{-1} = X \cap \Phi \subseteq \{1\}$ because of (4). Hence T is X-sparse.

Clearly the groups B mentioned at the outset meet the conditions of this corollary.

As mentioned above, the sparseness concept is implicit in [**5**], and it was the observation that this concept can be extended to periodic groups which led one of the authors to the embedding theorem [**3**]. Subsets with properties similar to sparseness have been applied by several authors to obtain embedding theorems, most notably by Philip Hall [**2**]. Our proof of Theorem 2 is a generalization of an argument of B. Hartley (unpublished) who first proved that every infinite group in which no element has an infinity of square roots possesses an infinite sparse subset. Theorem 2 does not generalize this result because of the assumption concerning the FC-center.

Theorem 1 leaves open the question whether there are $2^{\aleph_0}$ non-isomorphic 2-generated groups of exponent p (p a large prime). We feel that this result is doubtful in view of the extremely delicate subgroup structure of these Burnside groups.

2. A theorem on subgroups of wreath products.

LEMMA 1. *If S is a subset of a group G, the following conditions are equivalent:*

(i) *S is sparse in G.*
(ii) *For all $s \neq t \in S$, $s^{-1}S \cap t^{-1}S = \{1\}$.*
(iii) *For all $x \neq y \in G$, $|Sx \cap Sy| \leqq 1$.*
(iv) *For all $x \neq y \in G$, $|xS \cap yS| \leqq 1$.*

Proof. We will first show that (i) and (iv) are equivalent. Suppose that for some $x \neq y \in G$, $|xS \cap yS| > 1$. Then there are four distinct elements s_1, s_2, s_3, $s_4 \in S$ such that

$$xs_1 = ys_2 \neq xs_3 = ys_4.$$

Then $y^{-1}x = s_2s_1^{-1} = s_4s_3^{-1}$, which implies that S is not sparse. Thus (i) implies (iv). On the other hand, if S is not sparse, there are elements $s_2 \neq s_4$ and $s_1 \neq s_3$ of G such that $s_1s_2^{-1} = s_3s_4^{-1}$. Denoting this element by x, we see that $s_1, s_3 \in xS \cap S$, contrary to (iv). Thus (iv) implies (i).

By a similar argument, (ii) and (iii) are equivalent. Finally, it is immediate that (iii) implies (i) and (iv) implies (ii).

We will now establish some notation for wreath products. $G = A\mathrm{Wr}B$ is the *unrestricted wreath product of A by B*; and $\Omega =$ the base group of $G =$ the set of functions from B into A. If $Y \leqq \Omega$ and $b \in B$, we put $Y(b) = \{f(b) | f \in Y\}$. If $f \in \Omega$, the *support* of f, $\sigma(f)$, is the set $\{b \in B | f(b) \neq 1\}$. We identify A with the 1-coordinate of Ω, that is, $A = \{f \in \Omega | \sigma(f) \subseteq \{1\}\}$. Thus $A^B = \{f \in \Omega | \sigma(f) \text{ is finite}\}$, and this notation is consistent with that of the normal closure. Finally, if $f, g \in \Omega$ and $b \in B$, we note

$$\sigma(f^b) = \sigma(f)b, \quad \text{and} \quad \sigma([f, g]) \subseteq \sigma(f) \cap \sigma(g).$$

PROPOSITION 1. *Let A and $B = \langle x, y \rangle$ be groups with the properties*

(i) *A is countable and A is generated by each of its subsets having non-empty intersection with each conjugacy class of A,*
(ii) *$|x| < \infty$, and*
(iii) *B has an infinite subset S which is $\langle x \rangle$-sparse.*

Then, there is a 2-generator subgroup H of $G = A\mathrm{Wr}B$ such that

(1) *$H/H \cap \Omega \cong B$, and letting $N = H \cap \Omega$,*
(2) *for all $b \in B$, $N(b) = A$, and*
(3) *$N' = (A')^B$.*

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Proof. We first choose $f \in \Omega$ such that $\sigma(f) = S$ and, for all $a \in A$, $\{s \in S | f(s) = a\}$ is infinite.

We will note initially that, since S is sparse, it follows from (4.2.11) of [**8**] that

(2.1) $(f^B)' = (A')^B$.

We define the group $H = \langle xf, y \rangle$ and note that $H \leqq f^B B$. Since $G = H\Omega$, we have $H/H \cap \Omega \simeq B$, which is (i).

