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GENERALISATION OF KEITH'S CONJECTURE ON 9-REGULAR PARTITIONS AND 3-CORES

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Abstract

Recently, Keith used the theory of modular forms to study 9-regular partitions modulo 2 and 3. He obtained one infinite family of congruences modulo 3, and meanwhile proposed an analogous conjecture. In this note, we show that 9-regular partitions and 3-cores satisfy the same congruences modulo 3. Thus, we first derive several results on 3-cores, and then generalise Keith's conjecture and get a stronger result, which implies that all of Keith's results on congruences modulo 3 are consequences of our result.

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1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. For a positive integer ℓ, a partition is called ℓ-regular if none of its parts is divisible by ℓ. We denote the number of ℓ-regular partitions of n by $b_\ell(n)$, and follow the convention that $b_\ell(0) = 1$. The generating function of $b_\ell(n)$ satisfies

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell; q^\ell)_\infty}{(q; q)_\infty},$$

where $(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2)\cdots$.

In recent years, the divisibility and distribution of $b_\ell(n)$ modulo m has been widely studied in the literature; see [1, 2, 5–10, 13, 15–22]. Recently, infinite families of congruences modulo 2 and modulo 3 for $b_\ell(n)$ have received a great deal of attention. For example, Andrews *et al.* [2] gave some infinite families of congruences modulo 2 and modulo 3 for 4-regular partitions. In [22], Webb studied an infinite family of congruences modulo 3 for 13-regular partitions.

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More recently, Xia and Yao [24] derived several infinite families of congruences modulo 2 for $b_9(n)$. In [14], Keith considered congruences modulo 3 for $b_9(n)$ and obtained the following main result.

THEOREM 1.1 [14, Theorem 3]. *For all $n \geq 0$ and $a \geq 1$,*

$$b_9\Big(4^a n + \frac{10 \cdot 4^{a-1} - 1}{3}\Big) \equiv 0 \pmod{3}. \tag{1.1}$$

Furthermore, Keith proposed the following conjecture on congruences modulo 3 for $b_9(n)$.

CONJECTURE 1.2. For all $n \geq 0, a \geq 1$ and $k = 3, 13, 18$ or 23,

$$b_9\Big(5^{2a} n + \frac{5^{2a-2} - 1}{3} + 5^{2a-2} k\Big) \equiv 0 \pmod{3} \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} b_9(5n + 3) q^n \equiv q \frac{(q^{45}; q^{45})_\infty}{(q^5; q^5)_\infty} \pmod{3}. \tag{1.3}$$

Equating the coefficients of q^{5n+1} and $q^{5n}, q^{5n+2}, q^{5n+3}, q^{5n+4}$ on both sides of (1.3) respectively, we obtain that

$$b_9(25n + 8) \equiv b_9(n) \pmod{3}, \tag{1.4}$$

and for $k = 3, 13, 18$ or 23,

$$b_9(25n + k) \equiv 0 \pmod{3}. \tag{1.5}$$

From (1.4) and (1.5), and induction on a, it is not hard to obtain (1.2).

Thus to confirm the conjecture, we only need to prove (1.3). Recently, Xia and Yao [23] have found a proof of (1.3) by employing some q-series identities discovered by Ramanujan and Hirschhorn. An attempt to present an elementary proof of Keith's conjecture is the motivation for this note.

We aim to derive the following main result, which generalises (1.3) and leads to an affirmative answer to Keith's conjecture.

THEOREM 1.3. *Let $p \equiv 2 \pmod{3}$ be a prime. Then*

$$\sum_{n=0}^{\infty} b_9\Big(pn + \frac{2p - 1}{3}\Big) q^n \equiv q^{(p-2)/3} \frac{(q^{9p}; q^{9p})_\infty}{(q^p; q^p)_\infty} \pmod{3}. \tag{1.6}$$

REMARK. This result follows from Theorem 2.3, which can also be derived by two results of Hirschhorn and Sellers [12]. Based on a special case of Ramanujan's ${}_1\psi_1$ summation formula, Theorem 2.3 is deduced immediately. In contrast to Hirschhorn and Sellers' work, the techniques we use here are more elementary and direct.

2. Some results on 3-cores

Let $a_t(n)$ denote the number of partitions of n that are t-cores, where a partition is a t-core if it has no hook numbers that are divisible by t. Then it is well known that the generating function of $a_t(n)$ satisfies

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}.$$

First we present a simple relation concerning $b_{p^2}(n)$ and $a_p(n)$ for any prime p.

