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A MULTIPLICATIVE SCHWARZ ALGORITHM FOR THE NONLINEAR COMPLEMENTARITY PROBLEM WITH AN M-FUNCTION

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Abstract

A multiplicative Schwarz iteration algorithm is presented for solving the finite-dimensional nonlinear complementarity problem with an M-function. The monotone convergence of the iteration algorithm is obtained with special choices of initial values. Moreover, by applying the concept of weak regular splitting, the weighted max-norm bound is derived for the iteration errors.

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1. Introduction

We begin this section with the definitions of the M-matrix (see [3]) and M-function (see [17]).

DEFINITION 1.1. A matrix $A \in R^{n\times n}$ is called an M-matrix if it is nonsingular with nonpositive off-diagonals and nonnegative inverse $A^{-1} \geq 0$.

DEFINITION 1.2. Let F be a mapping from a closed convex set K to R^n. F is called an M-function if it satisfies the following conditions.

(1) *Inverse isotone*. For any $y, z \in K$, $y \geq z$ if $F(y) \geq F(z)$.
(2) *Off-diagonal antitone*. For every pair of indices $i \neq j$ and for every $y \in K$, the one-dimensional function $f_{ij} : X_i \to R$, defined by

$$f_{ij}(t) \equiv F_j(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_n), \tag{1.1}$$

is nonincreasing, where $X_i = \{t \in R \mid (y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_n)^T \in K\}$.

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We concern ourselves with the nonlinear complementarity problem (NCP): find $x \in K$ such that

$$x \geq \phi, \quad F(x) \geq 0 \quad \text{and} \quad (x - \phi)^T F(x) = 0, \tag{1.2}$$

where $\phi \in R^n$ is a given vector, and F is a continuous M-function from $K = \{x \in R^n \mid x \geq \phi\}$ to R^n. (If $F(y) = Ay + b$, where A is an M-matrix and $b \in R^n$ is a constant vector, the NCP (1.2) degenerates into a linear complementarity problem (LCP).)

Problem (1.2) can be obtained from the discretizations of some variational inequalities related to nonlinear elliptic operators and to nonlinear parabolic operators. For example, consider the following nonlinear elliptic variational inequalities: find $u \in K \subset H_0^1(\Omega)$ such that

$$a(u, v - u) \geq (f(u), v - u), \quad \forall v \in K, \tag{1.3}$$

where $K = \{v \geq \Phi\}$ (for $\Phi \in W^{2,s}(\Omega)$) is a closed subset of $H_0^1(\Omega)$, $\Omega \subset R^2$ is a bounded convex polygon, $(\cdot, \cdot)$ is the inner product of $L^2(\Omega)$, $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, and $f(v) \in W^{2,\infty}(R)$ such that $\partial f(v)/\partial v \geq 0$. By using the lumped mass finite element method (see [21]) to discretize the problem above, we obtain a finite-dimensional problem (1.2) as long as any angle of any triangle element in the triangulation is not larger than $\pi/2$.

In this paper we apply a multiplicative Schwarz iteration algorithm to solve the NCP (1.2). Schwarz algorithms are well-known iterative methods for solving partial differential equations (see, for example, [14, 20] and the references therein). Since the calculation of Schwarz algorithms can easily be implemented in parallel, the algorithms have been widely used to solve finite-dimensional variational inequalities and complementarity problems (see, for example, [1, 4, 9–13, 19, 22, 25]) for some of which the monotone convergent results are derived with special choices of the initial values (see, for example, [24, 25]). In this paper, the monotone convergence of the multiplicative Schwarz iteration algorithm is obtained with special choices of initial values. In addition, by applying the concept of weak regular splitting, the weighted max-norm bound is derived for the iteration errors.

The monotone results of our algorithm which we will obtain are based on the concepts of super-solution set and of lower-solution set. The super-solution set of the LCP (see [5]) is the set

$$S = \{y \in R^n \mid y \geq \phi, \, Ay + b \geq 0\}. \tag{1.4}$$

This set is also called the feasible set in the LCP literature (see, for example, [7]). It is well known that, if A is an M-matrix, the super-solution set has a minimal element which is just the unique solution of the LCP mentioned above.

For problem (1.2), its super-solution set and lower-solution set are respectively

$$\overline{S} = \{y \in R^n \mid y \geq \phi, \, F(y) \geq 0\} \tag{1.5}$$

and

$$\underline{S} = \{y \in R^n \mid y \geq \phi, \, F_i(y) \leq 0 \text{ or } y_i = \phi_i \text{ for } 1 \leq i \leq n\}. \tag{1.6}$$

The following two properties of F are useful for our results.

