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ENTIRE FUNCTIONS HAVING ASYMPTOTIC FUNCTIONS

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It is shown that an entire function having k distinct entire asymptotic functions of order less than $\frac{1}{4}$ is of lower order $\frac{1}{2}k$, mean type at least; further that if f is of lower order $\frac{1}{2}k$, mean type, then its order is $\frac{1}{2}k$.

1. Introduction

An entire function w is called an asymptotic function for the entire function f if $f(z) - w(z) \to 0$ as $z \to \infty$ along some curve joining 0 to ∞, called an asymptotic curve. A consequence of Wiman's theorem is that distinct asymptotic functions of order less than $\frac{1}{2}$ associated with the same function f cannot have the same asymptotic curve, and this together with the Denjoy-Carleman-Ahlfors theorem lends support to the following conjecture, which appears as Problem 2.3 in Hayman's book of problems [4].

If f is an entire function having k distinct asymptotic functions of order less than $\frac{1}{2}$ then f has order $\frac{1}{2}k$ at least.

When the asymptotic functions are constants this is a weak version of the Denjoy-Carleman-Ahlfors theorem. In a posthumous paper [7] Somorjai showed that if the orders of the asymptotic functions are less than about 1/30 then the stronger conclusion

$$\alpha = \liminf_{r\to\infty} r^{-k/2} \log M(r, f) > 0 \tag{1.1}$$

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holds, where $M(r, f)$ is the maximum modulus of f. The proof employs a differential inequality due to Carleman. Denjoy himself had shown earlier that if f is of finite order μ and all asymptotic curves are half-lines then the number of asymptotic functions of order less than $1/(2+\mu^{-1})$ is no more than 2μ; see [3] for details. In this note we show that the proof of the "regularity" version of the Denjoy-Carleman-Ahlfors theorem due to Heins [5] can be adapted to bear on the foregoing conjecture. We shall prove:

(1.2) *If f is an entire function having k distinct asymptotic functions of order less than $\frac{1}{4}$ then* (1.1) *holds. Moreover, if $\alpha < \infty$, then f has order $\frac{1}{2}k$.*

2. A lemma

It is apparent that we may assume that the asymptotic curves are not self-intersecting and also that they are composed of straight line segments, with only finitely many segments between any two of their points. Let us call such curves segmental.

Suppose that Γ_1 and Γ_2 are two segmental curves joining 0 to ∞ which are either identical or else intersect in the finite plane only at 0. In either case the complement of $\Gamma_1 \cup \Gamma_2$ contains an unbounded, simply-connected domain Ω with $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Given $R > 0$ let $\Omega(R)$ be the component of $\Omega \cap \Delta(R)$ which has 0 as a boundary point, and for each $r \in (0, R)$ let $\theta(r)$ be the angular measure of $\Omega(R) \cap C(r)$; here $\Delta(r)$ is the open disc centred at 0 with radius r, and $C(r)$ is its boundary. Suppose that $u(z)$ is a subharmonic function in $\Omega(R)$ which satisfies, for each $z \in \partial\Omega(R) \cap (\Gamma_1 \cup \Gamma_2)$,

$$\limsup_{\zeta \to z} u(\zeta) \le \phi(|z|), \tag{2.1}$$

where ϕ is increasing and continuously differentiable on $(0, R]$. The upper limit in (2.1) is to be taken as ζ approaches z from within $\Omega(R)$. We shall show that, for each $r \in (0, R)$,

$$(2.2) \quad \sigma(r,\ R) \le \phi(r) + 6\int_r^R \phi'(t)\exp\left[-\pi\int_r^t ds/s\theta(s)\right]dt + 6\sigma(R,\ R)\exp\left\{-\pi\int_r^R ds/s\theta(s)\right\} ,$$