We will note the following for future reference.

(2.2) *If* $b \in B$, then there exists $c \in f^B$ such that $cb \in H$.

The proofs of (2) and (3) involve computations with the element

(2.3) $g = (xf)^n = f^{x^{n-1}} \ldots f^x f$, where $n = |x|$.

Clearly $\sigma(g) \leqq S\langle x \rangle$ and $g^H \leqq H \cap \Omega = N$.

Let $a \in A$. Since $\{s \in S | f(s) = a\}$ is infinite and $|S \cap Sx^i| \leqq 1$ for all $1 \leqq i \leqq n$ (because S is sparse), there are infinitely many $s \in S$ such that

(2.4) $f(s) = a$, $s \notin \sigma(f^{x^i})$ for all $1 \leqq i < n$, and hence $g(s) = a$.

Let s be as in (2.4) and let $b \in B$. By (2.2) there exists $c \in f^B$ such that $cs^{-1}b \in H$. Since $g^{cs^{-1}b}(b) = g^c(s) = a^{c_1}$ where $c_1 = c(s)$, $g^H(b)$ contains a member of each conjugacy class of A. So, by assumption (i), $A = g^H(b) \leqq N(b)$ and (2) is now proved.

To prove part (3), we first show

(2.5) $A' \leqq (g^H)'$.

Let a and a_1 be distinct elements of A. By (2.4) there are elements $s, r \in S$ such that $g(s) = a$ and $g(r) = a_1$; and by (2.2) there exist $c, d \in f^B$ such that $cs^{-1} \in H$ and $dr^{-1} \in H$. Note that s depends on a and c depends on s; similarly d depends on r which depends on a_1. Let $h = [g^{cs^{-1}}, g^{dr^{-1}}]$. Thus $h \in (g^H)'$ and

$$\sigma(h) \subseteq \sigma(g^c)s^{-1} \cap \sigma(g^d)r^{-1} \subseteq S\langle x \rangle s^{-1} \cap S\langle x \rangle r^{-1} \subseteq \{1\}$$

since S is $\langle x \rangle$-sparse. Further, $h(1) = [g^c(s), g^d(r)] = [a^{c_1}, a_1^{d_1}]$ where $c_1 = c(s)$ and $d_1 = d(r)$. Thus c_1 depends on a and d_1 depends on a_1. We conclude that there is a subset D of A, which contains a member of each conjugacy class of A, such that $[D, D] \leqq (g^H)'$. Since $A = \langle D \rangle$ by (i), we have $A' = [D, D]^A \leqq (g^H)'$ by part (2), and this proves (2.5).

Now, to prove part (3), let $b \in B$ and choose $c \in f^B$ such that $cb \in H$ by (2.2). Letting $h = cb$, we have $(A')^b = (A')^h \leqq ((g^H)')^h = (g^H)'$ and hence

(2.6) $(A')^B \leqq (g^H)' \leqq N'$ where $N = H \cap \Omega$.

Finally, (2.1) and the fact that $N \leqq f^B$ imply that $N' \leqq (f^B)' = (A')^B$; together with (2.6) this completes the proof of (3).

3. Proof of Theorem 1.

LEMMA 2. *If p is an odd prime, then for each positive integer n, there is a group A_n such that*

(i) *A_n is a finite group of exponent p and nilpotence class $p - 1$,*
(ii) *the center of A_n has order p, and*
(iii) *if $n \neq m$, then $|A_n'| \neq |A_m'|$.*

Proof. Let H be the group of upper $p \times p$ unitriangular matrices over Z_p, the ring of integers mod p. It is well known that H has exponent p and class $p - 1$, that the center of H has order p, and that $|H'| = p^{(p-2)(p-1)/2}$. For each $n \geqq 1$ let A_n be the central product of n copies of H. The conclusions (i)–(iii) are easily verified.

Suppose I is any set of positive integers. We define

(3.1) $A(I) = \mathrm{Dr}\{A_n | n \in I\}$ (the direct product).