Lemma 2.1. *Given a prime p, for all $n \geq 0$,*

$$b_{p^2}(n) \equiv a_p(n) \pmod{p}. \tag{2.1}$$

Proof. It is straightforward to check that

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}.$$

Replacing q by q^p, we obtain

$$(q^p;q^p)_\infty^p \equiv (q^{p^2};q^{p^2})_\infty \pmod{p}$$

and

$$\frac{(q^{p^2};q^{p^2})_\infty}{(q;q)_\infty} \equiv \frac{(q^p;q^p)_\infty^p}{(q;q)_\infty} \pmod{p}.$$

This yields that $b_{p^2}(n) \equiv a_p(n) \pmod{p}$. □

As an immediate corollary of Lemma 2.1, we have

$$b_9(n) \equiv a_3(n) \pmod{3}, \tag{2.2}$$

which prompts us to shift our attention from 9-regular partitions to 3-cores.

We now introduce the following special case of Ramanujan's ${}_1\psi_1$ summation formula, which will play a key role throughout the note.

Lemma 2.2. *For $|q| < |x| < 1$,*

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{1-aq^n} = \frac{(q;q)_\infty^2(ax;q)_\infty(q/ax;q)_\infty}{(a;q)_\infty(q/a;q)_\infty(x;q)_\infty(q/x;q)_\infty}. \tag{2.3}$$

Proof. For a proof of (2.3), see [2]. □

Theorem 2.3. *Let $p \equiv 2 \pmod{3}$ be a prime. Then*

$$\sum_{n=0}^{\infty} a_3\Big(pn + \frac{2p-1}{3}\Big)q^n = q^{(p-2)/3}\frac{(q^{3p};q^{3p})_\infty^3}{(q^p;q^p)_\infty} \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} a_3\Big(pn + \frac{2p-1}{3}\Big)q^n = \sum_{n=0}^{\infty} a_3(n)q^{pn+(p-2)/3}. \tag{2.5}$$

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Proof. Employing the generating function of $a_3(n)$, the second statement follows immediately from the previous one, so it suffices to prove (2.4).

With q replaced by q^6, x by q^2, a by q in (2.3), we deduce that

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{6n+1}}. \tag{2.6}$$

We first consider the case $p \geq 5$. According to the congruence classes modulo p, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{6n+1}} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{p-1} \frac{q^{2(pn+k)}}{1-q^{6(pn+k)+1}} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{p-1} \sum_{j=0}^{\infty} q^{2(pn+k)+(6(pn+k)+1)j}. \tag{2.7}$$

Write $k_0 = (5p-1)/6$ and $\gamma(p) = (2p-1)/3$. Let $0 \leq k \leq p-1$. If $k \neq k_0$, then $6k+1$ is not divisible by p. Hence, $2k + (6k+1)j \equiv \gamma(p) \pmod{p}$ holds if and only if $j \equiv \gamma(p) \pmod{p}$. If $k = k_0$, it is clear that, for any $j \geq 0$,

$$2(pn+k_0) + (6(pn+k_0)+1)j \equiv \gamma(p) \pmod{p}.$$

Extracting those terms in (2.7) whose power of q is congruent to $\gamma(p)$ modulo p and using (2.6),

$$\begin{aligned}
\sum_{n=0}^{\infty} a_3(pn+\gamma(p))q^{pn+\gamma(p)} &= \sum_{n=-\infty}^{\infty} \sum_{\substack{k=0\\k\neq k_0}}^{p-1} \sum_{j=0}^{\infty} q^{2(pn+k)+(6(pn+k)+1)(pj+\gamma(p))} \\
&\quad + \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} q^{2(pn+k_0)+(6(pn+k_0)+1)j} \\
&= \sum_{n=-\infty}^{\infty} \sum_{\substack{k=0\\k\neq k_0}}^{p-1} \frac{q^{2(pn+k)+\gamma(p)(6(pn+k)+1)}}{1-q^{p(6(pn+k)+1)}} \\
&\quad + \sum_{n=-\infty}^{\infty} \frac{q^{2(pn+k_0)}}{1-q^{6(pn+k_0)+1}}.
\end{aligned}$$