where

$$\sigma(t,\ R) = \sup\{u(z)\ :\ z \in \overline{\Omega}(R) \cap C(t)\} .$$

Given $r \in (0,\ R)$ let $z_0 \in S_r = \Omega(R) \cap C(r)$. For $t \in (r,\ R)$ let $C_1(t)$ be an arc of S_t which is not separated from z_0 by any other arc of S_t . $C_1(t)$ cuts $\Omega(R)$ into two subdomains, $\Omega_1(t)$ containing z_0 , and $\Omega_1'(t)$. If there is an arc of S_t remaining in $\Omega_1(t)$ let $C_2(t)$ be one which is not separated from z_0 by any of the others, let $\Omega_2(t)$ be the subdomain of $\Omega_1(t)$ cut off by $C_2(t)$ which contains z_0 and let $\Omega_2'(t)$ be the other. And so on. After a finite number $(m = m(t)$ say$)$ of steps we arrive at a domain $\Omega(t) = \Omega_m(t) \subseteq \Delta(t)$, m domains $\Omega_1'(t),\ \ldots,\ \Omega_m'(t)$ and m arcs $C_1(t),\ \ldots,\ C_m(t)$. These domains and arcs are mutually disjoint and their union is $\Omega(R)$. Since $\Omega(t)$ is the unique component of $\Omega(R) \cap \Delta(t)$ which contains z_0 we have $\Omega(t) \subseteq \Omega(s)$ for all $r < t \le s < R$; thus if $\partial_i(t) = \{|z| > t\} \cap \partial\Omega_i'(t)$, $i = 1,\ \ldots,\ m$, and $\partial(t) = \cup\partial_i(t)$ then $\partial(t) \supseteq \partial(s)$ for $r < t \le s < R$.

It is clear that

$$\omega\big(z_0,\ \partial_i(t),\ \Omega(R)\big) \le \omega\big(z_0,\ C_i(t),\ \Omega(t)\big)$$

for $1 \le i \le m(t)$, where ω represents harmonic measure, and so

$$(2.3) \qquad \omega\big(z_0,\ \partial(t),\ \Omega(R)\big) \le 6\ \exp\left\{-\pi\int_r^t ds/s\theta(s)\right\}$$

from (for example) Kennedy [6, Lemma 4]. Given $r' \in (r,\ R)$ and a positive integer N let $t_j = r' + j(R-r')/N$, $0 \le j \le N$, and consider

$$h_N(z) = \phi(r') + \sum_{j=1}^{N} \{\phi(t_j)-\phi(t_{j-1})\}\omega(z, \partial(t_{j-1}), \Omega(R))$$
$$+ \sigma(R, R)\omega(z, \partial\Omega(R) \cap C(R), \Omega(R)) .$$

h_N is harmonic in $\Omega(R)$. Moreover if $z \in \partial\Omega(R)$ then $h_N(z) \geq \limsup_{\zeta\to z} u(\zeta)$. This is clear from the hypotheses of (1.2) if $|z| \leq r'$ or $|z| = R$, while otherwise $z \in \partial(t_{i-1}) - \partial(t_i)$ for exactly one i and therefore

$$h_N(z) \geq \phi(r') + \sum_{j=1}^{i} (\phi(t_j)-\phi(t_{j-1})) = \phi(t_i) \geq \phi(|z|) .$$

Allowing $N \to \infty$, $r' \to r$, after taking account of (2.3) we deduce (2.2).

3. Proof of the result

The first concern is to show that any two of the asymptotic curves are ultimately non-intersecting. Suppose this is not the case, that γ_1 and γ_2 are two asymptotic curves with points of intersection $z_n \to \infty$. γ_1 and γ_2 determine a sequence of bounded domains D_n , the union of the boundaries of which is $\gamma_1 \cup \gamma_2$. Let w_1 and w_2 be the asymptotic functions associated with γ_1 and γ_2 and let $W(z) = w_1(z) - w_2(z)$. Since γ_1 and γ_2 intersect at $z_n \to \infty$, W is not constant. Let

