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SUBNORMAL SUBGROUPS OF $E_n(R)$ HAVE NO FREE, NON-ABELIAN QUOTIENTS, WHEN $n \geqq 3$

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It is known that for certain rings R (for example $R=\mathbb{Z}$, the ring of rational integers) the group $GL_2(R)$ contains subnormal subgroups which have free, non-abelian quotients. When such a subgroup has finite index it follows that every countable group is embeddable in a quotient of $GL_2(R)$. (In this case $GL_2(R)$ is said to be *SQ-universal*.) In this note we prove that the existence of subnormal subgroups of $GL_2(R)$ with this property is a phenomenon peculiar to "$n=2$".

For a large class $\mathcal{S}$ of rings (which includes *all* commutative rings) it is shown that, for all $R \in \mathcal{S}$, *no* subnormal subgroup of $E_n(R)$ has a free, non-abelian quotient, when $n \geqq 3$. ($E_n(R)$ is the subgroup of $GL_n(R)$ generated by the elementary matrices.) In addition it is proved that, if $R \in \mathcal{S}$ is an SR_t-ring, for some $t \geqq 2$, then *no* subnormal subgroup of $GL_n(R)$ has a free, non-abelian quotient, when $n \geqq \max(t, 3)$. From the above these results are best possible since $\mathbb{Z}$ is an SR_3 ring.

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Introduction

Throughout this note we use "free" as an abbreviation for "free, non-abelian". Newman [10] has proved that if N is a normal subgroup of $SL_2(\mathbb{Z})$, where $\mathbb{Z}$ is the ring of rational integers, then, with finitely many exceptions, N/N_0 is free, where $N_0 = N \cap \{\pm I_2\}$. Grunewald and Schwermer [2] have proved that $SL_2(\mathcal{O}_d)$ has normal subgroups of finite index with a free quotient, where $\mathcal{O}_d$ is the ring of integers of the field $\mathbb{Q}(\sqrt{d})$ of discriminant $d<0$. (As usual $\mathbb{Q}$ denotes the set of rational numbers.) Let $\mathscr{C} = \mathscr{C}(C, k, P)$ be the coordinate ring of the affine curve obtained by removing a closed point P from a projective curve C over a field k. (The simplest case is the polynomial ring $k[x]$.) Lubotzky [4] has proved that $SL_2(\mathscr{C})$ has a normal subgroup of finite index with a free quotient when k is finite. The author [7], [8] has proved that "almost all" principal congruence subgroups of $SL_2(\mathscr{C})$ have a free quotient. (It follows from a result of P. M. Neumann [9] that $SL_2(R)$ is SQ-universal when $R = \mathbb{Z}, \mathcal{O}_d$ or $\mathscr{C}$ (with k finite).) All the above rings are Dedekind rings. One of the principal purposes of this note is to show that if D is such a ring then subnormal subgroups of $GL_n(D)$ with free quotients exist (if at all) only when $n=2$.

From now on we assume that R is a commutative ring, a von Neumann regular ring or a Banach algebra. (We assume also that R has an identity.) For each $n \geqq 2$ let $E_n(R)$ be the subgroup of $GL_n(R)$ generated by the elementary matrices. Suslin [12, Corollary

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1.4] has proved that $E_n(R)$ is normal in $GL_n(R)$, for all $n \geqq 3$. Vaserstein has proved that $E_n(R)$ is normal in $GL_n(R)$, for all $n \geqq 2$, when R is von Neumann regular or a Banach algebra. (See [**15**, Theorem 1], [**17**, Theorem 1.1].) Using results of Vaserstein [**16**], [**17**] we prove that, for all $n \geqq 3$, no subnormal subgroup of $E_n(R)$ has a free quotient. We also prove that if $GL_n(R)/E_n(R)$ is abelian for some $n \geqq 3$, then no subnormal subgroup of $GL_n(R)$ has a free quotient. For example $GL_n(R)/E_n(R)$ is abelian when R is a GE_n-ring, i.e. $GL_n(R)$ is generated by elementary and diagonal matrices. Moreover $GL_n(R)/E_n(R)$ is abelian when R is an SR_s-ring, for some $s \geqq 2$, and $n \geqq \max(s, 3)$. (See [**1**, p. 231].)