LEMMA 1.3. *If $F : K \mapsto R^n$ is an M-function, then for any $y \in K$, $f_{ii}(t)$ $(t \geq \phi_i;\ 1 \leq i \leq n)$ is strictly monotone increasing, where $f_{ii}(t)$ is defined by (1.1).*

PROOF. Suppose that $f_{ii}(t)$ is not strictly monotone increasing for some $y \in K$ and some i. Then there exist t_1, t_2 such that $t_2 > t_1 \geq \phi_i$ and $f_{ii}(t_2) \leq f_{ii}(t_1)$. By the off-diagonal antitone property of F, we have $f_{ij}(t_2) \leq f_{ij}(t_1)$ for $j \neq i$. Therefore $F(y_1, \ldots, y_{i-1}, t_2, y_{i+1}, \ldots, y_n) \leq F(y_1, \ldots, y_{i-1}, t_1, y_{i+1}, \ldots, y_n)$. We have $t_2 \leq t_1$ by the inverse isotone property of F. This is a contradiction, so the lemma holds. □

LEMMA 1.4. *Let $F : K \to R^n$ be an M-function and let I and J be sets satisfying $I \cup J = N = \{1, 2, \ldots, n\}$ and $I \cap J = \emptyset$. If vectors $y, z \in K$ satisfy $y_J \leq z_J$ and $F_I(y) \leq F_I(z)$, then $y \leq z$.*

PROOF. Let $J' = \{i \in N \mid y_i \leq z_i\}$, $I' = N \backslash J'$ (without loss of generality, we assume that $I' = \{1, 2, \ldots, k\}$ and $J' = \{k+1, \ldots, n\}$). Suppose that I' is not empty. By the definition of J', we know that $J \subset J'$ and $I' \subset I$. To be more precise,

$$F_{I'}(y) \leq F_{I'}(z), \quad y_{I'} > z_{I'}, \quad y_{J'} \leq z_{J'}. \tag{1.7}$$

Combining (1.7) with the off-diagonal antitone property of F leads to

$$F_{I'}(y_{I'}, z_{J'}) \leq F_{I'}(y_{I'}, y_{J'}) \leq F_{I'}(z_{I'}, z_{J'}). \tag{1.8}$$

Since $y_{I'} > z_{I'}$ by (1.7), we have, by the off-diagonal antitone property of F,

$$F_{J'}(y_{I'}, z_{J'}) \leq F_{J'}(z_{I'}, z_{J'}). \tag{1.9}$$

Combining (1.9) with (1.8), we have

$$F(y_{I'}, z_{J'}) \leq F(z_{I'}, z_{J'}). \tag{1.10}$$

By the inverse isotone property of F, we have $y_{I'} \leq z_{I'}$. This contradicts (1.7). Therefore $I' = \emptyset$ and $J' = N$, which means that $y \leq z$. □

In this paper, our statements assume that problem (1.2) has a unique solution. The rest of the paper is organized as follows: in Section 2 we propose a multiplicative Schwarz iteration algorithm for solving problem (1.2) and give some basic properties of the proposed algorithm; in Section 3 we obtain the monotone convergence of the iteration algorithm; and in Section 4 we estimate the weighted max-norm error bound for the iteration algorithm.

2. Multiplicative Schwarz iteration algorithm

In the rest of this paper, we assume that there exists an M-matrix A such that, for any $y, z \in K$ and $y \geq z$,

$$F(y) - F(z) \leq A(y - z). \tag{2.1}$$

Based on (2.1), we propose a multiplicative Schwarz iteration algorithm for solving problem (1.2). To be specific, let $V_i \subset R^n$, $i = 1, 2, \ldots, m$, be subspaces such that

$$\sum_{i=1}^{m} V_i \equiv \{v \in V : v = v_1 + \cdots + v_m,\ v_i \in V_i\ (i = 1, \ldots, m)\} = R^n. \tag{2.2}$$

That is, the bases of the subspaces V_i together span the whole space. Let $n_i = \dim(V_i)$ be the dimensions of subspaces V_i, $i = 1, 2, \ldots, m$. We consider both overlapping and nonoverlapping subdomains which correspond to the cases $\sum_{i=1}^{m} n_i > n$ and $\sum_{i=1}^{m} n_i = n$, respectively. For simplicity, we identify V_i with R^{n_i}. Let $R_i : R^n \to R^{n_i}$ be the restriction operator. In our context, R_i is an $n_i \times n$ matrix with $\operatorname{rank}(R_i) = n_i$. Its transpose $R_i^T : R^{n_i} \to R^n$ is a prolongation operator. Moreover, we choose the bases of V_i appropriately such that the images of the bases of V_i, under the prolongation operator R_i^T, are linearly independent unit elements in R^n. In other words, the columns of R_i^T consist of columns of the $n \times n$ identity matrix. Formally, such a matrix R_i can be expressed as

$$R_i = [I_i\ 0]\pi_i \geq 0, \tag{2.3}$$

where I_i is the $n_i \times n_i$ identity matrix and π_i is some $n \times n$ permutation matrix. In this case, matrix A_i is an $n_i \times n_i$ principal submatrix of A, which is also an M-matrix.

Given an initial value $x^0 \geq \phi$, $x^k \geq \phi$ is the approximation to the solution of (1.2) at the kth step. The multiplicative Schwarz algorithm consists of the following substeps.

ALGORITHM MSA (Multiplicative Schwarz algorithm). For $i = 1, 2, \ldots, m$, do the following.

(Restriction): restrict the matrix A and the vector $\phi - x^{k+(i-1)/m}$ so that

$$A_i = R_i A R_i^T, \tag{2.4}$$

$$\phi^{k,i} = R_i(\phi - x^{k+(i-1)/m}). \tag{2.5}$$

Solve the local problem of finding $x^{k,i} \in R^{n_i}$ such that

$$\begin{aligned} &x^{k,i} \geq \phi^{k,i}, \quad A_i x^{k,i} + R_i F(x^{k+(i-1)/m}) \geq 0, \\ &(x^{k,i} - \phi^{k,i})^T (A_i x^{k,i} + R_i F(x^{k+(i-1)/m})) = 0. \end{aligned} \tag{2.6}$$

$$x^{k+i/m} = x^{k+(i-1)/m} + R_i^T x^{k,i}, \quad i = 1, 2, \ldots, m. \tag{2.7}$$

REMARK 2.1. Since A is an M-matrix, A_i, the $n_i \times n_i$ principal submatrix of A, is also an M-matrix.

