Euler's limit for e^x and the exponential series.

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1948 is the bicentenary of Euler's discovery that[1]

$$(1) \qquad \lim_{n \to \infty} (1 + x/n)^n = \sum_0^\infty x^n/n! = e^x.$$

This note gives a brief account of the subsequent work on these relations and a proof of the equivalence of limit and series which appears to involve new features.

Cauchy in lectures published in 1821 followed Euler in regarding the argument

$$(2) \quad \left(1 + \frac{x}{n}\right)^n = 1 + x + \left(1 - \frac{1}{n}\right)\frac{x^2}{2!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{x^3}{3!} + \dots + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{n-1}{n}\right)\frac{x^n}{n!}$$

$$\to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

as adequate[2], but at about this time appreciated that the limiting operation required justification. He succeeded in giving this justification but only by a lengthy and roundabout method. In his lectures on calculus[3] published in 1823 the first lesson is devoted to a proof that Euler's limit exists and has the essential properties of e^x. It is not until the thirty-seventh lecture that the series is derived as an example of Taylor's theorem and the relation (1) completely justified. This method was essentially unaltered in the modified lectures[4] published in 1829.

The long delay in deriving the series is clearly a little unsatisfactory, especially as Cauchy used the inequality (5) below and the convergence of the series, from the commencement. In this "first

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[1] 7. According to 15 the special case $(1 + 1/n)^n \to e$ was mentioned by Euler in a letter to Goldbach, 25 Nov. 1731, and the exponential series itself was known to Newton: see for example "de Analysi per Aequationes infinitas" (1711), p. 16, and a letter from Newton to Leibnitz dated Oct. 1676 and published in *Leibnitzens Math. Schriften* (London and Berlin) I (1850) pp. 122-147, i.p. 143.

[2] 2. [3] 3. [4] 4. For the *limit* derived via Taylor's series see 6, 14, 20.

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lesson," Cauchy effectively proved the existence of Euler's limit and established the inequality

$$\lim_{n \to \infty} (1 + x/n)^n \leqq \sum_0^\infty x^n/n! \tag{3}$$

for positive values of x. It was the converse inequality which gave difficulty. Some eighty years after Cauchy's recognition of the problem, general theorems from which Euler's statement follows were discovered by Tannery[1], and in the special case of positive x by Beppo-Levi[2]. An interesting step was the discussion of the limit, given by Fort[3] in 1856 without any reference to the series. Writers now have the choice of a wide variety of methods, manipulation of elementary equalities giving rise to most of those not so far mentioned, but nearly all the accounts quoted in the list of references possess individual features.

We give below an elementary discussion which proves that if either the limit or the sum of the series in (1) exists then so does the other and with an equal value. In doing this we assume as already known that, for a fixed m and a positive x, $\lim (1 + x/n)^m = 1$ as $n \to \infty$. Our discussion starts by considering x to be positive and then extending the result to complex values.

Let

$$e_n(x) = (1 + x/n)^n, \qquad E_n(x) = 1 + x + x^2/2! + \ldots + x^n/n! \tag{4}$$

Then with $n \to 1$ and $x > 0$

$$\begin{aligned} e_n(x) = 1 + x + \left(1 - \frac{1}{n}\right)\frac{x^2}{2!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{x^3}{3!} + \ldots \\ + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)\frac{x^n}{n!} < E_n(x), \end{aligned} \tag{5}$$

$$\begin{aligned} e_n(x)\left(1 + \frac{x}{n}\right)^m = 1 + \left(1 + \frac{m}{n}\right)x + \left(1 + \frac{m}{n}\right)\left(1 + \frac{m-1}{n}\right)\frac{x^2}{2!} + \ldots \\ + \left(1 + \frac{m}{n}\right)\left(1 + \frac{m-1}{n}\right)\ldots\left(1 + \frac{m-m}{n}\right)\frac{x^m}{m!} + \ldots > E_m(x). \end{aligned} \tag{6}$$

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[1] 22. This method is used in 1, 8, 13.

[2] For the general theorem see 16, 17. For the application to Euler's limit see 10, 19.

[3] 9. Also 5 and 15.

The first of these inequalities is, of course, frequently used in this connection. We re-arrange these equalities to give

$$\frac{E_m(x)}{(1 + x/n)^m} < e_n(x) < E_n(x) \tag{7}$$

$$e_n(x) < E_n(x) < e_m(x)\,(1 + x/m)^n. \tag{8}$$

If the convergence of $E_n(x)$ to $E(x)$ be assumed, then by choice first of m and then for sufficiently large n, *depending* on m, $E_m(x)/(1 + x/n)^m$ and also $E_n(x)$ will be as near as we please to $E(x)$. The convergence of $e_n(x)$ to $E(x)$ follows from (7). Similarly if the convergence of $e_n(x)$ to $E(x)$ be assumed, then for sufficiently large n and m *depending* on n, both $e_n(x)$ and $e_m(x)(1 + x/m)^n$ are as near as we please to $E(x)$. The convergence of $E_n(x)$ to $E(x)$ then follows from (8). (If we tried to simplify this argument by taking a definite relation between m and n, say $m = n^2$, we should then have to prove that $(1 + x/n^2)^n \to 1$ instead of only $(1 + x/m)^n \to 1$ for fixed n as $m \to \infty$.)

To extend the result to negative, or to complex, values z of x we remark that

$$E_n(z) - e_n(z) = A_{n2}z^3 + A_{n3}z^3 + \ldots + A_{nn}z^n = R_n(z) \text{ (say)}. \tag{9}$$

All the numbers A_{pq} are positive so that

$$|R_n(z)| \leqq A_{n2}|z|^2 + A_{n3}|z|^3 \ldots + A_{nn}|z|^n = R_n(|z|). \tag{10}$$

Our statement for z follows then from the corresponding result for $x = |z|$.

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