Since every Dedekind ring is an SR_3-ring [**1**, Theorem 3.5, p. 239] the above examples show that the restriction "$n \geqq 3$" in our results is necessary. In addition the subnormality condition is necessary even for SR_2-rings. Let F be a field and suppose that either F has characteristic zero or that F has an element which is transcendental over a subfield. (Every field is an SR_2-ring by [**1**, Theorem 3.5, p. 239].) It is known [**18**, pp. 30–32] that $E_2(F)$ has free subgroups. By means of the natural embedding it follows that $E_n(F)$ has free subgroups, for all $n \geqq 2$. (Clearly this is not true when F is a finite field and it is not true for every infinite field. Suppose for example that F is an infinite algebraic extension of a finite field. Then $GL_n(F)$ is locally finite, for all $n \geqq 2$.)

1. Principal results

Let H, K be subgroups of a group L. We denote by $[H, K]$ the subgroup generated by all the commutators $[h,k] = h^{-1}k^{-1}hk$, where $h \in H$ and $k \in K$. We denote the t-th derived subgroup of H by H^t. We require the following properties of free groups.

Lemma 1.1. *Let F be a free group.*

(i) *If $x, y \in F$, where $x, y \neq 1$ and $[x, y] = 1$, then*

$$x^\alpha = y^\beta,$$

for some non-zero integers α, β.

(ii) *If $N \lhd F$ and $N \neq 1$, then N is not abelian (equivalently not cyclic).*

(iii) *If N_1, $N_2 \lhd F$, and $[N_1, N_2] = 1$, then either $N_1 = 1$ or $N_2 = 1$.*

Proof. Part (i) follows from [**5**, Proposition 2.17, p. 10] and part (ii) follows from [**5**, Proposition 2.19, p. 10].

For part (iii) suppose that $[N_1, N_2] = 1$ and that $N_1, N_2 \neq 1$. Choose $x_i \in N_i$, where $x_i \neq 1$ $(i = 1, 2)$. Then $x_1^\alpha = x_2^\beta$, for some non-zero α, β, by (i). Hence $M = N_1 \cap N_2 \neq 1$ and so $M^1 \neq 1$ by (ii). But $[N_1, N_2] = 1$. Part (iii) follows. □

We recall that throughout this note R is commutative, von Neumann regular or a Banach algebra. Let $\mathfrak{q}$ be a two-sided R-ideal. For each $n \geqq 2$, Let $E_n(R, \mathfrak{q})$ be the normal subgroup of $E_n(R)$ generated by the $\mathfrak{q}$-elementary matrices. Let $f: GL_n(R) \to GL_n(R/\mathfrak{q})$ be

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the natural map. We put $GL_n(R,\mathfrak{q})=\ker f$ and $H_n(R,\mathfrak{q})=f^{-1}(Z)$, where Z is the centre of $GL_n(R/\mathfrak{q})$. (By definition $E_n(R,R)=E_n(R)$ and $GL_n(R,R)=H_n(R,R)=GL_n(R)$.)

The order $o(S)$ of a subgroup S of $GL_n(R)$ is the *smallest* (two-sided) R-ideal $\mathfrak{q}_o$ with the property that $S\leqq H_n(R,\mathfrak{q}_o)$.

We record some basic properties.

Theorem 1.2. *For all R-ideals* $\mathfrak{q}$ *and for all* $n\geqq 3$,

(i) $E_n(R,\mathfrak{q})\lhd GL_n(R)$,

(ii) $[E_n(R),H_n(R,\mathfrak{q})]=E_n(R,\mathfrak{q})$.

Proof. For (i) see [**12**, Corollary 1.4], [**15**, Theorem 1] and [**17**, Theorem 1.1]. For part (ii) see [**14**, Corollary 14], [**15**, Theorem 1] and [**17**, Theorem 1.1]. □

Theorem 1.3. *Let N be a subnormal subgroup of a subgroup G of* $GL_n(R)$, *where* $E_n(R)\leqq G$ and $n\geqq 3$. *Then*

$$E_n(R,\mathfrak{q}^t)\leqq N,$$

for some positive integer t, where $\mathfrak{q}=o(N)$.

Proof. See [**16**, Theorem 1] and [**17**, Theorem 1.3]. □

We require a number of lemmas.