$A(I)$ is nilpotent of class $p - 1$ and has exponent p. Now, every normal subgroup of A_n, $n \in I$, is directly indecomposable as an $A(I)$-operator group where $A(I)$ acts on A_n by conjugation. This follows from the fact that the center of A_n has order p. It follows that, for distinct subsets I and J of positive integers, $A(I) \not\cong A(J)$. However, we need the slightly more involved

LEMMA 3. *Let p be an odd prime and $A(I)$ be one of the groups defined in* (3.1). *Let C be a cartesian power of $A(I)$; i.e., $C = \prod\{D_\alpha | \alpha \in \mathfrak{A}\}$ where each $D_\alpha \cong A(I)$. Let $L(I)$ be any subgroup of C satisfying*

(i) *$L(I)$ projects onto every coordinate of C, and*
(ii) *$L(I)' = \mathrm{Dr}\{D_\alpha' | \alpha \in \mathfrak{A}\}$.*

Then the set $\{F | F$ is a finite $L(I)$-indecomposable $L(I)$-direct factor of $L(I)'\}$ equals (up to isomorphism) the set $\{A_n' | n \in I\}$.

The proof of Lemma 3 is a standard application of the Remak-Krull-Schmidt theorem for operator groups (cf. Kurosh [**4**, Sec. 42]). Lemma 3 combined with part (iii) of Lemma 2 yields

LEMMA 4. *If I and J are distinct sets of positive integers and $L(I)$ and $L(J)$ satisfy the conditions of Lemma 3, then $L(I) \not\cong L(J)$.*

Before proving Theorem 1 we recall the results of Novikov-Adjan. For a prime p, let $B(2, p)$ be the 2-generated free group in the variety of groups of exponent p. Novikov and Adjan have shown [**7**] that $B(2, p)$ is infinite for large enough p. We refer to such primes as Novikov-Adjan primes. Let p be a Novikov-Adjan prime and define $B_p = B(2, p)/\mathscr{F}$ where $\mathscr{F}$ is the locally finite radical of $B(2, p)$. B_p is infinite since $B(2, p)$ is not locally finite, and we have

(3.2) *For each Novikov-Adjan prime p, there is an infinite 2-generated group B_p such that B_p has exponent p and trivial locally finite radical. Thus,* $\mathrm{Fit}(B_p)$ *and the FC-center of B_p are also trivial.*

Proof of Theorem 1. Let p be a Novikov-Adjan prime. We first display suitable groups A and B for use in Proposition 1. Let $A = A(I)$ where $A(I)$ is a group of exponent p given in (3.1), and let $B = B_p$, the group given in (3.2). Since $A(I)$ is nilpotent, it is easily shown that $A(I)$ satisfies the condition (i) of Proposition 1. Condition (ii) is trivial, while condition (iii) of Proposition 1 follows from (3.2) and the Corollary to Theorem 2.

Let $H(I) = \langle xf, y\rangle$ be the group obtained from Proposition 1. Since $H(I) \leqq A(I)\mathrm{Wr}B_p$, a group of exponent p^2, and $|xf| = p^2$, the exponent of $H(I)$ equals p^2.

It is an easy consequence of (3.2) and the conclusion (1) of Proposition 1 that $N(I) = H(I) \cap \Omega = \mathrm{Fit}(H(I))$ and that $\mathrm{Fit}(H(I))$ equals the locally finite radical of $H(I)$.

Finally, from the conclusions (2) and (3) of Proposition 1, we see that $N(I)$ satisfies the hypotheses of Lemma 3. Thus, by Lemma 4, if J is a set of positive integers distinct from I, then $N(J) \not\cong N(I)$. It follows that the set $\mathscr{H} = \{H(I)|\ I \text{ is a set of positive integers}\}$ has the properties required by Theorem 1.

4. Proof of Theorem 2. Before proving Theorem 2 we will make several definitions.

Definition. Let S be a subset of a group G.

(a) *S has infinite right (resp. left) index in G* if, for all finite subsets $F \subseteq G$, $G \neq SF$ (resp., $G \neq FS$). Clearly a left (right) coset has infinite left (right) index if and only if it is a coset of a subgroup of infinite index. If S is a normal subset of G, then the conditions that S has infinite right or left index in G are equivalent, and we simply say *S has infinite index in G*.

(b) If $H_1, \ldots, H_N$ are subgroups of G, we define

$$\varphi(H_1, \ldots, H_N) = \left\{ \bigcup_{i=1}^{N} H_i F_i | F_i \text{ is a finite subset of } G,\ 1 \leqq i \leqq N \right\},$$

and we put

$$\varphi_G = \bigcup\{\varphi(H_1, \ldots, H_N)|N \geqq 1 \text{ and } H_1, \ldots, H_N \text{ are subgroups of infinite index in } G\}$$

Thus, $S \in \varphi_G$ if and only if S is a finite union of right cosets of infinite index in G.