It will be convenient to denote

$$E_i = R_i^T R_i, \quad E = \sum_{j=1}^{m} E_j, \quad i = 1, 2, \ldots, m, \tag{2.8}$$

where for each i, R_i is defined by (2.3). It is easy to verify that

$$0 \leq E_i \leq I \tag{2.9}$$

and E is a diagonal matrix with positive diagonal elements, with I denoting the identify matrix.

We conclude this section by giving some notation that will be used in the analysis of the convergence of the algorithm. We denote the matrix $(|a_{ij}|)$ and the vector $(|x_i|)$ by $|A|$ and $|x|$, respectively. Similarly, the matrix inequalities $A \geq B$ and $A > B$, and the vector inequalities $x \geq y$ and $x > y$ are understood to be elementwise. Let $\pi \in R^{n\times n}$ be a permutation matrix. We set $A_\pi = \pi A \pi^T$, $x_\pi = \pi x$, $F_\pi = \pi F$, and $\phi_\pi = \pi\phi$. For $i = 1, \ldots, m$, since

$$A_i = R_i A R_i^T = \begin{bmatrix} I_i & 0 \end{bmatrix} A_{\pi_i} \begin{bmatrix} I_i \\ 0 \end{bmatrix}$$

is the $n_i \times n_i$ submatrix of A_{π_i}, we can represent the matrix A_{π_i} in the form

$$A_{\pi_i} = \begin{bmatrix} A_i & G_i \\ H_i & A_{i_c} \end{bmatrix}. \tag{2.10}$$

It will also be found convenient to represent vectors x_{π_i} and ϕ_{π_i} as

$$x_{\pi_i} = \begin{bmatrix} u_i \\ u_{i_c} \end{bmatrix}, \quad \phi_{\pi_i} = \begin{bmatrix} \varphi_i \\ \varphi_{i_c} \end{bmatrix},$$

where $u_i = R_i x \in R^{n_i}$ and $\varphi_i = R_i \phi \in R^{n_i}$.

3. Monotone convergence

In this section we show the monotone convergence of algorithm MSA. First, we prove some useful properties for the algorithm.

LEMMA 3.1. *Let $x^{k,i}$ be the solution of problem (2.6). If $x^{k+(i-1)/m} \in \overline{S}$ for $i \in \{1, 2, \ldots, m\}$, then $x^{k,i} \leq 0$.*

PROOF. We have $x^{k+(i-1)/m} \in \overline{S}$; that is, $F(x^{k+(i-1)/m}) \geq 0$ and $x^{k+(i-1)/m} \geq \phi$. It follows from (2.3) and (2.5) that $\phi^{k,i} \leq 0$. Then it is easy to verify that $0 \in R^{n_i}$ is a super-solution of problem (2.6). Since, for each $i = 1, \ldots, m$, A_i is also an M-matrix, $x^{k,i}$ is the minimal element of the super-solution set of LCP (2.6). Thus $x^{k,i} \leq 0$. □

LEMMA 3.2. *Let $\{x^k\}$ be generated by algorithm MSA. If $x^k \in \overline{S}$, then $x^{k+i/m} \in \overline{S}$ (for $i = 1, 2, \ldots, m$) and therefore $x^{k+1} \in \overline{S}$.*

PROOF. It suffices to show that $x^{k+i/m} \in \overline{S}$, $i \in \{1, 2 \ldots, m\}$, if $x^{k+(i-1)/m} \in \overline{S}$. Assume that $x^{k+(i-1)/m} \in \overline{S}$. Let R_i, E_i be defined by (2.3) and (2.8), respectively. By (2.5) and (2.6), $x^{k,i} \geq R_i(\phi - x^{k+(i-1)/m})$. Then

$$\begin{aligned} x^{k+i/m} &= x^{k+(i-1)/m} + R_i^T x^{k,i} \\ &\geq x^{k+(i-1)/m} + R_i^T R_i(\phi - x^{k+(i-1)/m}) \\ &= \phi + (I - E_i)(x^{k+(i-1)/m} - \phi) \\ &\geq \phi, \end{aligned} \tag{3.1}$$

where I is the identity matrix. The last inequality is obtained by (2.9) and $x^{k+(i-1)/m} \in \overline{S}$. By Lemma 3.1, we know that $x^{k,i} \leq 0$. Therefore, by (2.1),

$$\begin{aligned} F(x^{k+i/m}) - F(x^{k+(i-1)/m}) &= F(x^{k+(i-1)/m} + R_i^T x^{k,i}) - F(x^{k+(i-1)/m}) \\ &\geq A(R_i^T x^{k,i}). \end{aligned}$$