Lemma 1.4. (i) *Let S be a subgroup of* $GL_n(R)$ *and let* $\mathfrak{q}=o(S)$. *Then for all* $n\geqq 2$,

$$[GL_n(R),S]\leqq GL_n(R,\mathfrak{q}).$$

(ii) *Let* S_i *be a subgroup of* $GL_n(R.\mathfrak{q}_i)$, *where* $i=1.2$. *Then for all* $n\geqq 2$,

$$[S_1,S_2]\leqq GL_n(R,\mathfrak{q}^*),$$

where $\mathfrak{q}^*=\mathfrak{q}_1\mathfrak{q}_2+\mathfrak{q}_2\mathfrak{q}_1$.

Proof. See [**6**]. □

We denote by E_{ij} the $n\times n$ matrix where (i,j) entry is 1 and whose other entries are zero, where $1\leqq i,j\leqq n$.

Lemma 1.5. *Let* $\mathfrak{q}_1,\mathfrak{q}_2$ *be R-ideals. Then for all* $n\geqq 3$, *the quotient group*

$$\bar{E}_n(R,\mathfrak{q}_1)/H_n(R,\mathfrak{q}_2),$$

is not free, where $\bar{E}_n(R,\mathfrak{q}_1)=E_n(R,\mathfrak{q}_1)\cdot H_n(R,\mathfrak{q}_2)$.

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Proof. Since $E_n(R, \mathfrak{q}_1) \cdot E_n(R, \mathfrak{q}_2) = E_n(R, \mathfrak{q}_1 + \mathfrak{q}_2)$ we may assume therefore that $\mathfrak{q}_2 \leqq \mathfrak{q}_1$ and that $\mathfrak{q}_1 \neq \mathfrak{q}_2$. Suppose then that for some such $\mathfrak{q}_1, \mathfrak{q}_2$ and some $n \geqq 3$ the quotient group is free.

Choose $q \in \mathfrak{q}_1$, with $q \notin \mathfrak{q}_2$ and consider the matrices $S = I_n + qE_{12}$ and $T = I_n + qE_{13}$. Then by Lemma 1.1(i) it follows that $S^\alpha \equiv T^\beta \pmod{\mathfrak{q}_2}$, for some non-zero α, β. Hence $S^\alpha \equiv I_n \pmod{\mathfrak{q}_2}$. But $S \notin H_n(R, \mathfrak{q}_2)$. The result follows. □

In Lemma 1.5 the restriction "$n \geqq 3$" is necessary. For example, for all but two $\mathbb{Z}$-ideals $\mathfrak{q}$,

$$\bar{E}_2(\mathbb{Z}, \mathfrak{q})/H_2(\mathbb{Z}, \{0\}) \cong E_2(\mathbb{Z}, \mathfrak{q})$$

which is free by [**10**].

We now come to our principal result.

Theorem 1.6. *Let N be a subnormal subgroup of a subgroup G of $GL_n(R)$, where $E_n(R) \leqq G$ and $n \geqq 3$, and let $N_0 \lhd N$.*

If $N/N \cap E_n(R)$ is abelian, then N/N_0 is not free.

Proof. Let $\mathfrak{q}_0 = o(N_o)$, $M = N \cap E_n(R)$ and $P = N \cap H_n(R, \mathfrak{q}_o)$. By Theorems 1.2, 1.3 and Lemma 1.4 there exists a positive integer r such that

$$[M, P^r] \leqq N_o.$$

Suppose now that N/N_0 is free. Then by Lemma 1.1(iii) either $M \leqq N_o$ or $P = N_o$. The first possibility is excluded since N/N_o is not abelian. It follows that $\bar{N}/H_n(R, \mathfrak{q}_o)$ is free, where $\bar{N} = N \cdot H_n(R, \mathfrak{q})$.

Let $\mathfrak{q} = o(N)$. Then, for some positive integer t, $E_n(R, \mathfrak{q}^t) \leqq N$ and so $\mathfrak{q}^t \leqq \mathfrak{q}_o$, by Lemmas 1.1(ii) and 1.5. It follows that

$$[M, N^s] \leqq N_o,$$

for some integer s, by Theorems 1.2, 1.3 and Lemma 1.4, and hence that

$$\text{either} \quad M \leqq N_o \quad \text{or} \quad N^s \leqq N_o,$$

by Lemma 1.1(iii). Either possibility contradicts the fact that N/N_o is free. □

Corollary 1.7. *Let N be a normal subgroup of $E_n(R)$, where $n \geqq 3$, and let $N_o \lhd N$. Then N/N_o is not free.*

Proof. Obvious. □

We recall that R is said to be a GE_n-ring if and only if $GL_n(R)$ is generated by elementary and diagonal matrices.