We will need a sharp form of a basic and useful theorem of B. H. Neumann [**6**, 4.4].

NEUMANN'S THEOREM. *If $S \in \varphi_G$, then for some finite subset $F \subseteq G$, $G = (G - S)F$.*

The original theorem of Neumann follows easily from this. Since we do not know if this formulation occurs in the literature, we will prove it here.

We need two simple lemmas.

LEMMA 5. *If $S \in \varphi_G$, there is some $T \in \varphi_G$ such that $S \subseteq T$ and $T \in \varphi(H_1, \ldots, H_N)$ where, for all $1 \leqq i \neq j \leqq N$, $H_i \cap H_j$ has infinite index in H_i.*

Proof. Suppose $S \in \varphi_G$. Let N be minimal such that $S \subseteq T$ for some $T \in \varphi(H_1, \ldots, H_N)$. Suppose that, for some $1 \leqq i \neq j \leqq N$, $H_i \cap H_j$ has finite index in H_i. Then $H_i = (H_i \cap H_j)F$ for some finite $F \subseteq G$, and hence $H_i \subseteq H_jF$. It follows that $S \subseteq T \subseteq X$ for some $X \in \varphi(H_1, \ldots, \hat{H}_i, \ldots, H_n)$ with H_i deleted, contrary to the minimality of N.

LEMMA 6. *If H and K are subgroups of G and $x, y \in G$, then $Hx \cap Ky \subseteq (H \cap K)z$ for some $z \in G$.*

Proof. If $a, b \in Hx \cap Ky$, then $ab^{-1} \in H \cap K$.

Proof of Neumann's Theorem. In our proof, we can clearly replace $S \in \varphi_G$ by any $T \in \varphi_G$ such that $S \subseteq T$. Hence, using Lemma 5, we can assume $S \in \varphi(H_1, \ldots, H_N)$ where for all $1 \leqq i \neq j \leqq N$, $H_i \cap H_j$ has infinite index in H_i.

We will proceed by induction on N. If $N = 1$, the result is clear, since $G - S$ is the union of all but finitely many right cosets of H_1 in G.

Let $S = \bigcup_{i=1}^{N} H_iF_i$ where each $F_i \subseteq G$ is finite.

Choose $g \in G - H_1F_1$. We have

$$B = H_1g \cap S = H_1g \cap \left(\bigcup_{i=2}^{N} H_iF_i \right).$$

By Lemma 6,

$$Bg^{-1} = H_1 \cap \left(\bigcup_{i=2}^{N} H_iF_ig^{-1} \right) \subseteq J$$

for some $J \in \varphi(H_1 \cap H_2, \ldots H_1 \cap H_N)$. Clearly, we can assume $J < H_1$. Now, by induction, there is some finite $X \subseteq H_1$ such that $H_1 = (H_1 - J)X$. Hence,

$$H_1 = (H_1 - Bg^{-1})X = (H_1g - B)g^{-1}X \subseteq (G - S)g^{-1}X,$$

and $H_1F_1 \subseteq (G - S)g^{-1}XF_1 = (G - S)P_1$ where P_1 is finite.

Similarly, for each i, $1 \leqq i \leqq N$, there is a finite $P_i \subseteq G$ such that $H_iF_i \subseteq (G - S)P_i$.

Putting $F = (\bigcup_{i=1}^{N} P_i) \cup \{1\}$, we have $G = (G - S)F$ as desired.

COROLLARY 1. *Suppose G is a group and $I \subseteq G$ has infinite right (resp., left) index in G. Suppose further that $Y \subseteq G$ is a finite union of cosets of infinite index in G. Then $I \cup Y$ has infinite right (resp., left) index in G.*

Proof. Since every left coset of a subgroup H is a right coset of a conjugate of H, we have $Y \in \varphi_G$. Assume the "right" hypothesis and suppose $G = (I \cup Y)X = IX \cup YX$ for some finite $X < G$. Then $(G - YX) \subseteq IX$ and Neumann's Theorem implies that $G = IXF$ for some finite $F \subseteq G$, contrary

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to I having infinite right index in G. Hence, $I \cup Y$ has infinite right index in G.
By symmetry the corollary is correct under the "left" hypothesis.

The next corollary is an immediate result of the first and will be used in the proof of Theorem 2.