$$v(z) = \log\left|\left[f(z)-\tfrac{1}{2}(w_1(z)+w_2(z))\right]^2-\tfrac{1}{4}W(z)^2\right| ,$$

so that $v(z)$ is subharmonic. Under the hypotheses of (1.2) there is a constant $K_1 > 0$ such that $v(z) \leq \log^+|W(z)| + K_1$ for all z on γ_1 and γ_2 . We apply (2.2) with $\Gamma_1 = \Gamma_2 = \gamma_1$, $u(z) = u_R(z)$ - the harmonic function in $\Omega(R)$ with boundary values $\log^+|W(z)|$ - and $\phi(t) = t^{\rho+\varepsilon} + K_2$, where $\rho < \tfrac{1}{4}$ is the order of W , and ε and K_2 are positive with $\rho + \varepsilon < \tfrac{1}{2}$. This gives

$$u_R(z) \leq 6\sqrt{|z|/R}\, \log^+ M(R, W) + \frac{1+10(\rho+\varepsilon)}{1-2(\rho+\varepsilon)} |z|^{\rho+\varepsilon} ,$$

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using $\theta(s) \le 2\pi$. Since, for each fixed n , $D_n \subseteq \Omega(R)$ for all large R we deduce that

$$v(z) \le \frac{1+10(\rho+\varepsilon)}{1-2(\rho+\varepsilon)} |z|^{\rho+\varepsilon} + O(1)$$

for all z in UD_n . It follows that the subharmonic function

$$V(z) = \begin{cases} \log^+|W(z)| + K_1 & , \quad z \notin UD_n , \\ \max\{\log^+|W(z)|+K_1, v(z)\} & , \quad z \in UD_n , \end{cases}$$

has order ρ at most. Now Denjoy [2] has shown that if u is subharmonic of order $\mu \in [0, 1)$ and

$$A(r, u) = \inf\{u(z) : |z| = r\} , \quad B(r, u) = \max\{u(z) : |z| = r\} ,$$

then, given $\mu' \in (\mu, 1)$,

$$\int_x^\infty \{A(r, u) - \cos \pi\mu' B(r, u)\} r^{-\mu'-1} dr > K(\mu') x^{-\mu'} B(x, u)$$

for all $x > 0$. Here $K(\mu')$ is a positive constant. Since every circle about the origin intersects γ_1 and γ_2 we have $A(r, V) \le \log^+ M(r, W) + K_1$ for all r . Thus if $\rho' \in (\rho, \frac{1}{4})$ then

$$\int_x^\infty \{\log m(r, W) - \cos^2 \pi\rho' B(r, V)\} r^{-\rho'-1} dr > 0$$

for all large x , where $m(r, W)$ is the minimum modulus of W . It follows that, for a sequence $R_n \to \infty$,

$$B(R_n, V) < \sec^2 \pi\rho' \log m(R_n, W) < (2-\eta) \log m(R_n, W)$$

for some $\eta > 0$, since $\rho' < \frac{1}{4}$. The circle $|z| = R_n$ intersects one of the domains, D_N say, formed by γ_1 and γ_2 and so there is an arc of it in D_N , γ say, joining a point of γ_1 to a point of γ_2 . Taking account of the definition of V we deduce that since W is uniformly large on γ either

(3.1) $$f(z) - \tfrac{1}{2}\big(w_1(z)+w_2(z)\big) = W(z)\big(\tfrac{1}{2}+o(1)\big) \quad (z \in \gamma)$$

or the same except for a minus sign on the right. In either case a contradiction arises - in the case of (3.1) for instance take z to be the end-point of the arc on γ_2. We conclude that the asymptotic curves are ultimately non-intersecting and therefore may be altered near the origin so as to be intersecting only at 0. For the remainder of the proof we suppose this done.