Dividing both sides of the above equation by $q^{\gamma(p)}$ and replacing q^p by q yields

$$\begin{aligned}
\sum_{n=0}^{\infty} a_3(pn+\gamma(p))q^n &= \sum_{n=-\infty}^{\infty} \sum_{\substack{k=0\\k\neq k_0}}^{p-1} \frac{q^{4(pn+k)}}{1-q^{6(pn+k)+1}} + \sum_{n=-\infty}^{\infty} \frac{q^{2n+1}}{1-q^{6n+5}} \\
&= \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{6n+1}} - \sum_{n=-\infty}^{\infty} \frac{q^{4(pn+k_0)}}{1-q^{6(pn+k_0)+1}} + \sum_{n=-\infty}^{\infty} \frac{q^{2n+1}}{1-q^{6n+5}}.
\end{aligned}$$

Employing the fact that

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{6n+1}} = \sum_{n=-\infty}^{\infty} \frac{q^{-4n-4}}{1-q^{-6n-5}} = -\sum_{n=-\infty}^{\infty} \frac{q^{2n+1}}{1-q^{6n+5}},$$

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and applying (2.3) with $x = q^{4p}, a = q^{5p}, q = q^{6p}$,

$$\begin{aligned}\sum_{n=0}^{\infty} a_3(pn+\gamma(p))q^n &= -q^{4k_0}\sum_{n=-\infty}^{\infty}\frac{q^{4pn}}{1-q^{6pn+5p}}\\&= -q^{4k_0}\frac{(q^{6p};q^{6p})_\infty^2(q^{9p};q^{6p})_\infty(q^{-3p};q^{6p})_\infty}{(q^{5p};q^{6p})_\infty(q^{p};q^{6p})_\infty(q^{4p};q^{6p})_\infty(q^{2p};q^{6p})_\infty}\\&= -q^{4k_0}\frac{1-q^{-3p}}{1-q^{3p}}\frac{(q^{6p};q^{6p})_\infty^3(q^{3p};q^{6p})_\infty^3}{(q^p;q^p)_\infty}\\&= q^{(p-2)/3}\frac{(q^{3p};q^{3p})_\infty^3}{(q^p;q^p)_\infty}.\end{aligned}$$

We now consider the case $p = 2$. Since

$$\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{6n+1}} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} q^{2n}q^{(6n+1)k},$$

we have

$$\sum_{n=0}^{\infty} a_3(2n+1)q^{2n+1} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} q^{2n}q^{(6n+1)(2k+1)} = \sum_{n=-\infty}^{\infty}\frac{q^{8n+1}}{1-q^{12n+2}},$$

and thus

$$\begin{aligned}\sum_{n=0}^{\infty} a_3(2n+1)q^n &= \sum_{n=-\infty}^{\infty}\frac{q^{4n}}{1-q^{6n+1}}\\&= \frac{(q^6;q^6)_\infty^2(q^5;q^6)_\infty(q;q^6)_\infty}{(q;q^6)_\infty(q^5;q^6)_\infty(q^4;q^6)_\infty(q^2;q^6)_\infty}\\&= \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty}.\end{aligned}$$

This completes the proof. □

REMARK. Applying (2.2), we can immediately obtain our main result, Theorem 1.3, on 9-regular partitions which is analogous to Theorem 2.3 on 3-cores.

As a consequence of Theorem 2.3, we can obtain the following result by comparing the coefficients of q^n on both sides of (2.5).

THEOREM 2.4. *Let $p \equiv 2 \pmod 3$ be a prime and $0 \le r \le p-1$ be an integer with $r \neq (p-2)/3$. Then, for any $n \geq 0$,*

$$a_3\Big(p^2n + \frac{p^2-1}{3}\Big) = a_3(n) \tag{2.8}$$

and

$$a_3\Big(p^2n + pr + \frac{2p-1}{3}\Big) = 0. \tag{2.9}$$

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Based on Theorem 2.4 and by induction, we have the following corollary.

Corollary 2.5. *Let $p \equiv 2 \pmod 3$ be a prime and $0 \le r \le p-1$ be an integer with $r \neq (p-2)/3$. Then, for any $\alpha \ge 1$ and $n \ge 0$,*

$$a_3\Big(p^{2\alpha}n + \frac{p^{2\alpha}-1}{3}\Big) = a_3(n) \tag{2.10}$$

and

$$a_3\Big(p^{2\alpha}n + p^{2\alpha-1}r + \frac{2p^{2\alpha-1}-1}{3}\Big) = 0. \tag{2.11}$$

Remark. Hirschhorn and Sellers [12] derived (2.9) and (2.10) by establishing an explicit formula for $a_3(n)$ in terms of the factorisation of $3n+1$. Baruah and Nath [3, 4] presented two different proofs of (2.10).

