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The H and K Family of Mock Theta Functions

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Abstract. In his last letter to Hardy, Ramanujan defined 17 functions $F(q)$, $|q| < 1$, which he called mock θ-functions. He observed that as q radially approaches any root of unity ζ at which $F(q)$ has an exponential singularity, there is a θ-function $T_\zeta(q)$ with $F(q) - T_\zeta(q) = O(1)$. Since then, other functions have been found that possess this property. These functions are related to a function $H(x, q)$, where x is usually q^r or $e^{2\pi ir}$ for some rational number r. For this reason we refer to H as a "universal" mock θ-function. Modular transformations of H give rise to the functions K, K_1, K_2. The functions K and K_1 appear in Ramanujan's lost notebook. We prove various linear relations between these functions using Appell–Lerch sums (also called generalized Lambert series). Some relations (mock theta "conjectures") involving mock θ-functions of even order and H are listed.

1 Introduction

In Ramanujan's last letter to Hardy ([22, pp. 354–355], [23, pp. 127–131], [26, pp. 56–61]) he observes that the asymptotic expansions of certain q-series with exponential singularities at roots of unity "close" in a striking manner. For example, let

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

(where the last equality is the first Rogers-Ramanujan identity). If $q = e^{-t}$ and $t \to 0^+$ (so that q approaches 1 radially from inside the unit circle), then

$$G(q) = \sqrt{\frac{2}{5-\sqrt{5}}} \exp\left(\frac{\pi^2}{15t} - \frac{t}{60}\right) + o(1).$$

In the same letter Ramanujan notes that it is only for some special q-series $f(q)$ that the exponential closes, *i.e.,* its argument terminates with some power t^N. If $f(q)$ is not the sum of a theta function and a function which is $O(1)$ at all roots of unity ζ, and if for each such ζ there is an approximation of the form

$$f(q) = \sum_{\mu=1}^{M} t^{k_\mu} \exp\left(\sum_{\nu=-1}^{N} c_{\mu\nu} t^{\nu}\right) + O(1)$$

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as $t \to 0^+$ with $q = \zeta e^{-t}$, he calls $f(q)$ a *mock θ-function*. It appears from his letter, however, that he was actually concerned with functions having the (possibly) more restrictive property that for every root of unity ζ, there are modular forms $h_j^{(\zeta)}(q)$ and rational numbers α_j, $1 \le j \le J(\zeta)$, such that

$$f(q) = \sum_{j=1}^{J(\zeta)} q^{\alpha_j} h_j^{(\zeta)}(q) + O(1)$$

as q radially approaches ζ. For a further description of mock theta functions see [13].

The most well-known infinite family of mock θ-functions is defined by $M(q^r, q)$, where r is a noninteger rational number and

$$M(x, q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x)_{n+1}(q/x)_{n+1}}.$$

In this paper we will use the standard notation for the q-shifted factorial:

$$(a\,;q^k)_0 = 1, \quad (a\,;q^k)_n = \prod_{m=0}^{n-1}(1 - aq^{km}), \quad (a\,;q^k)_\infty = \prod_{m=0}^{\infty}(1 - aq^{km}),$$

where k is a positive integer. When $k = 1$ it is customary to write $(a)_n$ instead of $(a\,;q)_n$.

The functions $M(q, q^5)$ and $M(q^2, q^5)$ appear in the celebrated Mock Theta Conjectures stated by Ramanujan in the lost notebook [23] and later proved by Hickerson [15]. These conjectures are linear relations involving the fifth order mock θ-functions.

The function

$$N(y, q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(yq)_n(y^{-1}q)_n}$$

is related to $M(x, q)$ by a modular transformation law, proved in [12] and restated in Section 4. This function is also known as the rank generating function (see, for example [4]).

In [12] the functions $M(q^r, q)$ and $N(e^{2\pi ir}, q)$ are denoted by $M(r, q)$ and $N(r, q)$, respectively. The product $(e^{2\pi ir})_n(e^{-2\pi ir})_n$ in the definition of $N(r, q)$ in [12] should be $(e^{2\pi ir}q)_n(e^{-2\pi ir}q)_n$. The function $N_1(r, q)$ in [12] is equal to our $M(e^{2\pi ir}q, q^2)$.

In this paper we study another infinite family of mock θ-functions defined by $H(q^r, q)$, where r is a noninteger rational number and

$$H(x, q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}(-q)_n}{(x)_{n+1}(q/x)_{n+1}}.$$

The function $H(x,q)$ is defined in a different way by Choi [6, eq. (2.34)]. It is closely related to the functions

$$K(y,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(yq^2;q^2)_n (y^{-1}q^2;q^2)_n},$$

$$K_1(y,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(yq;q^2)_{n+1} (y^{-1}q;q^2)_{n+1}},$$

$$K_2(y,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)} (-1)_n}{(yq)_n (y^{-1}q)_n}.$$

Of particular interest are the linear and modular relations connecting these functions. One such linear relation involving the functions K and K_1 appears on page 8 of the lost notebook (see also [1, pp. 264–267]). We prove several more in Section 3. The modular relations are studied in Section 4.

For every classical mock θ-function $f(q)$ explicit linear relations involving f, H (or M), and ordinary θ-functions are known [13]. These relations are usually referred to as mock theta "conjectures", even when their proofs are known. The "conjectures" for the functions of even order involve H and are listed in Section 5.

In Section 2 we show that the function H is a normalized level 2 Appell function (see Section 6 for the definition of an Appell function), whereas the function M is a normalized level 3 Appell function. Appell functions of higher level can often be expressed in terms of those with lower level. A linear relation expressing M in terms of H and a θ-function is given in Section 3. By this relation and the mock theta "conjectures", every classical mock θ-function is related to H. For this reason we refer to H as a "universal" mock θ-function.

A preprint of this paper was circulated during a conference at the University of Florida in 2004. Subsequently, several results of the preprint were cited by Bringmann, Ono, and Rhoades [5]. Their desire to see a published version is fulfilled here.

2 Appell-Lerch Sums

To prove linear relations and construct transformation laws for these functions it is more convenient to work with the Appell-Lerch sums (also called generalized Lambert series) studied in [18, 19]. Two of these sums defined for positive integers k are

$$U_k(x,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1 - xq^n}, \tag{2.1}$$

$$V_k(y,q) = \frac{1}{1-y^{-1}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{1 - yq^n}. \tag{2.2}$$

It is not difficult to show that

$$U_k(x,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-q^{n+1}/x)}, \tag{2.3}$$

$$V_k(y,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)}. \tag{2.4}$$

Observe that

$$\begin{aligned}
U_k(x,q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-xq^n} + \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-xq^n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-xq^n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n-1)}}{1-xq^{-n}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-xq^n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{\frac{1}{2}kn(n+1)}}{1-xq^{-n-1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-xq^n} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)} q^{n+1}/x}{1-q^{n+1}/x} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}(1-q^{2n+1})}{(1-xq^n)(1-q^{n+1}/x)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-q^{n+1}/x)} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)} q^{2n+1}}{(1-xq^n)(1-q^{n+1}/x)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-q^{n+1}/x)} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n-1)} q^{2n-1}}{(1-xq^{n-1})(1-q^n/x)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-q^{n+1}/x)} + \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)} q^{-2n-1}}{(1-xq^{-n-1})(1-q^{-n}/x)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-q^{n+1}/x)} + \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(q^{n+1}-x)(q^n-1/x)} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-q^{n+1}/x)},
\end{aligned}$$

which is (2.3). Similarly,

$$\begin{aligned}
V_k(y,q) &= \frac{1}{(1-y)(1-y^{-1})} \\
&\quad + \frac{1}{1-y^{-1}} \sum_{n=1}^{\infty} \left(\frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{1-yq^n} + \frac{(-1)^{kn} q^{\frac{1}{2}n(kn-1)}}{1-yq^{-n}} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-y)(1-y^{-1})} \\
&\quad + \frac{1}{1-y^{-1}} \sum_{n=1}^{\infty} \left(\frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{1-yq^n} - \frac{(-1)^{kn} q^{\frac{1}{2}n(kn-1)} y^{-1} q^n}{1-y^{-1}q^n} \right) \\
&= \frac{1}{(1-y)(1-y^{-1})} \\
&\quad + \frac{1}{1-y^{-1}} \sum_{n=1}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)} (1-y^{-1})(1+q^n)}{(1-yq^n)(1-y^{-1}q^n)} \\
&= \frac{1}{(1-y)(1-y^{-1})} + \sum_{n=1}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)} (1+q^n)}{(1-yq^n)(1-y^{-1}q^n)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)} + \sum_{n=1}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)} q^n}{(1-yq^n)(1-y^{-1}q^n)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)} + \sum_{n=-\infty}^{-1} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn-1)} q^{-n}}{(1-yq^{-n})(1-y^{-1}q^{-n})} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)} + \sum_{n=-\infty}^{-1} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)} q^{-2n}}{(1-yq^{-n})(1-y^{-1}q^{-n})} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)} + \sum_{n=-\infty}^{-1} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(q^n-y)(q^n-y^{-1})} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)},
\end{aligned}
$$

which is (2.4).