Let $w_1, \ldots, w_k$ be the asymptotic functions with corresponding asymptotic curves $\gamma_1, \ldots, \gamma_k$ in clockwise order, let Ω_i be the unbounded simply-connected domain between γ_i and γ_{i+1} $(\gamma_{k+1} \equiv \gamma_1)$ and let $\theta_i(t)$ be the angular measure of $\Omega_i \cap C(t)$. Let $W_i(z) = w_i(z) - w_{i+1}(z)$ $(w_{k+1} \equiv w_1)$, let

$$v_i(z) = \log\left|\left[f(z)-\tfrac{1}{2}\left(w_i(z)+w_{i+1}(z)\right)\right]^2-\tfrac{1}{4}W_i(z)^2\right|$$

and let

$$\sigma_i(t) = \max\{v_i(z) : z \in \Omega_i \cap C(t)\} .$$

For each i there is a constant $K_i > 0$ such that $v_i(z) \leq \log^+|W_i(z)| + K_i$ on γ_i and γ_{i+1}. We apply (2.2) with $\Gamma_1 = \gamma_i$, $\Gamma_2 = \gamma_{i+1}$, $\Omega = \Omega_i$ and $u(z)$ the harmonic function in $\Omega(R)$ with boundary values $\log^+|W_i(z)|$ on Γ_1 and Γ_2 and $\sigma_i(R)$ on $\partial\Omega(R) \cap C(R)$. With ϕ as above we deduce that, for all large R,

$$\sigma_i(r) < 6\sigma_i(R)\exp\left\{-\pi \int_r^R ds/s\theta_i(s)\right\} + O\left(r^{\rho+\varepsilon}\right) . \tag{3.2}$$

It follows that $\liminf R^{-\frac{1}{2}}\sigma_i(R) > 0$. For if this were not so the first term on the right of (3.2) would tend to zero as $R \to \infty$ through a sequence of values and we should thus arrive at $\sigma_i(r) = O\left(r^{\rho+\varepsilon}\right)$ for each $\varepsilon > 0$; hence

$$V_i(z) = \begin{cases} \log^+|W_i(z)| + K_i & , \quad z \notin \Omega_i , \\ \max\{\log^+|W_i(z)|+K_i,\ v_i(z)\} & , \quad z \in \Omega_i , \end{cases}$$

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would have order ρ and just as before this leads to a contradiction when $\rho < \frac{1}{4}$, provided W_i is not constant. If W_i is constant then the earlier argument is inapplicable; however, it is easy to see that v_i is bounded in Ω_i (take ϕ constant in (2.2)). But then $f(z) - w_i(z)$ is bounded in Ω_i and has distinct limits on γ_i and γ_{i+1}, which violates the Phragmén-Lindelöf principle [1, p. 3] and again we arrive at a contradiction. We rearrange (3.2) to give

$$(3.3)\qquad \big(1+o(1)\big)\sigma_i(r) \le \sigma_i(R)\exp\left\{-\pi \int_r^R ds/s\theta_i(s)\right\} .$$

This is the equivalent of Heins' (2.2) and (3.2) [5]. The remainder of the proof simply reproduces Heins' arguments and we content ourselves to demonstrate this only so far as proving (1.1), which can be quickly done. As we have noticed $\sigma_i(r) > 0$ for all large r and so, from (3.3),

$$\big(1+o(1)\big) \prod_1^k \sigma_i(r) \le \exp\left\{-\pi \int_r^R \left[\sum_1^k \frac{1}{\theta_i(s)}\right] \frac{ds}{s}\right\} \prod_1^k \sigma_i(R) .$$

Applying Hölder's inequality to the identity $k = \sum_1^k \theta_i^{\frac{1}{2}}\theta_i^{-\frac{1}{2}}$ we obtain $\sum_1^k \theta_i^{-1} \ge k^2/2\pi$ and hence

$$\big(1+o(1)\big) r^{-k^2/2} \prod_1^k \sigma_i(r) \le R^{-k^2/2} \prod_1^k \sigma_i(R) .$$

From this it follows that $\liminf\left\{R^{-k^2/2} \prod_1^k \sigma_i(R)\right\} > 0$ which implies (1.1).

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