Correspondence

DEAR EDITOR,

Whilst playing with the roots p and q of the quadratic equation $x^2 + bx + c = 0$, I thought I would try to express $p^{1/3} + q^{1/3}$ in terms of b and c. Accordingly I wrote $(p^{1/3} + q^{1/3})^3 = p + q + 3p^{1/3}q^{1/3}(p^{1/3} + q^{1/3}) = -b + 3c^{1/3}(p^{1/3} + q^{1/3})$. Then I wrote $y = (p^{1/3} + q^{1/3})$ so that

$$y^3 - 3c^{1/3}y + b = 0. \tag{1}$$

I at once realised that I had accidentally derived the formula for the solution of the general cubic equation since any cubic can be put in the form (1). (I dimly remembered that the term in x^2 can always be removed from the general cubic $x^3 + a_1x^2 + a_2x + a_3 = 0$ by a change of variable and I soon discovered this to be $x = x' - a_1/3$.)

Tidying up by writing $-3c^{1/3} = d$, I ended up with the result that $y = p^{1/3} + q^{1/3}$ is the solution of $y^3 + dy + b = 0$ if p and q are the roots of $x^2 + bx - d^3/27 = 0$.

On consulting a textbook I found this to be the celebrated formula published by Cardan in 1573, though it is really due to Tartaglia. I wondered if Tartaglia found it by accident, as I did.

Yours sincerely,

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DEAR EDITOR,

I wish to add a note to the article 'Triangles, surds, and Pell's equation' by Robert J. Clarke (*The Mathematical Gazette*, Vol. 83: No. 497, July 1999, pp. 221-225).

In preparing our students for their university entrance examination, we show them the following technique for determining the principal square root of expressions of the form

$$a \pm b\sqrt{m}$$

where a, b and m are positive integers.

The technique has the advantage of relying on the students' existing background in algebra and therefore requires the memorisation of no new formula.

The first step is to transform the given radical expression to read

$$a \pm 2\sqrt{pm}$$

where p is a positive integer.

This transformation is straightforward if b is even; p is simply the square of half of b. If b is odd, we look for a factor of 4 in m and remove it from the radical. See the third example below. If m has no factor of 4, then we must first multiply and divide by 2, as is illustrated in the final example below.

https://doi.org/10.1017/S0025557200208231 Published online by Cambridge University Press

The key now is to look for two positive integers, x and y, whose sum is a and whose product is pm. If such integers exist, then the principal square root is

$$\left|\sqrt{x} \pm \sqrt{y}\right|$$

Here are some examples:

$$\sqrt{9 - 4\sqrt{5}} = \sqrt{9 - 2\sqrt{20}} = \sqrt{(\sqrt{5} - \sqrt{4})^2} = \sqrt{5} - 2$$

$$\sqrt{23 + 6\sqrt{10}} = \sqrt{23 + 2\sqrt{90}} = \sqrt{(\sqrt{18} + \sqrt{5})^2} = 3\sqrt{2} + \sqrt{5}$$

$$\sqrt{28 - 3\sqrt{12}} = \sqrt{28 - 2(3)\sqrt{3}} = \sqrt{28 - 2\sqrt{27}}$$

$$= \sqrt{(\sqrt{27} - \sqrt{1})^2} = \sqrt{27} - \sqrt{1} = 3\sqrt{3} - 1$$

In an expression such as

$$\sqrt{181 - 44\sqrt{15}}$$

the numbers are so large as to make us reach for a calculator:

$$\sqrt{181 - 44\sqrt{15}} = \sqrt{181 - 2\sqrt{7260}} = \sqrt{(\sqrt{121} - \sqrt{60})^2}$$

$$= 11 - \sqrt{60} = 11 - 2\sqrt{15}.$$

In the expression

$$\sqrt{6 + 2\sqrt{10}}$$

there do not exist two positive integers whose sum is 6 and whose product is 10. Thus we cannot simplify this expression to the form

$$\sqrt{x} + \sqrt{y}$$

where x and y are positive integers.

In the expression

$$\sqrt{2 - \sqrt{3}}$$

we must first multiply and divide by 2 in order to apply the above technique

$$\sqrt{2 - \sqrt{3}} = \sqrt{\frac{4 - 2\sqrt{3}}{2}} = \sqrt{\frac{(\sqrt{3} - \sqrt{1})^2}{2}} = \frac{\sqrt{3} - 1}{\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{2}.$$

Yours sincerely,

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https://doi.org/10.1017/S0025557200208231 Published online by Cambridge University Press