By (2.3) and (2.4) we see that $U_k(x,q) = U_k(q/x,q)$ and $V_k(y,q) = V_k(y^{-1},q)$. Also, $V_k(e^{2\pi i r},q)$ is real when r is a noninteger rational number and q is real with $0 < |q| < 1$. (This function plays an important role in equation (4.2).)

Many of our identities involve the Jacobi θ-function defined by

$$
j(x,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} x^n = (x)_\infty (q/x)_\infty (q)_\infty
$$

(where the last equality is the well-known Jacobi triple-product identity; see, for example [7, p. 12]). Following Hickerson [15], we define a *θ-product* (or *θ-quotient*) to be an expression of the form

$$
Cq^e x_1^{f_1} \cdots x_r^{f_r} L_1^{g_1} \cdots L_s^{g_s},
$$

where C is a complex number, e and f_i are rational numbers, g_j are integers, and each L_j has the form

$$
j(Dq^h x_1^{k_1} \cdots x_r^{k_r}, \pm q^m)
$$

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for some complex number D (usually $D = \pm 1$) and rational numbers h, k_i, m with $m > 0$. A *θ-function* is a finite sum of θ-products. Thus $(q)_\infty = j(q, q^3)$ is a θ-function, even though it lacks the factor $q^{\frac{1}{24}}$ needed to make it a modular form.

The sums U_1 and V_1 (multiplied by $1 - y$) turn out to be θ-functions, since

$$U_1(x,q) = \frac{(q)_\infty^2}{(x)_\infty (q/x)_\infty} = \frac{(q)_\infty^3}{j(x,q)}, \tag{2.5}$$

$$V_1(y,q) = \frac{U_1(y,q)}{1 - y^{-1}} = \frac{(q)_\infty^2}{(y)_\infty (y^{-1})_\infty}. \tag{2.6}$$

Equation (2.5) is the expansion for the reciprocal of a θ-function and is equivalent to the next to last formula on page 1 of the lost notebook (see also [1, p. 264, eq. (12.2.9)]). The function $(1 - z)U_1(z, q)/(q)_\infty$ is the crank statistic of Garvan [8, eq. (1.25)].

At this point we introduce two more θ-functions: Jacobi's $\theta_4(0, q)$ defined by

$$\theta_4(0,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = j(q,q^2) = \frac{(q)_\infty^2}{(q^2;q^2)_\infty} = \frac{(q)_\infty}{(-q)_\infty}$$

and the Gauss function $\psi(q)$ defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} = \frac{1}{2} j(-1,q) = j(-q,q^4) = \frac{(q^2;q^2)_\infty^2}{(q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Using these functions we can express the relations between the H and K family and the sums U_2 and V_2 as follows:

$$H(x,q) = \frac{U_2(x,q)}{\theta_4(0,q)} \tag{2.7}$$

$$K(y,q) = (1-y)(1-y^{-1}) \frac{V_2(y,q^2)}{\psi(q)}, \tag{2.8}$$

$$K_1(y,q) = \frac{1}{(1-y^{-1})\psi(q)} \sum_{n=-\infty}^{\infty} \frac{q^{(n+1)(2n+1)}}{1 - yq^{2n+1}} = \frac{V_2(y,q^2) - V_1(y,q)}{\psi(q)}, \tag{2.9}$$

$$\begin{aligned} K_2(y,q) &= \frac{(1-y)(1-y^{-1})}{\theta_4(0,q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1-yq^n)(1-y^{-1}q^n)} \\ &= \frac{1-y}{1+y} \left(1 + 2y \frac{U_2(y,q)}{\theta_4(0,q)} \right). \end{aligned} \tag{2.10}$$

Equation (2.8) is equivalent to the last formula on page 1 of the lost notebook. Other identities of this type are given in [1, Chapter 12].

Equations (2.5)–(2.10) can be proved by the Watson–Whipple transformation [7, p. 242, eq. (III.17)]:

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$$(2.11)\quad {}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & f \\ & a^{\frac12}, & -a^{\frac12}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & \frac{aq}{f}, \end{matrix} ;q;\frac{a^2q^2}{bcdef}\right] = \frac{(aq)_\infty(\frac{aq}{de})_\infty(\frac{aq}{df})_\infty(\frac{aq}{ef})_\infty}{(\frac{aq}{d})_\infty(\frac{aq}{e})_\infty(\frac{aq}{f})_\infty(\frac{aq}{def})_\infty}\,{}_4\phi_3\left[\begin{matrix} \frac{aq}{bc}, & d, & e, & f \\ \frac{aq}{b}, & \frac{aq}{c}, & \frac{def}{a} \end{matrix};q;q\right].$$

We now prove (2.7). (The proofs of the other identities are similar.) Observe that

$$\frac{(qa^{\frac12})_n(-qa^{\frac12})_n}{(a^{\frac12})_n(-a^{\frac12})_n} = \frac{(1-aq^2)(1-aq^4)\cdots(1-aq^{2n})}{(1-a)(1-aq^2)\cdots(1-aq^{2n-2})} = \frac{1-aq^{2n}}{1-a}.$$

Let e and f tend to infinity (or equivalently, put $e = 1/e'$, $f = 1/f'$, simplify and then let $e' = f' = 0$). Then $(aq/e)_n$ and $(aq/f)_n$ tend to 1. Also,

$$\begin{aligned}(e)_n &= (1-e)(1-eq)\cdots(1-eq^{n-1}) \\ &= (-e)^n\Big(-\frac{1}{e}+1\Big)\Big(-\frac{1}{e}+q\Big)\cdots\Big(-\frac{1}{e}+q^{n-1}\Big) \\ &\sim (-e)^n q^{\frac12 n(n-1)}\end{aligned}$$

as $e \to \infty$. Similarly, $(f)_n \sim (-f)^n q^{\frac12 n(n-1)}$ as $f \to \infty$. Hence in the limit, (2.11) becomes

$$\sum_{n=0}^{\infty}\Big(\frac{1-aq^{2n}}{1-a}\Big)\frac{(a)_n(b)_n(c)_n(d)_n}{(q)_n\big(\frac{aq}{b}\big)_n\big(\frac{aq}{c}\big)_n\big(\frac{aq}{d}\big)_n}\Big(\frac{a^2}{bcd}\Big)^n q^{n(n+1)} = \frac{(aq)_\infty}{\big(\frac{aq}{d}\big)_\infty}\sum_{n=0}^{\infty}\frac{\big(\frac{aq}{bc}\big)_n(d)_n}{(q)_n\big(\frac{aq}{b}\big)_n\big(\frac{aq}{c}\big)_n}\Big(-\frac{a}{d}\Big)^n q^{\frac12 n(n+1)}.$$

Now put $a = q$, $b = x$, $c = q/x$, and $d = -q$ to get

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}(1-q^{2n+1})}{(1-xq^n)(1-q^{n+1}/x)} = \frac{(q)_\infty}{(-q)_\infty}\sum_{n=0}^{\infty}\frac{q^{\frac12 n(n+1)}(-q)_n}{(x)_{n+1}(q/x)_{n+1}}.$$

Therefore

$$\begin{aligned}\theta_4(0,q)\,H(x,q) &= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\Big(\frac{xq^n}{1-xq^n}+\frac{1}{1-q^{n+1}/x}\Big) \\ &= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\Big(\frac{1}{1-q^{n+1}/x}-\frac{1}{1-q^{-n}/x}\Big) \\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)}}{1-q^{n+1}/x} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)}}{1-xq^n} = U_2(x,q),\end{aligned}$$

since $U_2(x,q) = U_2(q/x,q)$. This completes the proof of (2.7).