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Corollary 1.8. *Suppose that R is a GE_n-ring for some $n \geqq 3$ and that N is a subnormal subgroup of a subgroup G of $GL_n(R)$, where $E_n(R) \leqq G$. If $N_o \lhd N$ then N/N_0 is not free.*

Proof. By the Whitehead lemma [1, Proposition 1.7] it follows that $GL_n(R)/E_n(R)$ is abelian. □

By virtue of a result of Suslin [12, Corollary 7.9] Corollary 1.8 holds for all $n \geqq 3$ when R is a polynomial (or a Laurent polynomial) ring over a field or $\mathbb{Z}$.

Corollary 1.9. *Suppose that R is an SR_s-ring for some $s \geqq 2$. Let N be a subnormal subgroup of a subgroup G of $GL_n(R)$, where $E_n(R) \leqq G$ and let $N_o \lhd N$. If $n \geqq \max(s,3)$ then N/N_o is not free.*

Proof. By a well-known theorem of Bass and Vaserstein [13, Theorem 3.2] $GL_n(R)/E_n(R)$ is isomorphic to $K_1(R)$ which is abelian. □

In particular if R is a Dedekind ring and hence an SR_3-ring by [1, Theorem 3.5, p. 239] it follows that when $n \geqq 3$ no subnormal subgroup of $GL_n(R)$ has a free quotient.

2. Concluding remarks

For some R no subnormal subgroup of a subgroup G of $GL_2(R)$, where G contains $E_2(R)$, has a free quotient. We record two examples (both of which are SR_3-rings by [1, Theorem 3.5, p. 239]).

Theorem 3.1. *Let k be a field. Let N be a subnormal subgroup of a subgroup G of $GL_2(k)$, where $E_2(k) \leqq G$, and let $N_0 \lhd N$. Then N/N_0 is not free.*

Proof. We may assume that k is infinite. It is well-known that $SL_2(k) = E_2(k)$ and that $PSL_2(k)$ is simple. In addition $SL_2(k)$ is perfect (by, for example, [1, Proposition 9.4, p. 268]).

Let S be any subgroup of $GL_2(k)$, normalized by $E_2(k)$. Then by the above either $E_2(k) \leqq S$ or $S \cap SL_2(k) \leqq \{\pm I_2\}$. In the former case it follows that $S^1 = E_2(k)$ and in the latter case that S consists of scalar matrices.

It is clear that this holds for the case where $S = N$ or N_0. If N consists of scalar matrices N is abelian and hence N/N_0 is not free. We may assume therefore that $N^1 = E_2(k)$. If $N_0^1 = E_2(k)$, then N/N_0 is abelian. If N_0 consists of scalar matrices, then $M^1 = M$ and $M^1 \neq 1$, where $M = \bar{N}_0/N_0$ with $\bar{N}_0 = N_0 \cdot N^1$. In either case N/N_0 is not free. □

Suppose now that D is the ring of S-integers in a global field K, where S is a finite set of places of K containing all the archimedean places (if any) of K. Then D is a Dedekind ring sometimes called a *Dedekind ring of arithmetic type*. (See [11, p. 489].) By a theorem originally due to Dirichlet D has (at most) finitely many units if and only if $D = \mathbb{Z}$, $D = \mathcal{O}_d$ or $D = \mathcal{C}(C, k, p)$, for some d, C, P and *finite* k.

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Suppose in addition that D has infinitely many units. Liehl [3] has proved that $SL_2(D)=E_2(D)$. Let N be a subnormal subgroup of a subgroup G of $GL_2(D)$, where $G \geqq E_2(D)$. It follows from the proof (slightly adapted) of a result of Serre [**11**, Proposition 2, p. 492] that, either N consists of scalar matrices or $E_2(D, \mathfrak{q}) \leqq N$, for some *non-zero* D-ideal $\mathfrak{q}$. In the former case N is abelian. In the latter case N^1 is non-scalar and, so repeating the above argument for N^1, N/N^1 is finite by [3] and [**11**, Corollaire 1, p. 499]. In either case N has no free quotient. Combining this with the results discussed in the introduction we have:

Theorem 3.2. *If D is a Dedekind ring of arithmetic type then $GL_2(D)$ has subnormal subgroups with free quotients if and only if D has finitely many units.*

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