Since the equalities $\pi^T \pi = \pi \pi^T = I$ hold for any permutation matrix π,

$$\begin{aligned} F(x^{k+i/m}) &\geq F(x^{k+(i-1)/m}) + A R_i^T x^{k,i} \\ &= F(x^{k+(i-1)/m}) + A \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} x^{k,i} \\ &= F(x^{k+(i-1)/m}) + \pi_i^T A_{\pi_i} \begin{bmatrix} I_i \\ 0 \end{bmatrix} x^{k,i} \\ &= F(x^{k+(i-1)/m}) + \pi_i^T \begin{bmatrix} A_i \\ H_i \end{bmatrix} x^{k,i}, \end{aligned} \tag{3.2}$$

where the last equality follows from (2.10). Obviously, A_{π_i} is an M-matrix. This implies that $H_i \leq 0$. It follows from (3.2) and $x^{k,i} \leq 0$ that

$$\begin{aligned} F(x^{k+i/m}) &\geq F(x^{k+(i-1)/m}) + \pi_i^T \begin{bmatrix} A_i \\ 0 \end{bmatrix} x^{k,i} \\ &= F(x^{k+(i-1)/m}) + \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} A_i x^{k,i} \\ &= F(x^{k+(i-1)/m}) + R_i^T A_i x^{k,i} \\ &\geq F(x^{k+(i-1)/m}) - R_i^T R_i F(x^{k+(i-1)/m}) \\ &= (I - E_i) F(x^{k+(i-1)/m}) \\ &\geq 0, \end{aligned} \tag{3.3}$$

where the second equality is obtained by (2.3), the second inequality by (2.6) and the last inequality by (2.9) and $F(x^{k+(i-1)/m}) \geq 0$. □

The following theorem shows the monotone convergence of algorithm MSA.

THEOREM 3.3. *Let $\{x^k\}$ be generated by algorithm MSA. If $x^0 \in \overline{S}$, then $\{x^k\}$ converges to the solution x^* of problem (1.2). Moreover, for any $k \geq 0$,*

$$x^k \in \overline{S} \quad \textit{and} \quad x^* \leq x^{k+1} \leq x^k.$$

PROOF. Since $x^0 \in \overline{S}$, $x^k \in \overline{S}$ holds for any k by Lemma 3.2 (to be more precise, $x^{k+i/m} \in \overline{S}$ for $i \in \{1, 2, \ldots, m\}$). Moreover, by (2.7) and Lemma 3.1, $x^{k+1} \leq \cdots \leq x^{k+i/m} \leq x^{k+(i-1)/m} \leq \cdots \leq x^k$ holds for any k. So $\{x^k\}$ is bounded below by ϕ and convergent. Let $x^k \to \bar{x}$. Clearly, $\bar{x} \leq x^k$ for all k and $x^{k+i/m} \to \bar{x}$. Taking limits on both sides of (2.7) yields $x^{k,i} \to 0$ as $k \to \infty$ by $\operatorname{rank}(R_i^T) = n_i$. Therefore, taking limits in (2.6), we deduce that

$$-R_i(\phi - \bar{x}) \geq 0, \quad R_i F(\bar{x}) \geq 0$$

and

$$(-R_i(\phi - \bar{x}))^T R_i F(\bar{x}) = 0.$$

It then follows that, for each $i = 1, 2, \ldots, m$,

$$R_i^T R_i(\bar{x} - \phi) \geq 0, \quad R_i^T R_i F(\bar{x}) \geq 0 \tag{3.4}$$

and

$$(\bar{x} - \phi)^T R_i^T R_i F(\bar{x}) = 0. \tag{3.5}$$

Let $E = \sum_{i=1}^m E_i = \sum_{i=1}^m R_i^T R_i$. Summing (3.4) and (3.5) over $i = 1, 2, \ldots, m$, respectively, we get

$$E(\bar{x} - \phi) \geq 0, \quad EF(\bar{x}) \geq 0 \tag{3.6}$$

and

$$(\bar{x} - \phi)^T EF(\bar{x}) = 0. \tag{3.7}$$

Since E is a diagonal matrix with positive diagonals, (3.6) and (3.7) are equivalent to

$$\bar{x} - \phi \geq 0, \quad F(\bar{x}) \geq 0, \quad (\bar{x} - \phi)^T F(\bar{x}) = 0.$$

This shows that $\bar{x}$ is the solution of (1.2), and the lemma follows from the uniqueness of problem (1.2). □

Similarly, we can obtain the following theorem.

THEOREM 3.4. *Let $\{x^k\}$ be generated by algorithm MSA and x^* be the solution of problem (1.2). If $x^0 \in \underline{S}$, then $\{x^k\}$ converges to the solution of (1.2). Moreover, for any $k \geq 0$,*

$$x^k \in \underline{S} \quad \text{and} \quad x^* \geq x^{k+1} \geq x^k.$$

4. Weighted max-norm bound

In this section we assume that there exists an M-matrix $\dot{A}$ such that for any $y, z \in K\,(y \geq z)$,

$$\dot{A}(y - z) \leq F(y) - F(z). \tag{4.1}$$

We will obtain the weighted max-norm bound for the iteration errors of algorithm MSA. We first introduce the concepts of the weighted max-norm (see [6]) and weak regular splitting of a matrix (see, for example, [18, 23]) and some relevant results.

Let $w \in R^n$ be a positive vector. For a vector $y \in R^n$, the weighted max-norm is defined by

$$\|y\|_w = \max_{1 \leq j \leq n} \left| \frac{y_j}{w_j} \right|.$$

For a matrix $A \in R^{n\times n}$, the weighted max-norm is defined by

$$\|A\|_w = \sup_{\|y\|_w=1} \{\|Ay\|_w \mid y \in R^n\}.$$

Obviously, if $w = (1, \ldots, 1)^T$, the weighted max-norm reduces to the usual maximum norm of matrices.