Subtracting $V_2(y,q^2)$ from $V_1(y,q)$ removes the even terms in $V_1(y,q)$. The second equality in (2.9) easily follows from this observation. The second equality in (2.10) is a consequence of the two identities:

$$\text{(2.12)}\quad yU_2(y,q) + y^{-1}U_2(y^{-1},q) = -\theta_4(0,q),$$

$$\text{(2.13)}\quad yU_2(y,q) - y^{-1}U_2(y^{-1},q) = (y - y^{-1}) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1-yq^n)(1-y^{-1}q^n)}.$$

The proof of (2.13) is straightforward. We now prove (2.12). By (2.3) we get $U_2(x,q) = U_2(q/x,q)$. Hence,

$$\begin{aligned}
yU_2(y,q) + y^{-1}U_2(y^{-1},q) &= yU_2(y,q) + y^{-1}U_2(yq,q)\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)} y}{1-yq^n} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)} y^{-1}}{1-yq^{n+1}}\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)} y}{1-yq^n} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)} y^{-1}}{1-yq^n}\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2} (yq^n - y^{-1}q^{-n})}{1-yq^n}\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2} (1+yq^n)(1-y^{-1}q^{-n})}{1-yq^n}\\
&= -\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2} (1+yq^n)(1-yq^n)y^{-1}q^{-n}}{1-yq^n}\\
&= -\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} (1+y^{-1}q^{-n})\\
&= -\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} - y^{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)}\\
&= -\theta_4(0,q),
\end{aligned}$$

since the last sum vanishes.

The M and N analogues of (2.7) and (2.8) are ([12])

$$\text{(2.14)}\qquad M(x,q) = \frac{U_3(x,q)}{(q)_\infty},$$

$$\text{(2.15)}\qquad N(y,q) = \frac{(1-y)(1-y^{-1})}{(q)_\infty} V_3(y,q).$$

We see that the H and K family is related to U_2 and V_2, and the M and N family is related to U_3 and V_3. In his proof of the Mock Theta Conjectures, Hickerson denotes the function M by g. In view of these observations, in the forthcoming paper *A survey of classical mock theta functions* [13], the functions H, K, M, N are denoted by g_2, h_2, g_3, h_3, respectively.

3 Linear Relations

In this section we give linear relations for mock θ-functions, where the coefficients are usually θ-functions. Unlike the modular transformation laws in Section 4, convergence of the functions in these relations is not required. Equality is to be interpreted as equality of formal q-series (or Laurent series in q after replacing q by a suitable power of q if necessary).

Since

$$U_2(x,q)+U_2(-x,q)=2U_1(x^2,q^2)=\frac{2(q^2;q^2)_\infty^3}{j(x^2,q^2)},$$

it follows by (2.7) that

$$H(x,q)+H(-x,q)=\frac{2\psi^2(q)}{j(x^2,q^2)}. \tag{3.1}$$

From (2.10) and (2.7) we obtain

$$\frac{1+y}{1-y}K_2(y,q)=1+2yH(y,q). \tag{3.2}$$

Since $K_2(x,q)=K_2(x^{-1},q)$, it follows by (3.2) that

$$xH(x,q)+x^{-1}H(x^{-1},q)=-1. \tag{3.3}$$

Substituting $H(x^{-1},q)=H(q/x^{-1},q)=H(xq,q)$ into (3.3), we get

$$xH(x,q)+x^{-1}H(xq,q)=-1,$$

which is equivalent to the functional equation

$$H(xq,q)=-x^2H(x,q)-x.$$

Combining (3.2) and (3.1) gives

$$\frac{1+y}{1-y}K_2(y,q)-\frac{1-y}{1+y}K_2(-y,q)=\frac{4y\psi^2(q)}{j(y^2,q^2)},$$

or equivalently,

$$\frac{K_2(y,q)}{(1-y)(1-y^{-1})}+\frac{K_2(-y,q)}{(1+y)(1+y^{-1})}=\frac{4V_1(y^2,q^2)}{\theta_4(0,q)}.$$

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By (2.8) and (2.9),

$$(3.4)\quad (1-y)^{-1}K(y,q)-(1-y^{-1})K_1(y,q)=\frac{(1-y^{-1})V_1(y,q)}{\psi(q)}=\frac{(q)_\infty^3}{\psi(q)\,j(y,q)}.$$

When $y=-a$ we obtain the fifth formula on page 8 of the lost notebook (see also [1, p. 265, eq. (12.3.2)]:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2}}{(-a;q^2)_{n+1}(-q^2/a;q^2)_n}-(1+1/a)\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{(n+1)^2}}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}}=\frac{(q;q^2)_\infty\,\theta_4(0,q)}{(-a;q)_\infty(-q/a;q)_\infty}.$$

Generalizations of equations (3.1) and (3.4) were later given in [5, Theorem 1.3] and in [17, eq. (1.7), (1.11)].

The functions H and K are related by the identity [14]

$$\frac{qH(x,q)}{x}+\frac{K(-x^2/q,q^2)}{1+x^2/q}=\frac{(q^2;q^2)_\infty^3}{j(x,q)\,j(-x^2/q,q^4)}.$$

The analogous relation between M and N is

$$xM(x,q)+1=\frac{N(x,q)}{1-x},$$

which is equivalent to

$$(3.5)\qquad x^2U_3(x,q)+(1-x)V_3(x,q)+x(q)_\infty=0$$

by (2.14) and (2.15).

We now prove (3.5). By (2.1) and (2.2) we have

$$\begin{aligned}
&x^2U_3(x,q)+(1-x)V_3(x,q)+x(q)_\infty\\
&=x^2\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac32 n(n+1)}}{1-xq^n}+\frac{1-x}{1-x^{-1}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac12 n(3n+1)}}{1-xq^n}+xj(q^2,q^3)\\
&=\sum_{n=-\infty}^{\infty}\frac{(-1)^nx^2q^{\frac32 n(n+1)}}{1-xq^n}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nxq^{\frac12 n(3n+1)}}{1-xq^n}+\sum_{n=-\infty}^{\infty}(-1)^nxq^{\frac12 n(3n+1)}\\
&=\sum_{n=-\infty}^{\infty}\frac{(-1)^nxq^{\frac12 n(3n+1)}(xq^n-1+(1-xq^n))}{1-xq^n}=0.
\end{aligned}$$

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The functions M and N can be expressed in terms of H. More precisely,

$$M(x^4,q^4) = \frac{qH(x^6q,q^6)}{x^2} + \frac{x^2H(x^6/q,q^6)}{q} - \frac{x^2(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty j(x^2q,q^2)j(x^{12}q^6,q^{12})}{q(q^4;q^4)_\infty(q^6;q^6)_\infty^2 j(x^4,q^2)j(x^6/q,q^2)}.$$

This identity is a special case of more general identities expressing each U_k as a combination of $k-1$ copies of U_2 and a θ-function [13, eq. (6.7), (6.8)]. Identities (6.7) and (6.8) in [13] were discovered by the author and proved by Gordon [14].

All of the functions K, K_1, K_2, M, and N can be expressed in terms of H and θ-functions. In [13] a case is made for considering H as a "universal" mock θ-function.

Other linear relations involving H can be constructed using the transformation laws in Section 4 and the hyperbolic function identity

$$\frac{\cosh ax}{\cosh x} = \frac{\sinh(1+a)x}{\sinh 2x} + \frac{\sinh(1-a)x}{\sinh 2x}.$$

Some special values of H, K_2 and K are

$$H(-1,q) = 1/2, \tag{3.6}$$

$$H(q,-q^2) = \psi(q^4), \tag{3.7}$$

$$H(iq,q^2) = \psi(q^4), \tag{3.8}$$

$$H(i,q) = \theta_4(0,-q)/2 + i/2, \tag{3.9}$$

$$K_2(i,q) = \theta_4(0,-q), \tag{3.10}$$

$$K_2(1,q) = 1/\theta_4(0,q), \tag{3.11}$$

$$K(1,q) = 1/\psi(q). \tag{3.12}$$

Equation (3.6) is obtained from (3.3) with $x=-1$. Observe that (3.7) is (3.8) with q replaced by $-iq$, and (3.10) follows from (3.9) by (3.2). We will now prove (3.8) and (3.9).