COROLLARY 2. *No set of the form $I \cup Y$, as described in Corollary 1, contains any subgroup of finite index in G.*

Proof of Theorem 2. We first recall the hypothesis of this theorem (see Introduction): namely, that $\Phi_2 = \{g \in G | g^2 \in \Phi = \text{the } FC\text{-center of } G\}$ has infinite index in G. Φ_2 is, of course, a normal subset of G.

Let $G = \bigcup_{i=1}^{\infty} X_i$ where $X_i \subseteq X_{i+1}$ is finite and $X_i = X_i^{-1}$.

We will construct $S = \{z_1, \ldots, z_N, \ldots\}$ so that the following three conditions hold for all $N \geqq 1$. Let $S_N = \{z_1, \ldots, z_N\}$.

$(A)_N$: For all $u \neq v \in S - S_N$, $SX_N u^{-1} \cap SX_N v^{-1} = X_N \cap \Phi$,
$(B)_N$: $z_{N+1} \notin \Phi S_N X_N^2$, and
$(C)_N$: $z_{N+1} \in C_G(X_N \cap \Phi)$.

First we will show that these conditions imply the theorem.

Suppose $X \subseteq G$ is finite. Then $X \subseteq X_N$ for some N and we put $T = S - S_N$. So (1) holds, and since $(C)_N$ implies $T \subseteq C_G(X_N \cap \Phi) \subseteq C_G(X \cap \Phi)$, (2) holds also. To prove (3), we suppose that $g = tx = sy$ where $t \in T$, $s \in S$, and $x, y \in X_N$. Assume $t \neq s$. Since $t \in S - S_N$, then for some $M \geqq N$, either

$$t \in S_M \text{ and } s \in S_{M+1} - S_M \quad \text{or} \quad s \in S_M \text{ and } t \in S_{M+1} - S_M.$$

In the first case $s = txy^{-1} \in S_M X_M^2$, contrary to $(B)_M$, and the other case is identical. Hence, $t = s$, $x = y$ and (3) holds. To prove (4), first note that $I = SXu^{-1} \cap SXv^{-1} \supseteq X \cap \Phi$ because of (2). To prove the reverse inclusion, suppose $x \in I$. Then $x \in X_N \cap \Phi$ by $(A)_N$, and $x = uxu^{-1}$ since $u \in C_G(X_N \cap \Phi)$. Now $x \in I$ also implies $x = syu^{-1}$ for some $s \in S$ and $y \in X$. Hence, $ux = sy$, and the previous argument shows $x = y \in X$, proving (4).

In order to construct the set S inductively we will replace the condition $(A)_N$ by

$(A)^N$: $S_{N+1} X_i u^{-1} \cap S_{N+1} X_i v^{-1} = X_i \cap \Phi$ for all $1 \leqq i < N$, and for all $u \neq v \in S_{N+1} - S_i$.

Clearly, if $(A)^N$ holds for all N, then $(A)_N$ holds for all N, as required.

Assume that $S_N = \{z_1, \ldots, z_N\}$ has been constructed so that $(A)^i$, $(B)_i$, and $(C)_i$ hold for all $i < N$ (z_1 can be chosen arbitrarily). In order to choose $z = z_{N+1}$ so that $(A)^N$ and $(B)_N$ will hold, we will divide these conditions into several cases and show that, to satisfy the condition of each case, it is sufficient to choose $z = z_{N+1}$ either

(α): outside of a certain coset of infinite index in G, or
(β): outside of the set $I = \Phi_2 S_N (X_N \cup X_N^2)$.

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Thus, we will need to choose $z = z_{N+1} \in G$ in accordance with finitely many conditions of type (α); the condition (β); and, finally, $(C)_N$. Suppose Y is the union of all the cosets occurring in the conditions of type (α). Because of our hypothesis that Φ_2 has infinite index in G, it follows that I has infinite index in G, and Corollary 2 above implies that there exists $z = z_{N+1} \in G$ which will meet all the conditions of type (α); the condition (β); and $(C)_N$, since $C_G(X_N \cap \Phi)$ is a subgroup of finite index in G. That is, there exists $z \in C_G(X_N \cap \Phi) - (I \cup Y)$.

Clearly the condition $(B)_N$ will be satisfied if the condition (β) is met. So we must only attend to the condition $(A)^N$.