LEMMA 4.1 [2]. *Let P be a matrix, w a positive vector and γ a positive scalar such that*

$$|P|w \le \gamma w. \tag{4.2}$$

Then $\|P\|_w \le \gamma$. In particular, $\|Px\|_w \le \gamma\|x\|_w$ for all x. Moreover, if strict inequality holds in (4.2), then $\|P\|_w < \gamma$.

DEFINITION 4.2. For a matrix $A \in R^{n\times n}$, we call $A = M - N$ a weak regular splitting of A if $M^{-1} \ge 0$ and $M^{-1}N \ge 0$.

The concept of weak regular splitting has been widely used to analyze the convergence of various splitting algorithms [15, 16]. We will use this concept later in the estimation of the weighted max-norm error bound for the proposed multiplicative Schwarz algorithm.

LEMMA 4.3 [2]. *Let $A = M_i - N_i$ $(i = 1, \ldots, m)$ be weak regular splittings of an M-matrix A, E_i $(i = 1, \ldots, m)$ be defined as in (2.8) and*

$$T = (I - E_m M_m^{-1} A)(I - E_{m-1} M_{m-1}^{-1} A) \cdots (I - E_1 M_1^{-1} A)$$

where I is the unit matrix. For any vector $R^n \ni e > 0$, $w = A^{-1}e > 0$, there exists a positive $\gamma \in (0, 1)$ such that $\rho(T) \le \|T\|_w < \gamma$.

For algorithm MSA, we have the following lemmas.

LEMMA 4.4. *Let x^* be the solution of (1.2) and $x^{k,i}$ be the solution of problem (2.6). If $x^{k+(i-1)/m} = x^*$ for $i \in \{1, \ldots, m\}$, then $x^{k,i} = 0$.*

PROOF. Since $x^* - \phi \ge 0$, $F(x^*) \ge 0$ and $(x^* - \phi)^T F(x^*) = 0$, it follows from the nonnegativity of R_i that

$$\begin{cases} 0 - \phi^{k,i} = R_i(x^* - \phi) \ge 0, \\ A_i 0 + R_i F(x^*) = R_i F(x^*) \ge 0. \end{cases} \tag{4.3}$$

Multiplying these two inequalities and noting (2.8) and (2.9), we have

$$\begin{aligned} 0 &\le (0 - \phi^{k,i})^T (A_i 0 + R_i F(x^*)) \\ &= (x^* - \phi)^T R_i^T R_i F(x^*) \\ &\le (x^* - \phi)^T F(x^*) \\ &= 0, \end{aligned}$$

and hence

$$(0-\phi^{k,i})^T(A_i 0 + R_i F(x^*)) = 0. \tag{4.4}$$

It follows from (4.3) and (4.4) that $x^{k,i}=0$ is a solution of (2.6), which is unique since A_i is an M-matrix. The proof is complete. □

LEMMA 4.5. *Let $x^{k,i}$ be the solution of problem (2.6) and $x_{\pi_i}^{k+(i-1)/m} = \begin{bmatrix} u^{k,i} \\ u^{k,i_c} \end{bmatrix}$ with $u^{k,i} = R_i x^{k+(i-1)/m} \in R^{n_i}$. Let $y^{k,i} \in R^{n_i}$ be given by $y^{k,i} = x^{k,i} + u^{k,i}$. Then $y^{k,i}$ is the solution of the following LCP on R^{n_i}:*

$$\begin{aligned} y &\geq \varphi^i, \\ A_i y + R_i F(x^{k+(i-1)/m}) - A_i u^{k,i} &\geq 0, \\ (y-\varphi^i)^T(A_i y + R_i F(x^{k+(i-1)/m}) - A_i u^{k,i}) &= 0, \end{aligned} \tag{4.5}$$

where $\varphi^i = R_i\phi$.

PROOF. The lemma is obvious by (2.6). □

Lemmas 4.4 and 4.5 imply the following useful corollary.

COROLLARY 4.6. *Let x^* be the solution of problem (1.2). Then $u^{*,i} = R_i x^*$ is the unique solution of the following NCP in R^{n_i}:*

$$\begin{aligned} &y \geq \varphi^i, \quad A_i y + R_i F(x^*) - A_i u^{*,i} \geq 0, \\ &(y-\varphi^i)^T(A_i y + R_i F(x^*) - A_i u^{*,i}) = 0, \end{aligned} \tag{4.6}$$

where $\varphi^i = R_i\phi$.

LEMMA 4.7. *Let $\{x^k\}$ and $\{\bar{x}^k\}$ be generated by algorithm MSA. If $x^k \geq \bar{x}^k$ at step k, then $x^{k+i/m} \geq \bar{x}^{k+i/m}$ for any $1 \leq i \leq m$. Therefore $x^{k+1} \geq \bar{x}^{k+1}$.*