By (2.7) and (2.5),

$$\begin{aligned} H(iq,q^2) &= \frac{U_2(iq,q^2)}{\theta_4(0,q^2)} = \frac{1}{\theta_4(0,q^2)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+1)}}{1-iq^{2n+1}} \\ &= \frac{1}{\theta_4(0,q^2)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+1)}(1+iq^{2n+1})}{1+q^{4n+2}} \\ &= \frac{1}{\theta_4(0,q^2)} \left(U_1(-q^2,q^4) + iq \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+2)}}{1+q^{4n+2}} \right) = \psi(q^4). \end{aligned} \tag{3.13}$$

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The last sum in (3.13) vanishes, since

$$\sum_{n=-\infty}^{-1} \frac{(-1)^n q^{2n(n+2)}}{1+q^{4n+2}} = -\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n+2)}}{1+q^{4n+2}}.$$

By (3.1) we have

$$H(i,q) + H(-i,q) = \frac{2\psi^2(q)}{j(-1,q^2)} = \theta_4(0,-q), \tag{3.14}$$

and by (3.3) we have

$$iH(i,q) - iH(-i,q) = -1,$$

which is equivalent to

$$H(i,q) - H(-i,q) = i. \tag{3.15}$$

Adding (3.14) and (3.15) we obtain (3.9).

To prove (3.11), we begin with the first identity of (2.10):

$$\begin{aligned}\theta_4(0,q)K_2(y,q) &= (1-y)(1-y^{-1}) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1-yq^n)(1-y^{-1}q^n)} \\ &= 1 + (1-y)(1-y^{-1}) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1-yq^n)(1-y^{-1}q^n)}.\end{aligned}$$

When $y = 1$, this becomes $\theta_4(0,q)K_2(1,q) = 1$, which is (3.11).

The proof (3.12) is similar to the proof of (3.11). By (2.8) and (2.2) we obtain

$$\begin{aligned}\psi(q)K(y,q) &= (1-y)(1-y^{-1})V_2(y,q^2) = (1-y) \sum_{n=-\infty}^{\infty} \frac{q^{n(2n+1)}}{1-yq^{2n}} \\ &= 1 + (1-y) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{q^{n(2n+1)}}{1-yq^{2n}}.\end{aligned}$$

When $y = 1$, this becomes $\psi(q)K(1,q) = 1$, which is (3.12).

In the transformation law for $H(q^r, -q)$ (see Section 4) the integral vanishes when $r = 1/2$. This implies that $H(q^{1/2}, -q)$ and $K_2(i, -q)$ are θ-functions. Using computer algebra we found identities (3.7)–(3.10).

4 Transformation Laws

In discussing the approximation of mock θ-functions near roots of unity, we have adhered to the notation $q = e^{-\alpha}$, employed by Ramanujan and his early successors. This maps the right half-plane $\mathrm{Re}(\alpha) > 0$ onto the punctured disc $0 < |q| < 1$. In the classical theory of θ-functions, as expounded for example in [25, 27], it is customary to write instead $q = e^{\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$. Thus $\alpha = -\pi i\tau$. The transformations of mock θ-functions are more complicated than those of θ-functions; they involve Mordell integrals [21]. For example, the θ-functions $(q)_\infty$, $\theta_4(q)$, and $\psi(q)$ satisfy the transformation laws:

(4.1)
$$q^{\frac{1}{24}}(q)_\infty = \sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{1}{6}}(q_1^4; q_1^4)_\infty,$$

$$q^{\frac{1}{24}}(-q; -q)_\infty = \sqrt{\frac{\pi}{\alpha}}\, q_1^{\frac{1}{24}}(-q_1; -q_1)_\infty,$$

$$\theta_4(0, q) = \sqrt{\frac{4\pi}{\alpha}}\, q_1^{\frac{1}{4}}\psi(q_1^2),$$

$$\theta_4(0, -q) = \sqrt{\frac{\pi}{\alpha}}\,\theta_4(0, -q_1),$$

$$q^{\frac{1}{8}}\psi(q) = \sqrt{\frac{\pi}{2\alpha}}\,\theta_4(0, q_1^2),$$

$$q^{\frac{1}{8}}\psi(-q) = \sqrt{\frac{\pi}{\alpha}}\, q_1^{\frac{1}{8}}\psi(-q_1),$$

where $q = e^{-\alpha}$ and $q_1 = e^{-\beta}$ with $\alpha\beta = \pi^2$. Here

$$(a; -q)_n = \prod_{m=0}^{\infty}\left(1 - a(-q)^m\right) = (a; q^2)_\infty(-aq; q^2)_\infty.$$

Observe that (4.1) is the functional equation for the Dedekind η-function (see, for example [3, p. 48]); the other five identities above can easily be deduced from it.

The corresponding laws for the mock θ-function $H(q^r, q)$ are

$$q^{r(1-r)}H(q^r, q) = \sqrt{\frac{\pi}{4\alpha}}\csc(\pi r)\, q_1^{-\frac{1}{4}} K(e^{2\pi i r}, q_1^2) - \sqrt{\frac{\alpha}{\pi}}\int_0^\infty e^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,dx,$$

$$q^{r(1-r)}H(-q^r, q) = -\sqrt{\frac{4\pi}{\alpha}}\sin(\pi r)\, q_1^{-\frac{1}{4}} K_1(e^{2\pi i r}, q_1^2) + \sqrt{\frac{\alpha}{\pi}}\int_0^\infty e^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,dx,$$

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$$q^{r(1-r)}H(q^r,-q) = \sqrt{\frac{\pi}{4\alpha}}\cot\Big(\frac{\pi r}{2}\Big)K_2(e^{\pi i r},-q_1) + \sqrt{\frac{\alpha}{\pi}}\int_0^\infty e^{-\alpha x^2}\frac{\sinh(2r-1)\alpha x}{\sinh \alpha x}dx,$$

$$q^{r(1-r)}H(-q^r,-q) = \sqrt{\frac{\pi}{4\alpha}}\tan\Big(\frac{\pi r}{2}\Big)K_2(-e^{\pi i r},-q_1) - \sqrt{\frac{\alpha}{\pi}}\int_0^\infty e^{-\alpha x^2}\frac{\sinh(2r-1)\alpha x}{\sinh \alpha x}dx.$$

Observe that the first two transformation laws for H involve the same Mordell integral. Using (3.1), (3.4), and transformation laws for the above θ-functions one can show that these laws are equivalent. Since $H(-q^r,-q) = H(q^{1-r},-q)$, the last two transformation laws for H are also equivalent. A complete transformation theory of H is found in [5, Theorem 4.3] proved by the same method of contour integration used to prove (4.2) below. This method extends back to the work of Watson [26].

The analogous transformation laws for $M(q^r,q)$ are [12]

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}M(q^r,q) = \sqrt{\frac{\pi}{2\alpha}}\csc(\pi r)q_1^{-\frac{1}{6}}N(e^{2\pi i r},q_1^4) - \sqrt{\frac{3\alpha}{2\pi}}J(r,\alpha),$$

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}M(-q^r,q) = -\sqrt{\frac{2\pi}{\alpha}}q_1^{\frac{4}{3}}M(e^{2\pi i r}q_1^2,q_1^4) - \sqrt{\frac{3\alpha}{2\pi}}J_1(r,\alpha),$$

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}M(q^r,-q) = \sqrt{\frac{\pi}{4\alpha}}\csc\Big(\frac{\pi r}{2}\Big)q_1^{-\frac{1}{24}}N(e^{\pi i r},-q_1) - \sqrt{\frac{3\alpha}{2\pi}}J_2(r,\alpha),$$

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}M(-q^r,-q) = \sqrt{\frac{\pi}{4\alpha}}\sec\Big(\frac{\pi r}{2}\Big)q_1^{-\frac{1}{24}}N(-e^{\pi i r},-q_1) - \sqrt{\frac{3\alpha}{2\pi}}J_2(1-r,\alpha),$$

where the Mordell integrals J, J_1, J_2 are defined by

$$J(r,\alpha) = \int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-2)\alpha x + \cosh(3r-1)\alpha x}{\cosh\frac{3}{2}\alpha x}dx,$$

$$J_1(r,\alpha) = \int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\sinh(3r-2)\alpha x - \sinh(3r-1)\alpha x}{\sinh\frac{3}{2}\alpha x}dx,$$

$$J_2(r,\alpha) = \int_0^\infty e^{-\frac{3}{2}\alpha x^2}\Big(\cosh\Big(3r-\frac{7}{2}\Big)\alpha x + \cosh\Big(3r-\frac{5}{2}\Big)\alpha x + \cosh\Big(3r-\frac{1}{2}\Big)\alpha x - \cosh\Big(3r+\frac{1}{2}\Big)\alpha x\Big)\Big/\cosh 3\alpha x\,dx.$$

A complete transformation theory of M is found in [4, Theorems 2.1, 2.2].