In order to satisfy $(A)^N$ we must choose $z = z_{N+1}$ to avoid satisfying all relations of the form

$$(*) \qquad sxu^{-1} = tyv^{-1} \notin X_i \cap \Phi$$

where $1 \leqq i < N$; $s, t \in S_N \cup \{z\}$; $x, y \in X_i$ and u, v are distinct elements of $(S_N \cup \{z\}) - S_i$.

We will consider all possible cases (up to symmetry) where one or more of the elements s, t, u, v equals z, and in each case show that (*) can be avoided by choosing z in accord with finitely many conditions of type (α) or in accord with (β).

Case I. Exactly one of the elements s, t, u, v equals z. If s or $t = z$, then (*) implies $z \in S_N X_N S_N^{-1} S_N X_N$, while if u or $v = z$, a similar condition holds. These z all lie in a finite set, which can be avoided by trivial conditions of type (α).

Case II. $s = t = z$. Then (*) implies $xu^{-1} = yv^{-1}$. If $u \neq z$ and $v \neq z$, then u and v are distinct elements of $S_N - S_i$. Thus, $i < N - 1$ and we have a contradiction to $(A)^{N-1}$. So suppose $u = z$. Thus, $z \in S_N X_N^2$, contrary to (β), and the case $v = z$ is similar.

Case III. $s = u = z$. Then (*) becomes $zxz^{-1} = tyv^{-1} \notin X_i \cap \Phi$ where $v \in S_N$ and $x, y \in X_i$.
We assume $t \neq z$ since otherwise Case II applies. Since z will satisfy $(C)_N$, z will centralize $X_i \cap \Phi$. Hence $zxz^{-1} \notin X_i \cap \Phi$ is possible only if $x \notin \Phi$. So we can assume $x \notin \Phi$, that is, $C_G(x)$ has infinite index in G. Now $zxz^{-1} = tyv^{-1} \in S_N X_i S_N^{-1}$ if and only if z belongs to a certain finite set of left cosets of $C_G(x)$. Since only finitely many $x \in X_N$ can occur in (*), conditions of type (α) will imply that (*) is not satisfied in this case.

Case IV. $u = t = z$. In this case (*) implies

$$\text{(i)} \qquad sxz^{-1} = zyv^{-1}.$$

Suppose $w \in G$ also satisfies this equation in place of z; then

$$\text{(ii)} \qquad sxw^{-1} = wyv^{-1}.$$

Computing $(\mathrm{ii})^{-1}(\mathrm{i})$ gives

$$wz^{-1} = (w^{-1}z)^{yv^{-1}},$$

while $(\mathrm{i})(\mathrm{ii})^{-1}$ gives

$$(z^{-1}w)^{(sx)^{-1}} = zw^{-1}.$$

Since these equations are inverses, we have

$$(w^{-1}z)^{yv^{-1}} = (w^{-1}z)^{(sx)^{-1}}.$$

Thus, $w^{-1}z$ commutes with $yv^{-1}sx$, that is, w and z belong to the same left coset of $C_G(yv^{-1}sx)$. If $yv^{-1}sx \notin \Phi$, then z must be chosen to meet a condition of type (α) for each of finitely many sets $\{y, v, s, x\}$.

Thus, we can assume $yv^{-1}sx \in \Phi$.

Subcase 1. $v \neq s$. We can assume that $v \neq z$ and $s \neq z$, since otherwise, previous cases apply. Writing $v = z_k$ and $s = z_j$, we see that $i < k$ and $\max\{k, j\} \leqq N$. Note also that $x, y \in X_i$. Thus, if $k < j$, we have $s \in \Phi S_k X_i^2 \subseteq \Phi S_k X_k^2$, which is contrary to $(B)_{j-1}$. Similarly, if $j < k$ we have $v \in \Phi S_j X_i^2 \subseteq \Phi S_\tau X_\tau^2$ where $\tau = \max\{i, j\} < k$. This contradicts $(B)_{k-1}$.

Subcase 2. $v = s$. Thus, $yx \in \Phi$, $y = x^{-1} \pmod{\Phi}$, and (i) becomes $sxz^{-1} = zx^{-1}s^{-1} \pmod{\Phi}$. Since these elements are also inverses they have order 2 $(\mathrm{mod}\ \Phi)$, and hence $z \in \Phi_2 S_N X_N$. So we will choose $z \notin \Phi_2 S_N X_N$, a condition implied by (β).

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Michigan State University,
East Lansing, Michigan