PROOF. It suffices to show that $x^{k+i/m} \geq \bar{x}^{k+i/m}$, $i \in \{1, 2, \ldots, m\}$, if $x^{k+(i-1)/m} \geq \bar{x}^{k+(i-1)/m}$. Suppose that $x^{k+(i-1)/m} \geq \bar{x}^{k+(i-1)/m}$ for $i \in \{1, 2, \ldots, m\}$. Let $x_{\pi_i}^{k+(i-1)/m} = \begin{bmatrix} u^{k,i} \\ u^{k,i_c} \end{bmatrix}$ and $\bar{x}_{\pi_i}^{k+(i-1)/m} = \begin{bmatrix} \bar{u}^{k,i} \\ \bar{u}^{k,i_c} \end{bmatrix}$. Let $y^{k,i}$ and $\bar{y}^{k,i}$ be defined in the same way as $y^{k,i}$ in Lemma 4.5. Since $\bar{y}^{k,i} \geq \varphi^i$ (φ^i defined in Lemma 4.5), the components of $\bar{y}^{k,i}$ can be divided into the following two parts: $I = \{j \mid \bar{y}_j^{k,i} = \varphi_j^i\}$ and $J = \{j \mid \bar{y}_j^{k,i} > \varphi_j^i\}$. Obviously $y_I^{k,i} \geq \bar{y}_I^{k,i}$. By Lemma 4.5, for $j \in J$,

$$(A_i \bar{y}^{k,i} + R_i F(\bar{x}^{k+(i-1)/m}) - A_i \bar{u}^{k,i})_j = 0 \tag{4.7}$$

and

$$(A_i y^{k,i} + R_i F(x^{k+(i-1)/m}) - A_i u^{k,i})_j \geq 0. \tag{4.8}$$

By (2.1) and the condition $x^{k+(i-1)/m} \geq \bar{x}^{k+(i-1)/m}$, we have $F(x^{k+(i-1)/m}) - F(\bar{x}^{k+(i-1)/m}) \leq A(x^{k+(i-1)/m} - \bar{x}^{k+(i-1)/m})$. It follows from the subtraction of (4.7) and (4.8) that

$$\begin{aligned} 0 &\le (A_i(y^{k,i} - \bar{y}^{k,i}) + R_i(F(x^{k+(i-1)/m}) - F(\bar{x}^{k+(i-1)/m})) - A_i(u^{k,i} - \bar{u}^{k,i}))_j \\ &\le (A_i(y^{k,i} - \bar{y}^{k,i}) + R_i A(x^{k+(i-1)/m} - \bar{x}^{k+(i-1)/m}) - A_i(u^{k,i} - \bar{u}^{k,i}))_j \\ &= (A_i(y^{k,i} - \bar{y}^{k,i}) + G_i(u^{k,i_c} - \bar{u}^{k,i_c}))_j \\ &\le (A_i(y^{k,i} - \bar{y}^{k,i}))_j, \end{aligned} \tag{4.9}$$

where G_i is defined by (2.10), the last inequality by $x^{k+(i-1)/m} \ge \bar{x}^{k+(i-1)/m}$ and $G_i \le 0$. Therefore

$$\begin{aligned} 0 &\le (A_i(y^{k,i} - \bar{y}^{k,i}))_J \\ &= (A_i)_{J,J}(y_J^{k,i} - \bar{y}_J^{k,i}) + (A_i)_{J,I}(y_I^{k,i} - \bar{y}_I^{k,i}) \\ &\le (A_i)_{J,J}(y_J^{k,i} - \bar{y}_J^{k,i}), \end{aligned}$$

where the second inequality is obtained from $(A_i)_{J,I} \le 0$ and $y_I^{k,i} \ge \bar{y}_I^{k,i}$. Since $(A_i)_{J,J}$ is an M-matrix, we have $y_J^{k,i} - \bar{y}_J^{k,i} \ge 0$. That is, $y^{k,i} \ge \bar{y}^{k,i}$. Therefore, $x^{k+i/m} \ge \bar{x}^{k+i/m}$. □

Let x^* be the solution of problem (1.2). By Lemma 4.4, we know that if $x^k = x^*$, then $x^{k+i/m} = x^*$ for $i \in \{1, 2, \ldots, m\}$. So Lemma 4.7 has the following two corollaries.

COROLLARY 4.8. *Let $\{x^k\}$ be generated by algorithm MSA. If $x^k \ge x^*$ at step k, then $x^{k+i/m} \ge \bar{x}^*$ for any $1 \le i \le m$; therefore $x^{k+1} \ge x^*$.*

COROLLARY 4.9. *Let $\{x^k\}$ be generated by algorithm MSA. If $x^k \le x^*$ at step k, then $x^{k+i/m} \le \bar{x}^*$ for any $1 \le i \le m$; therefore $x^{k+1} \le x^*$.*

LEMMA 4.10. *Let x^* be the unique solution of (1.2) and let $\{x^k\}$ be generated by algorithm MSA. Let $x^*_{\pi_i} = \begin{bmatrix} u^{*,i} \\ u^{*,i_c} \end{bmatrix}$, $i = 1, \ldots, m$ and $u^{k,i}$, u^{k,i_c} be as in Lemma 4.5. Let $y^{k,i} = x^{k,i} + u^{k,i}$ and $y^{*,i} = u^{*,i} = R_i x^*$. If $x^k \ge x^*$(or $x^k \le x^*$), then*

$$A_i|y^{k,i} - y^{*,i}| \le -(\dot{A}_i - A_i)|u^{k,i} - u^{*,i}| - \dot{G}_i|u^{k,i_c} - u^{*,i_c}|, \tag{4.10}$$

where the submatrices $\dot{A}_i$, $\dot{G}_i$ of $\dot{A}$ are defined in the same way as A_i, G_i in (2.10).

PROOF. We show that (4.10) holds elementwise. Because of the similarity of the proofs, we only consider the case $x^k \ge x^*$. By Lemma 4.7, we know that $y^{k,i} \ge y^{*,i}$. We consider the following two cases.