In [13] we deduce the first transformation laws for H and M from the following transformation law for U_k:

(4.2)

$$
\begin{aligned}
q^{\frac{1}{2}kr(1-r)}U_k(q^r,q) &= \frac{4\pi}{\alpha}\sin(\pi r)\,V_k(e^{2\pi ir},q_1^4) \\
&\quad - \sum_{m=1}^{k-1}\theta_1\Big(\frac{m\pi}{k},q_1^{\frac{2}{k}}\Big)\int_0^\infty e^{-\frac{1}{2}k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac{1}{2}k\alpha x}dx \\
&= \frac{4\pi}{\alpha}\sin(\pi r)\,V_k(e^{2\pi ir},q_1^4) \\
&\quad - \sqrt{\frac{k\alpha}{2\pi}}\sum_{m=1}^{k-1}q^{\frac{(k-2m)^2}{8k}}j(q^m,q^k)\int_0^\infty e^{-\frac{1}{2}k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac{1}{2}k\alpha x}dx,
\end{aligned}
$$

where the Jacobi θ-function θ_1 is defined by

$$\theta_1(z;\tau)=\theta_1(z,q)=2\sum_{n=0}^{\infty}(-1)^n q^{\frac{(2n+1)^2}{4}}\sin(2n+1)z.$$

As usual for Jacobi θ-functions $q=e^{\pi i\tau}$.

We will now prove (4.2) by contour integration and the saddle-point method. By analytic continuation, it suffices to prove the identity for real $\alpha>0$. Put $q=e^{-\alpha}$ and consider the contour integral

$$
\begin{aligned}
I = I_1+I_2 &= \frac{1}{2\pi i}\int_{-\infty-\epsilon i}^{+\infty-\epsilon i}\frac{\pi}{\sin\pi z}\frac{e^{-\frac{1}{|}2}k\alpha z(z+1)}{1-e^{-\alpha(z+r)}}dz \\
&\qquad + \frac{1}{2\pi i}\int_{+\infty+\epsilon i}^{-\infty+\epsilon i}\frac{\pi}{\sin\pi z}\,e^{-\frac{1}{2}k\alpha z(z+1)}1-e^{-\alpha(z+r)}dz,
\end{aligned}
$$

where $\epsilon>0$ is sufficiently small. By Cauchy's residue theorem, I is equal to the sum of the residues of the poles of the integrand inside the contour. Now $\pi/\sin\pi z$ has a simple pole of residue $(-1)^n$ at each integer n and $1/(1-e^{-\alpha(z+r)})$ has a simple pole of residue $1/\alpha$ at $z=-r$. If ϵ is sufficiently small, there are no other poles inside the contour. Hence

$$I=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-q^{n+r}}+\frac{\pi}{\sin(-\pi r)}\frac{q^{-\frac{1}{2}kr(1-r)}}{\alpha}=U_k(q^r,q)+\frac{\pi q^{-\frac{1}{2}kr(1-r)}}{\alpha\sin(-\pi r)}. \tag{4.3}$$

We now consider I_2. In the upper half plane we have

$$\frac{1}{\sin\pi z}=-2i\sum_{n=0}^{\infty}e^{(2n+1)\pi iz},$$

so

$$I_2=\sum_{n=0}^{\infty}\int_{-\infty+\epsilon i}^{+\infty+\epsilon i}\frac{e^{(2n+1)\pi iz-\frac{1}{2}k\alpha z(z+1)}}{1-e^{-\alpha(z+r)}}dz=\sum_{n=0}^{\infty}J_n,$$

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say. The integrand of J_n has poles in the upper half plane at the points z where $1 - e^{-\alpha(z+r)} = 0$, that is, at the points

$$z_m = -r + \frac{2\pi i m}{\alpha}$$

for $m = 1, 2, \ldots$. The residue at z_m (multiplied by $2\pi i$) is

$$\begin{aligned}\mu_{n,m} &= 2\pi i \frac{e^{(2n+1)\pi i z_m - \frac{1}{2}k\alpha z_m(z_m+1)}}{\alpha} \\ &= \frac{2\pi i}{\alpha} e^{-(2n+1)\pi i r} q_1^{(2n+1)2m} q^{-\frac{1}{2}kr(1-r)} e^{-k(1-2r)\pi i m} q_1^{-2km^2},\end{aligned}$$

where $q_1 = e^{-\pi^2/\alpha}$. Next, we symmetrize the denominator of the integrand of J_n by using the identity

$$\frac{1}{1-t} = \frac{t^{-\frac{1}{2}k} + t^{-\frac{1}{2}k+1} + t^{-\frac{1}{2}k+2} + \cdots + t^{\frac{1}{2}k-1}}{t^{-\frac{1}{2}k} - t^{\frac{1}{2}k}}.$$

Applying this with $t = e^{-\alpha(z+r)}$, we find that the integrand of J_n is

$$\frac{e^{\frac{1}{2}k\alpha(z+r)} + e^{(\frac{1}{2}k-1)\alpha(z+r)} + \cdots + e^{(-\frac{1}{2}k+1)\alpha(z+r)}}{e^{\frac{1}{2}k\alpha(z+r)} - e^{-\frac{1}{2}k\alpha(z+r)}} \, e^{-\frac{1}{2}k\alpha z} e^{(2n+1)\pi i z - \frac{1}{2}k\alpha z^2}.$$

To find the saddle point, we set the derivative of the last factor equal to 0, getting $(2n+1)\pi i - k\alpha z = 0$ or

$$z = \frac{(2n+1)\pi i}{k\alpha} = w_n,$$

say. We move the upper contour of J_n up to the horizontal line through w_n, getting J_n'. By the residue theorem,

$$J_n = J_n' + \text{ sum of residues of poles of integrand between the two contours.}$$

These poles are the points $z_m = -r + \frac{2\pi i m}{\alpha}$ for which $0 < 2m < \frac{2n+1}{k}$, or equivalently, $0 < m \leq \frac{n}{k}$. Hence,

$$J_n = J_n' + \sum_{0<m\leq \frac{n}{k}} \mu_{n,m}.$$

Summing over n, we obtain

$$I_2 = \sum_{n=0}^{\infty} J_n' + \sum_{m=1}^{\infty} \sum_{n=km}^{\infty} \mu_{n,m}.$$

Now

$$\mu_{n+1,m} = e^{2\pi i z_m} \mu_{n,m} = e^{-2\pi i r} q_1^{4m} \mu_{n,m}.$$

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Hence,

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=km}^{\infty}\mu_{n,m} &= \sum_{m=1}^{\infty}\frac{\mu_{km,m}}{1-e^{-2\pi ir}q_1^{4m}}\\ &= \sum_{m=1}^{\infty}\frac{2\pi i}{\alpha}\frac{e^{-(2km+1)\pi ir}q_1^{(2km+1)2m}q^{-\frac{1}{2}kr(1-r)}e^{-k(1-2r)\pi im}q_1^{-2km^2}}{1-e^{-2\pi ir}q_1^{4m}}\\ &= \frac{2\pi i}{\alpha}q^{-\frac{1}{2}kr(1-r)}e^{-\pi ir}\sum_{m=1}^{\infty}\frac{(-1)^{km}q_1^{2km^2+2m}}{1-e^{-2\pi ir}q_1^{4m}},\end{aligned}$$

so

$$I_2 = \frac{2\pi i}{\alpha}q^{-\frac{1}{2}kr(1-r)}e^{-\pi ir}\sum_{m=1}^{\infty}\frac{(-1)^{km}q_1^{2km^2+2m}}{1-e^{-2\pi ir}q_1^{4m}} + \sum_{n=0}^{\infty}J_n'. \tag{4.4}$$

Before going on to evaluate the integral J_n', we remark that the integral I_1 over the lower contour can be handled similarly. This time the expansion

$$\frac{1}{\sin\pi z} = 2i\sum_{n=0}^{\infty}e^{-(2n+1)\pi iz}$$

is employed. Note that this is just the complex conjugate of the expansion used in the upper half plane. Thus $I_1 = \sum_{n=0}^{\infty}K_n$, where

$$K_n = \int_{-\infty-\epsilon i}^{+\infty-\epsilon i}\frac{e^{-(2n+1)\pi iz-\frac{1}{2}k\alpha z(z+1)}}{1-e^{-\alpha(z+r)}}dz.$$

