ON SOME POLYNOMIALS OF TOUCHARD

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In the preceding paper Touchard considers a set of polynomials $Q_n(x)$ defined by

$$(1) \quad Q_{n+1}(x) = (2x+1)\,Q_n(x) + \frac{n^4}{4n^2-1}\,Q_{n-1}(x), \quad Q_{-n}(x) = 0, \quad Q_0(x) = 1.$$

Touchard uses (1) to compute $Q_n(x)$ for $0 \leqslant n \leqslant 9$ and also finds $Q_n(-\frac{1}{2})$. He remarks however "l'expression générale des polynômes $Q_n(x)$ nous echappe." The object of this note is to derive an explicit expression for $Q_n(x)$.

Under the substitution

$$(2) \qquad Q_n = 2^n \binom{2n}{n}^{-1} W_n$$

the conditions (1) become

$$(3) \quad (n+1)W_{n+1} = (2x+1)(2n+1)\,W_n + n^3 W_{n-1}, \quad W_{-n}(x)=0,\ W_0(x)=1.$$

Now define the generating function

$$(4) \qquad W(t) = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!}.$$

The conditions (3) then imply

$$(5) \quad t(t^2-1)\frac{d^2W}{dt^2} + \{3t^2 + 2(2x+1)t - 1\}\frac{dW}{dt} + (t+2x+1)W = 0,$$
$$W(0) = 1.$$

Equation (5) is a special case of Heun's equation. Its solution can be obtained in the following way: Let

$$(6) \qquad W = (1-t)^{-(2x+1)}w, \quad z = t^2.$$

Then (5) becomes

$$(7) \qquad z(z-1)\frac{d^2w}{dz^2} + \{1-(1-2x)z\}\frac{dw}{dz} - x^2 w = 0, \qquad w(0) = 1.$$

This is the well-known hypergeometric equation. The only solution regular at $z = 0$ and satisfying the boundary condition is

$$(8) \qquad w = F(-x, -x, 1, +z).$$

Hence from (6) we obtain

$$(9) \qquad W = (1-t)^{-(2x+1)}F(-x, -x, -1, t^2).$$

Received January 11, 1956.

https://doi.org/10.4153/CJM-1956-035-9 Published online by Cambridge University Press

Since

$$W_n = \left.\frac{d^n W}{dt^n}\right|_{t=0}$$

(9) implies

$$(10) \quad W_n = (\Gamma(-(2x+1))\,\Gamma^2(-x))^{-1} \sum_{r=0}^{[\frac{1}{2}n]} \binom{2n}{n} \frac{\Gamma(2x+n-2r+1)\,\Gamma^2(r-x)\,(2r)!}{(r!)^2}$$

$$= n! \sum_{r=0}^{[\frac{1}{2}n]} \binom{2x+n-2r}{n-2r} \binom{x}{r}^2$$

By (2) and (10) an explicit expression for $Q_n(x)$ is

$$(11) \qquad Q_n(x) = 2^n n! \binom{2n}{n}^{-1} \sum_{r=0}^{[\frac{1}{2}n]} \binom{2x+n-2r}{n-2r} \binom{x}{r}^2.$$

This of course checks with the values of $Q_n(x)$ computed by Touchard for $0 \leqslant n \leqslant 9$, and also gives his value of $Q_n(-\frac{1}{2})$. Finally, the expression (11) simplifies considerably for x a positive or negative integer and for $2x$ a negative integer. Thus for example

$$(12) \qquad Q_n(0) = 2^n n! \binom{2n}{n}^{-1}$$

and

$$(13) \qquad Q_n(1) = 2^n n! (n^2 + n + 1) \binom{2n}{n}^{-1}.$$

Equation (13) provides still another simple check on values of $Q_n(x)$ computed from the recurrence formula.

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