Case I: $(y^{k,i})_j = (y^{*,i})_j$. Here, A_i is a matrix with nonpositive off-diagonal elements, $\dot{G}_i \le 0$ and $\dot{A}_i - A_i \le 0$ (this can be easily verified by (2.1) and (4.1)). Inequality (4.10) follows from the fact that its left-hand side is nonpositive while its right-hand side is nonnegative.

Case II: $(y^{k,i})_j > (y^{*,i})_j$. Since $(y^{*,i})_j \geq (\varphi^i)_j$, it is obvious that $(y^{k,i})_j > (\varphi^i)_j$, where $\varphi^i = R_i \phi$. By Lemma 4.5 and Corollary 4.6,

$$(A_i y^{k,i} + R_i F(x^{k+(i-1)/m}) - A_i u^{k,i})_j = 0$$

and

$$(A_i y^{*,i} + R_i F(x^*) - A_i u^{*.i})_j \geq 0.$$

It then follows that

$$(A_i(y^{k,i} - y^{*,i}) + R_i(F(x^{k+(i-1)/m}) - F(x^*)) - A_i(u^{k,i} - u^{*,i}))_j \leq 0. \quad (4.11)$$

Since $x^{k+(i-1)/m} \geq x^*$ for any k and $i \in \{1, 2, \ldots, m\}$ by Corollary 4.8, we have $F(x^{k+(i-1)/m}) - F(x^*) \geq \dot{A}(x^{k+(i-1)/m} - x^*)$ by (4.1). Then by (4.11),

$$\begin{aligned} 0 &\geq (A_i(y^{k,i} - y^{*,i}) + R_i \dot{A}(x^{k+(i-1)/m} - x^*) - A_i(u^{k,i} - u^{*,i}))_j \\ &= (A_i(y^{k,i} - y^{*,i}) + (\dot{A}_i - A_i)(u^{k,i} - u^{*,i}) + \dot{G}_i(u^{k,i_c} - u^{*,i_c}))_j. \end{aligned}$$

Therefore

$$(A_i(y^{k,i} - y^{*,i}))_j \leq (-(\dot{A}_i - A_i)(u^{k,i} - u^{*,i}) - \dot{G}_i(u^{k,i_c} - u^{*,i_c}))_j. \quad (4.12)$$

Since $y^{k,i} - y^{*,i} \geq 0$, $-(\dot{A}_i - A_i) \geq 0$, and $-\dot{G}_i \geq 0$, we obtain (4.10). □

LEMMA 4.11. *Let $\{x^k\}$ be generated by algorithm MSA, let x^* be the solution of problem (1.2) and suppose that $\epsilon^{k+i/m} = x^{k+i/m} - x^*$. If $x^k \geq x^*$(or $x^k \leq x^*$), then, for any $1 \leq i \leq m$,*

$$|\epsilon^{k+i/m}| \leq (I - E_i M_i^{-1} \dot{A})|\epsilon^{k+(i-1)/m}|, \quad (4.13)$$

where

$$M_i = \pi_i^T \begin{bmatrix} A_i & 0 \\ 0 & A_{i_c} \end{bmatrix} \pi_i, \quad i = 1, \ldots, m. \quad (4.14)$$

PROOF. By (2.7),

$$\begin{aligned} 0 \leq |\epsilon^{k+i/m}| &= |\epsilon^{k+(i-1)/m} + R_i^T x^{k,i}| \\ &= \left| \pi_i^T \begin{bmatrix} y^{k,i} - y^{*,i} \\ u^{k,i_c} - u^{*,i_c} \end{bmatrix} \right| \\ &\leq \pi_i^T \begin{bmatrix} |y^{k,i} - y^{*,i}| \\ |u^{k,i_c} - u^{*,i_c}| \end{bmatrix} \\ &\leq \pi_i^T \begin{bmatrix} -A_i^{-1}((\dot{A}_i - A_i)|u^{k,i} - u^{*,i}| + \dot{G}_i|u^{k,i_c} - u^{*,i_c}|) \\ |u_{i_c}^k - u_{i_c}^*| \end{bmatrix}, \end{aligned}$$

where $\dot{A}_i$, $\dot{G}_i$, $u^{k,i}$, $y^{k,i}$, $u^{*,i}$ and $y^{*,i}$ are as in Lemma 4.10, and the last inequality follows from Lemma 4.10 and $A_i^{-1} \geq 0$. Therefore,