The lower contour is moved down to the horizontal line through $\overline{w}_n$, giving

$$K_n = \overline{J}_n' + \sum_{0<m\le\frac{n}{3}}\overline{\mu}_{n,m}.$$

The sum here is just the complex conjugate of the one evaluated above, so from (4.4) it follows that

$$I_1 = -\frac{2\pi i}{\alpha}q^{-\frac{1}{2}kr(1-r)}e^{\pi ir}\sum_{m=1}^{\infty}\frac{(-1)^{km}q_1^{2km^2+2m}}{1-e^{2\pi ir}q_1^{4m}} + \sum_{n=0}^{\infty}\overline{J}_n'. \tag{4.5}$$

Adding (4.4) and (4.5), we obtain

$$\begin{aligned}I &= I_1 + I_2\\ &= \frac{2\pi i}{\alpha}q^{-\frac{1}{2}kr(1-r)}\sum_{m=1}^{\infty}(-1)^{km}q_1^{2km^2+2m}\left[\frac{e^{-\pi ir}}{1-e^{-2\pi ir}q_1^{4m}} - \frac{e^{\pi ir}}{1-e^{2\pi ir}q_1^{4m}}\right]\\ &\quad + \sum_{n=0}^{\infty}(J_n'+\overline{J}_n')\\ &= \frac{4\pi}{\alpha}q^{-\frac{1}{2}kr(1-r)}\sin(\pi r)\sum_{m=1}^{\infty}\frac{(-1)^{km}q_1^{2km^2+2m}(1+q_1^{4m})}{(1-e^{2\pi ir}q_1^{4m})(1-e^{-2\pi ir}q_1^{4m})} + \sum_{n=0}^{\infty}(J_n'+\overline{J}_n').\end{aligned}$$

It now follows from equation (4.3) that

(4.6)

$$U_k(q^r,q) = I - \frac{\pi}{\sin(-\pi r)}\frac{q^{-\frac12 kr(1-r)}}{\alpha}$$
$$= \frac{4\pi\sin(\pi r)q^{-\frac12 kr(1-r)}}{\alpha}\left[\frac{1}{4\sin^2(\pi r)} + \sum_{m=1}^{\infty}\frac{(-1)^{km}q_1^{2km^2+2m}(1+q_1^{4m})}{(1-e^{2\pi ir}q_1^{4m})(1-e^{-2\pi ir}q_1^{4m})}\right]$$
$$+ \sum_{n=0}^{\infty}(J_n' + \overline{J}_n')$$
$$= \frac{4\pi\sin(\pi r)q^{-\frac12 kr(1-r)}}{\alpha}V_k(e^{2\pi ir},q_1^4) + \sum_{n=0}^{\infty}(J_n' + \overline{J}_n').$$

We now evaluate $\sum_{n=0}^{\infty}(J_n' + \overline{J}_n')$. In the integral J_n' put $z = -r + p + x$, where $p = \frac{(2n+1)\pi i}{k\alpha}$ and x is a real variable running from $-\infty$ to ∞. This gives

$$J_n' = q^{-\frac12 kr}\int_{-\infty}^{\infty} ABC\,dx\,,$$

where

$$A = e^{(2n+1)\pi i(-r+p+x)},$$
$$B = \frac{1 + e^{-\alpha(p+x)} + e^{-2\alpha(p+x)} + \cdots + e^{(-k+1)\alpha(p+x)}}{e^{\frac12 k\alpha(p+x)} - e^{-\frac12 k\alpha(p+x)}},$$
$$C = e^{-\frac12 k\alpha(-r+p+x)^2}.$$

Simplifying, we obtain

$$J_n' = q^{-\frac12 kr(1-r)}q_1^{\frac{(2n+1)^2}{2k}}\int_{-\infty}^{\infty}\frac{e^{k\alpha rx-\frac12 k\alpha x^2}}{2i(-1)^n\cosh\frac12 k\alpha x}\sum_{m=1}^{k-1}e^{-\frac{m(2n+1)\pi i}{k}}e^{-m\alpha x}\,dx$$
$$+ q^{-\frac12 kr(1-r)}q_1^{\frac{(2n+1)^2}{2k}}\int_{-\infty}^{\infty}\frac{e^{k\alpha rx-\frac12 k\alpha x^2}}{2i(-1)^n\cosh\frac12 k\alpha x}dx$$
$$= P_n + Q_n,$$

say. Since Q_n is purely imaginary, we have $J_n' + \overline{J}_n' = P_n + \overline{P}_n$. Hence,

$$J_n' + \overline{J}_n' = -q^{-\frac12 kr(1-r)}\sum_{m=1}^{k-1}(-1)^n q_1^{\frac{(2n+1)^2}{2k}}\sin\left(\frac{m(2n+1)\pi}{k}\right)\int_{-\infty}^{\infty}\frac{e^{(kr-m)\alpha x-\frac12 k\alpha x^2}}{\cosh\frac12 k\alpha x}\,dx$$
$$= -2q^{-\frac12 kr(1-r)}\sum_{m=1}^{k-1}(-1)^n q_1^{\frac{(2n+1)^2}{2k}}\sin\left(\frac{m(2n+1)\pi}{k}\right)$$
$$\cdot\int_0^{\infty}e^{-\frac12 k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac12 k\alpha x}\,dx$$

and so

(4.7)
$$\sum_{n=0}^{\infty}(J_n' + \overline{J}_n') = -q^{-\frac{1}{2}kr(1-r)}\sum_{m=1}^{k-1}\sum_{n=0}^{\infty}2(-1)^n q_1^{\frac{(2n+1)^2}{2k}}\sin\Big(\frac{m(2n+1)\pi}{k}\Big) \cdot \int_0^{\infty} e^{-\frac{1}{2}k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac{1}{2}k\alpha x}dx$$
$$= -q^{-\frac{1}{2}kr(1-r)}\sum_{m=1}^{k-1}\theta_1\Big(\frac{m\pi}{k}, q_1^{\frac{2}{k}}\Big)\int_0^{\infty} e^{-\frac{1}{2}k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac{1}{2}k\alpha x}dx,$$

where the Jacobi θ-function θ_1 is defined by
$$\theta_1(z;\tau) = \theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{\frac{(2n+1)^2}{4}}\sin(2n+1)z = ie^{-iz}q^{\frac{1}{4}}j(e^{2iz}, q^2)$$
and satisfies the transformation law
$$-i\sqrt{-i\tau}\exp\Big(\frac{iz^2}{\pi\tau}\Big)\theta_1(z;\tau) = \theta_1\Big(\frac{z}{\tau}; -\frac{1}{\tau}\Big).$$
Hence,
$$-i\sqrt{-i\tau}\exp\Big(\frac{iz^2}{\pi\tau}\Big)\theta_1(z,q) = \theta_1\Big(\frac{z}{\tau}, q_1\Big).$$
Replacing q by $q^{\frac{k}{2}}$ (so $\tau \to \frac{k}{2}\tau$, $q_1 \to q_1^{\frac{2}{k}}$), we obtain
$$-i\sqrt{\frac{-ik\tau}{2}}\exp\Big(\frac{2iz^2}{k\pi\tau}\Big)\theta_1(z, q^{\frac{k}{2}}) = \theta_1\Big(\frac{2z}{k\tau}, q_1^{\frac{2}{k}}\Big).$$
When $z = m\pi\tau/2$, this becomes

(4.8)
$$\theta_1\Big(\frac{m\pi}{k}, q_1^{\frac{2}{k}}\Big) = -i\sqrt{\frac{-ik\tau}{2}}\exp\Big(\frac{m^2\pi i\tau}{2k}\Big)\theta_1\Big(\frac{m\pi\tau}{2}, q^{\frac{k}{2}}\Big)$$
$$= -i\sqrt{\frac{-ik\tau}{2}}\,q^{\frac{m^2}{2k}}\theta_1\Big(\frac{m\pi\tau}{2}, q^{\frac{k}{2}}\Big)$$
$$= -i\sqrt{\frac{k\alpha}{2\pi}}q^{\frac{m^2}{2k}}\theta_1\Big(\frac{m\pi\tau}{2}, q^{\frac{k}{2}}\Big) = \sqrt{\frac{k\alpha}{2\pi}}q^{\frac{(k-2m)^2}{8k}}j(q^m, q^k).$$
Substituting (4.8) into (4.7) gives

(4.9)
$$\sum_{n=0}^{\infty}(J_n' + \overline{J}_n') = -q^{-\frac{1}{2}kr(1-r)}\sum_{m=1}^{k-1}\theta_1\Big(\frac{m\pi}{k}, q_1^{\frac{2}{k}}\Big)\int_0^{\infty} e^{-\frac{1}{2}k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac{1}{2}k\alpha x}dx$$
$$= -q^{-\frac{1}{2}kr(1-r)}\sqrt{\frac{k\alpha}{2\pi}}\sum_{m=1}^{k-1}q^{\frac{(k-2m)^2}{8k}}j(q^m, q^k) \cdot \int_0^{\infty} e^{-\frac{1}{2}k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac{1}{2}k\alpha x}dx.$$

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Finally, by (4.6) and (4.9) we get the transformation law

$$
\begin{aligned}
q^{\frac{1}{2}kr(1-r)} U_k(q^r, q) &= \frac{4\pi}{\alpha} \sin(\pi r)\, V_k(e^{2\pi i r}, q_1^4) \\
&\qquad - \sum_{m=1}^{k-1} \theta_1\Big(\frac{m\pi}{k}, q_1^{\frac{2}{k}}\Big) \int_0^\infty e^{-\frac{1}{2}k\alpha x^2} \frac{\cosh(kr-m)\alpha x}{\cosh \frac{1}{2}k\alpha x} dx \\
&= \frac{4\pi}{\alpha} \sin(\pi r)\, V_k(e^{2\pi i r}, q_1^4) \\
&\qquad - \sqrt{\frac{k\alpha}{2\pi}} \sum_{m=1}^{k-1} q^{\frac{(k-2m)^2}{8k}} j(q^m, q^k) \int_0^\infty e^{-\frac{1}{2}k\alpha x^2} \frac{\cosh(kr-m)\alpha x}{\cosh \frac{1}{2}k\alpha x} dx,
\end{aligned}
$$

which completes the proof of (4.2). ■

5 Mock Theta Conjectures for Functions of Even Order

Hickerson [15,16] proved that Ramanujan's fifth and seventh order mock θ-functions are related to the function M. The third order mock θ-function $\omega(q)$ is $M(q, q^2)$, and the third order mock θ-function $\psi(q)$ is equal to $qM(q, q^4)$. Ramanujan gave relations, later proved by Watson [26], between $\omega(q)$ or $\psi(q)$ and some of the other third order mock θ-functions. A complete list of relations between all of the third order mock θ-functions and the function M is given in [13].

It turns out that the mock θ-functions of even order are related to the function H. Lists of all of these relations (referred to as mock theta "conjectures" even after their proofs are known) are found in [13]. We will discuss some of these relations.

The second order mock θ-function $B(q)$ is $H(q, q^2)$ (see [20]). The function $V_1(q)$ in [11, 20] is equal to $qH(q, q^4)$.

The sixth order mock θ-functions $\phi(q)$ and $\psi(q)$ defined by

$$
\phi(q) = \sum_{n=0}^\infty \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q; q)_{2n}}, \quad \psi(q) = \sum_{n=0}^\infty \frac{(-1)^n q^{(n+1)^2} (q; q^2)_n}{(-q; q)_{2n+1}}
$$

are related to H by

$$
\begin{aligned}
\phi(q^4) &= \frac{(q^2; q^2)_\infty^3 (q^3; q^3)_\infty^2 (q^{12}; q^{12})_\infty^3}{(q)_\infty^2 (q^6; q^6)_\infty^3 (q^8; q^8)_\infty (q^{24}; q^{24})_\infty} - 2qH(q, q^6), \\
\psi(q^4) &= \frac{q^3 (q^2; q^2)_\infty^2 (q^4; q^4)_\infty (q^{24}; q^{24})_\infty^2}{(q)_\infty (q^3; q^3)_\infty (q^8; q^8)_\infty^2} - q^3 H(q^3, q^6).
\end{aligned}
$$

These identities were discovered using transformation laws and computer algebra. Some proofs for these and similar identities for the eighth and tenth order functions below will appear in a forthcoming paper. Relations between other sixth order mock θ-functions and $\phi(q)$ or $\psi(q)$ are found in [2].

Gordon and the author [11] discovered the eighth order mock θ-functions by applying the half-shift method to the θ-functions appearing in the Göllnitz–Gordon identities [9],[10], [24, eq. (36), (34)]. Two of the eighth order mock θ-functions are

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}, \quad S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}.$$

They are related to H by

$$S_0(-q^2) = \frac{j(-q,q^2)\,j(q^6,q^{16})}{j(q^2;q^8)} - 2qH(q,q^8),$$
$$S_1(-q^2) = \frac{j(-q,q^2)\,j(q^2,q^{16})}{j(q^2,q^8)} - 2qH(q^3,q^8).$$

On page 9 of the lost notebook Ramanujan defined four functions that came to be known as the tenth order mock θ-functions. These functions are

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(q;q^2)_{n+1}}, \qquad \psi(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n+1)(n+2)}}{(q;q^2)_{n+1}},$$
$$X(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, \qquad \chi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}.$$

The mock theta "conjectures" of order 10 are

$$\phi(q) = \frac{(q^{10};q^{10})_\infty^2\, j(-q^2,q^5)}{(q^5;q^5)_\infty\, j(q^2,q^{10})} + 2qH(q^2,q^5),$$
$$\psi(q) = -\frac{q(q^{10};q^{10})_\infty^2\, j(-q,q^5)}{(q^5;q^5)_\infty\, j(q^4,q^{10})} + 2qH(q,q^5),$$
$$X(-q^2) = \frac{(q^4;q^4)_\infty^2\big(j(-q^2,q^{20})^2 j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3\big)}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty\, j(q^8,q^{40})} - 2qH(q,q^{20}) + 2q^5H(q^9,q^{20}),$$
$$\chi(-q^2) = \frac{q^2(q^4;q^4)_\infty^2\big(2q(q^{40};q^{40})_\infty^3 - j(-q^6,q^{20})^2 j(q^4,q^{40})\big)}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty\, j(q^{16},q^{40})} - 2q^3H(q^3,q^{20}) - 2q^5H(q^7,q^{20}).$$

The first two were stated and proved by Choi [6, pp. 533–534], and the last two were discovered by the author [13] by matching the Mordell integrals in their transformation laws (obtained using computer algebra) with the Mordell integrals in the transformation laws for $H(q^r,q)$. A rigorous proof has yet to be worked out.

6 Concluding Remarks

Unlike that for θ-functions, transformation laws for mock θ-functions are not unique. For example,

$$q^{\frac{3}{2}r(1-r)}U_3(q^r,q) = \frac{4\pi}{\alpha}\sin(\pi r)\,V_3(e^{2\pi ir},q_1^4) + \text{ an integral} \tag{6.1}$$

$$= \frac{-2\pi i}{\alpha}\,e^{3\pi ir}U_3(e^{2\pi ir},q_1^4) + \text{ some integrals.} \tag{6.2}$$

Equation (6.1) is (4.2) with $k = 3$. To prove (6.2) we first introduce the Appell function.

The Appell function of level l (not to be confused with the level of a modular form) is defined by (see, for example [30])

$$A_l(u,v;\tau) = e^{l\pi iu}\sum_{n=-\infty}^{\infty}\frac{(-1)^{ln}e^{l\pi i(n^2+n)\tau+2\pi inv}}{-e^{2\pi in\tau+2\pi iu}},\ \tau\in\mathcal{H},\ v\in\mathbf{C},\ u\in\mathbf{C}\setminus(\mathbf{Z}\tau+\mathbf{Z}),$$

and satisfies the transformation law [30]

$$A_l(u,v;\tau) = \frac{e^{\pi i(lu-2v)u/\tau}}{\tau}A_l(u/\tau,v/\tau;-1/\tau) + \text{ some integrals.}$$

When $l = 3$ and $v = 0$, this law becomes

$$e^{3\pi iu}U_3(e^{2\pi iu},e^{2\pi i\tau}) = \frac{e^{3\pi iu^2/\tau}}{\tau}e^{3\pi iu/\tau}U_3(e^{2\pi iu/\tau},e^{-2\pi i/\tau}) + \text{ some integrals.} \tag{6.3}$$

Recall that $q = e^{-\alpha} = e^{\pi i\tau}$ and $q_1 = e^{-\pi^2/\alpha}$. So $\alpha = -\pi i\tau$ and $q_1 = e^{-\pi i/\tau}$. If we put $u = r\tau$, then (6.3) simplifies to

$$q^{3r}U_3(q^{2r},q^2) = \frac{-\pi i}{\alpha}q^{3r^2}e^{3\pi ir}U_3(e^{2\pi ir},q_1^2) + \text{ some integrals.}$$

Replacing q by $q^{\frac{1}{2}}$ (hence $\alpha\to\frac{1}{2}\alpha$, $q_1\to q_1^2$), this becomes (6.2).