$$\begin{aligned}
0 &\leq |\epsilon^{k+i/m}| \\
&\leq \pi_i^T \begin{bmatrix} I_i - A_i^{-1}\dot{A}_i & -A_i^{-1}\dot{G}_i \\ 0 & I_{i_c} \end{bmatrix} |\epsilon_{\pi_i}^{k+(i-1)/m}| \\
&= |\epsilon^{k+(i-1)/m}| - \pi_i^T |\epsilon_{\pi_i}^{k+(i-1)/m}| + \pi_i^T \begin{bmatrix} I_i - A_i^{-1}\dot{A}_i & -A_i^{-1}\dot{G}_i \\ 0 & I_{i_c} \end{bmatrix} |\epsilon_{\pi_i}^{k+(i-1)/m}| \\
&= |\epsilon^{k+(i-1)/m}| + \pi_i^T \begin{bmatrix} -A_i^{-1}\dot{A}_i & -A_i^{-1}\dot{G}_i \\ 0 & 0 \end{bmatrix} |\epsilon_{\pi_i}^{k+(i-1)/m}| \\
&= |\epsilon^{k+(i-1)/m}| - R_i^T A_i^{-1} [\dot{A}_i \;\; \dot{G}_i] |\epsilon_{\pi_i}^{k+(i-1)/m}| \\
&= |\epsilon^{k+(i-1)/m}| - R_i^T A_i^{-1} [I_i \;\; 0] \dot{A}_{\pi_i} |\epsilon_{\pi_i}^{k+(i-1)/m}| \\
&= |\epsilon^{k+(i-1)/m}| - R_i^T A_i^{-1} [I_i \;\; 0] \pi_i \dot{A} \pi_i^T \pi_i |\epsilon^{k+(i-1)/m}| \\
&= |\epsilon^{k+(i-1)/m}| - R_i^T A_i^{-1} R_i \dot{A} |\epsilon^{k+(i-1)/m}| \\
&= (I - E_i M_i^{-1} \dot{A}) |\epsilon^k|,
\end{aligned}$$

where the third and the sixth equalities follow from (2.3), the fourth equality from (2.10), and the last equality from the relation $R_i^T A_i^{-1} R_i = E_i M_i^{-1}$. Therefore, when $x^k \geq \phi$ (or $x^k \leq \phi$),

$$|\epsilon^{k+i/m}| \leq (I - E_i M_i^{-1} \dot{A}) |\epsilon^{k+(i-1)/m}|. \qquad \square$$

THEOREM 4.12. *Let x^* be the solution of (1.2), $\{x^k\}$ be generated by algorithm MSA, and $\epsilon^k \triangleq x^k - x^*$ be the iteration error. Then there is a positive $\gamma < 1$ such that*

$$\|\epsilon^{k+1}\|_w \leq \gamma \|\epsilon^k\|_w. \tag{4.15}$$

PROOF. Let E_i be defined by (2.8) and M_i be defined by (4.14). It is easy to verify that M_i, $N_i = \dot{A} - M_i$, $i = 1, 2, \ldots, m$, are weak regular splittings of $\dot{A}$. Then by Lemma 4.3, there exists a positive scalar $\gamma < 1$ such that $\|\dot{T}\|_w \leq \gamma$, where

$$\dot{T} = (I - E_m M_m^{-1} \dot{A})(I - E_{m-1} M_{m-1}^{-1} \dot{A}) \cdots (I - E_1 M_1^{-1} \dot{A}).$$

On the other hand, we known from Lemma 4.11 that when $x^k \geq x^*$(or $x^k \leq x^*$),

$$\begin{aligned}
0 &\leq |\epsilon^{k+1}| = |\epsilon^{k+m/m}| \\
&\leq (I - E_m M_m^{-1} \dot{A}) |\epsilon^{k+(m-1)/m}| \\
&\leq \cdots \\
&\leq (I - E_m M_m^{-1} \dot{A})(I - E_{m-1} M_{m-1}^{-1} \dot{A}) \cdots (I - E_1 M_1^{-1} \dot{A}) |\epsilon^k| \\
&= \dot{T} |\epsilon^k|.
\end{aligned}$$

Therefore, when $x^k \geq x^*$(or $x^k \leq x^*$),

$$\|\epsilon^{k+1}\|_w = \||\epsilon^{k+1}|\|_w \leq \|T_\theta|\epsilon^k|\|_w \leq \gamma\||\epsilon^k|\|_w = \gamma\|\epsilon^k\|_w.$$

For any initial value $x^0 \geq \phi$, $x^k \geq \phi$ holds for any k by algorithm MSA. Let $\overline{x}^k = \max(x^k, x^*)$ and $\underline{x}^k = \min(x^k, x^*)$, where for $y, z \in R^n$, $\max(y, z)$, $\min(y, z) \in R^n$ are defined by $(\max(y, z))_i = \max(y_i, z_i)$ and $(\min(y, z))_i = \min(y_i, z_i)$, $i = 1, 2, \ldots, n$. Then $\overline{x}^k \geq x^k \geq \underline{x}^k$ and $\overline{x}^k \geq x^* \geq \underline{x}^k$. Let $\overline{x}^{k+1}$ and $\underline{x}^{k+1}$ be generated by algorithm MSA with $\overline{x}^k, \underline{x}^k$ as the approximation of the solution of problem (1.2) at step k, respectively. Then by Lemma 4.7, $\overline{x}^{k+1} \geq x^{k+1} \geq \underline{x}^{k+1}$, and by Corollaries 4.8 and 4.9, $\overline{x}^{k+1} \geq x^* \geq \underline{x}^{k+1}$. From the above proof, we know that $\|\overline{\epsilon}^{k+1}\|_w \leq \gamma\|\overline{\epsilon}^k\|_w$ and $\|\underline{\epsilon}^{k+1}\|_w \leq \gamma\|\underline{\epsilon}^k\|_w$, where $\underline{\epsilon}^k = \underline{x}^k - x^*$ and $\overline{\epsilon}^k = \overline{x}^k - x^*$. Therefore,

$$\|\epsilon^{k+1}\|_w \leq \max(\|\overline{\epsilon}^{k+1}\|_w, \|\underline{\epsilon}^{k+1}\|_w) \leq \gamma \max(\|\overline{\epsilon}^k\|_w, \|\underline{\epsilon}^k\|_w) = \|\epsilon^k\|_w.$$

The proof is complete. □

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