Comparing (6.1) and (6.2), we cannot conclude that

$$\frac{4\pi}{\alpha}\sin(\pi r)\,V_3(e^{2\pi ir},q_1^4) = \frac{-2\pi i}{\alpha}\,e^{3\pi ir}U_3(e^{2\pi ir},q_1^4),$$

which is equivalent to

$$2i\sin(\pi r)\,V_3(e^{2\pi ir},q_1^4) = e^{3\pi ir}U_3(e^{2\pi ir},q_1^4)\,.$$

By (3.5) (with $x\to e^{2\pi ir}$, $q\to q_1^4$, then divide by $e^{\pi ir}$) we obtain

$$2i\sin(\pi r)\,V_3(e^{2\pi ir},q_1^4) - e^{3\pi ir}U_3(e^{2\pi ir},q_1^4) = e^{\pi ir}(q_1^4;q_1^4)_\infty.$$

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Hence $2i\sin(\pi r)\,V_3(e^{2\pi ir}, q_1^4)$ and $e^{3\pi ir}U_3(e^{2\pi ir}, q_1^4)$ differ by a θ-function.

In general, the mock θ-functions on the right-hand sides of transformation laws similar to (6.1) and (6.2) (with the same left side) differ by a θ-function, because they have the same shadow [28].

The presence of a nonzero Mordell integral in a transformation formula for a function $f(q)$ does not always indicate that $f(q)$ is a mock θ-function. We provide an example using Zwegers' μ-function [29] (this function is a normalized level 1 Appell function):

$$\mu(a,b,q) = \mu(u,v;\tau) = ib^{\frac{1}{2}}q^{-\frac{1}{8}}A_1(u,v;\tau)/j(b,q),$$

where $a = e^{2\pi iu}$, $b = e^{2\pi iv}$, and $q = e^{2\pi i\tau}$.

If we put $u = 1/2 + \tau/2$ and $v = 1/2$, then

$$\mu(u,v;\tau) = \mu(-q^{\frac{1}{2}}, -1, q) = q^{\frac{1/8}{2i}}, \quad \mu\Big(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\Big) = \mu(-q_1^{-\frac{1}{2}}, q_1^{-\frac{1}{2}}, q_1) = 0,$$

where $q_1 = e^{-2\pi i/\tau}$. The transformation law for $f(q) = \mu(-q^{\frac{1}{2}}, -1, q)$ becomes

$$q^{1/8} = e^{-\alpha/4} = 2\int_0^\infty e^{-\alpha x^2}\frac{\cos\alpha x}{\cosh\pi x}dx,$$

where $\alpha = -\pi i\tau$.

Equation (2.7) expresses the function $H(x,q)$ as a normalized level 2 Appell function. In particular,

$$H(x,q) = \frac{U_2(x,q)}{j(q,q^2)} = \frac{\tilde{A}_2(x,-1,q)}{j(q,q^2)},$$

where

$$\tilde{A}_l(a,b,q) = \sum_{n=-\infty}^{\infty}\frac{(-1)^{ln}q^{\frac{1}{2}ln(n+1)}b^n}{1-aq^n}.$$

By the identity

$$\frac{1}{1-x} = \frac{1+x+x^2+\cdots+x^{l-1}}{1-x^l},$$

where $x = aq^n$, it is not difficult to show that

$$\tilde{A}_l(a,b,q) = \sum_{m=0}^{l-1}a^m\tilde{A}_1(a^l, (-1)^{l-1}bq^m, q^l). \tag{6.4}$$

Kang [17] used this to prove that

$$iaH(a,q) = \frac{\eta^4(2\tau)}{\eta^2(\tau)\vartheta(2u;2\tau)} + aq^{-\frac{1}{4}}\mu(2u,\tau;2\tau) \tag{6.5}$$

and

$$ia^{\frac{3}{2}}q^{-\frac{1}{24}}M(a,q) = \frac{\eta^3(3\tau)}{\eta(\tau)\vartheta(3u;3\tau)} + aq^{-\frac{1}{6}}\mu(3u,\tau;3\tau) + a^2q^{-\frac{2}{3}}\mu(3u,2\tau;3\tau),$$

where $a = e^{2\pi i u}$, $q = e^{2\pi i \tau}$, and $\eta(\tau) = q^{\frac{1}{24}}(q)_\infty$ is the Dedekind η-function.

The transformation laws for H and μ can be combined to eliminate the Mordell integrals. This resulting transformation law is

$$q^{-\frac{1}{2}r^2}\left(q^{\frac{1}{8}}H(a^{\frac{1}{2}}b^{-\frac{1}{2}}q^{\frac{1}{4}}, q^{\frac{1}{2}}) + i\mu(u, v; \tau)\right) = \frac{1}{\sqrt{-i\tau}}\left(\frac{1}{2}\sec(\pi r)\, q_1^{-\frac{1}{8}}K(-q_1^{v-u}, q_1) - i\mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right)\right),$$

where $q = e^{2\pi i\tau}$, $q_1 = e^{-2\pi i/\tau}$, $a = e^{2\pi i u}$, $b = e^{2\pi i v}$, and $r = (u - v)/\tau$. Hence,

$$i\mu(u, v; \tau) + q^{\frac{1}{8}}H(a^{\frac{1}{2}}b^{-\frac{1}{2}}q^{\frac{1}{4}}, q^{\frac{1}{2}}) \tag{6.6}$$

and

$$i\mu(u, v; \tau) - \frac{a^{\frac{1}{2}}b^{-\frac{1}{2}}q^{-\frac{1}{8}}K(-a/b, q)}{1 + a/b} \tag{6.7}$$

are Jacobi forms; they behave like θ-functions. A proof that (6.6) vanishes when $u + v = \tau/2$ and (6.7) vanishes when $u + v = 1/2$ is given in [14]. Therefore

$$H(a, q) = -iq^{-\frac{1}{4}}\mu(u, \tau - u; 2\tau) \tag{6.8}$$

and

$$K(a, q) = (a^{\frac{1}{2}} - a^{-\frac{1}{2}})q^{\frac{1}{8}}\mu\left(\frac{u}{2}, \frac{1-u}{2}; \tau\right) = 2i\sin(\pi u)q^{\frac{1}{8}}\mu\left(\frac{u}{2}, \frac{1-u}{2}; \tau\right),$$

or equivalently,

$$H(x, q) = -iq^{-\frac{1}{4}}\mu(x, q/x, q^2) = \frac{\tilde{A}_1(x, q/x, q^2)}{j(q/x, q^2)}$$

and

$$\frac{K(y, q)}{1 - y} = -y^{-\frac{1}{2}}q^{\frac{1}{8}}\mu(y^{\frac{1}{2}}, -y^{-\frac{1}{2}}, q) = \frac{\tilde{A}_1(y^{\frac{1}{2}}, -y^{-\frac{1}{2}}, q)}{y^{\frac{1}{2}}j(-y^{-\frac{1}{2}}, q)}.$$

Observe that (6.8) removes the θ-quotient from (6.5). This has a nice extension to higher level Appell functions. It follows from (6.4) and (2.5) that

$$\begin{aligned}\tilde{A}_l(a, (-1)^{l-1}, q) &= \frac{(q^l; q^l)_\infty^3}{j(a^l, q^l)} + \sum_{m=1}^{l-1} a^m \tilde{A}_1(a^l, q^m, q^l) \\ &= \frac{(q^l; q^l)_\infty^3}{j(a^l, q^l)} - i\sum_{m=1}^{l-1} a^{m-\frac{l}{2}} q^{\frac{l}{8}-\frac{m}{2}} j(q^m, q^l)\mu(lu, m\tau; l\tau).\end{aligned} \tag{6.9}$$

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The θ-quotient in (6.9) is removed by the conjectured identity

$$\tilde{A}_l(a,(-1)^{l-1},q) = \sum_{m=1}^{l-1} \frac{j(q^m,q^l)}{j(q^m/a,q^l)} a^{m-1} \tilde{A}_l(a^{l-1},q^m/a,q^l)$$
$$= -i \sum_{m=1}^{l-1} a^{m-\frac{l}{2}} q^{\frac{l}{8}-\frac{m}{2}} j(q^m,q^l)\mu(lu-u,m\tau-\tau;l\tau)$$

for $l \geq 2$.

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