Another application of Lemma 2.2 to the study of 3-cores is as follows.

Theorem 2.6. *For all $n \ge 0$,*

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \tag{2.12}$$

where $d_{r,3}(n)$ denotes the number of positive divisors of n congruent to r modulo 3.

Proof. Replacing q with q^3, x with q, a with $-q$ in (2.3),

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1+q^{3n+1}} = \sum_{n=0}^{\infty}\Big(\frac{q^{2n+1}}{1+q^{3n+2}} + \frac{q^n}{1+q^{3n+1}}\Big). \tag{2.13}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} a_3(n)q^{3n+1} &= \sum_{n=0}^{\infty}\Big(\frac{q^{6n+4}}{1+q^{9n+6}} + \frac{q^{3n+1}}{1+q^{9n+3}}\Big) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}((-1)^k q^{(3n+2)(3k+2)} + (-1)^k q^{(3n+1)(3k+1)}) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(q^{(3n+2)(6k+2)} - q^{(3n+2)(6k+5)} + q^{(3n+1)(6k+1)} - q^{(3n+1)(6k+4)}) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(q^{(3n+1)(3k+1)} - q^{(3n+2)(3k+2)}).
\end{aligned}$$

This yields that

$$a_3(n) = \sum_{\substack{0<d\mid(3n+1)\\ d\equiv 1 \pmod 3}} 1 - \sum_{\substack{0<d\mid(3n+1)\\ d\equiv 2 \pmod 3}} 1 = d_{1,3}(3n+1) - d_{2,3}(3n+1),$$

which completes the proof. □

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REMARK. Theorem 2.6 was first proved by Granville and Ono [11] using the theory of modular forms. Hirschhorn and Sellers [12] presented an elementary proof of (2.12). Recently Baruah and Nath [4] gave a new proof of Theorem 2.6 with the help of a classical result by Lorenz.

3. Generalisation of Keith's conjecture

We now present a stronger result, which includes Keith's conjecture as a special case.

THEOREM 3.1. *Suppose that R is a positive integer and $R = \prod_{i=1}^{n} r_i^{2\gamma_i}$, where the r_i are distinct primes congruent to 2 modulo 3 and $\gamma_i \geq 1$. Then, for all $n \geq 0$,*

$$b_9\Big(Rn + \frac{R-1}{3}\Big) \equiv b_9(n) \pmod 3 \tag{3.1}$$

and

$$b_9(Rn + t) \equiv 0 \pmod 3, \tag{3.2}$$

provided that there exists $1 \leq i \leq n$ such that $\mathrm{ord}_{r_i}(3t+1)$ is odd and less than $2\gamma_i$.

PROOF. Applying (2.2) and (2.10), for any $\alpha \geq 1$ and $n \geq 0$,

$$b_9\Big(p^{2\alpha} n + \frac{p^{2\alpha}-1}{3}\Big) \equiv b_9(n) \pmod 3, \tag{3.3}$$

where $p \equiv 2 \pmod 3$ is a prime.

From (3.3),

$$b_9\Big(r_i^{2\gamma_i} n + \frac{r_i^{2\gamma_i}-1}{3}\Big) \equiv b_9(n) \pmod 3.$$

It follows that if $j \neq i$ then

$$\begin{aligned} b_9(n) &\equiv b_9\Big(r_j^{2\gamma_j}\Big(r_i^{2\gamma_i} n + \frac{r_i^{2\gamma_i}-1}{3}\Big) + \frac{r_j^{2\gamma_j}-1}{3}\Big) \\ &\equiv b_9\Big(r_j^{2\gamma_j} r_i^{2\gamma_i} n + \frac{r_j^{2\gamma_j} r_i^{2\gamma_i}-1}{3}\Big) \pmod 3. \end{aligned}$$

Repeating this argument yields (3.1).

Assume that $\mathrm{ord}_{r_i}(3t+1) = 2\beta_i - 1$ with $1 \leq \beta_i \leq \gamma_i$. Then $3t+1$ can be written as

$$3t + 1 = d \cdot r_i^{2\beta_i - 1}$$

where d is not divisible by r_i. It is easy to see that d must be congruent to 2 modulo 3 and

$$\frac{d-2}{3} \not\equiv \frac{r_i - 2}{3} \pmod{r_i}.$$

Let $(d-2)/3 = lr_i + s$ with $0 \leq s < r_i$. Then we can rewrite $Rn + t$ as

$$Rn + t = r_i^{2\beta_i}\Big(\frac{Rn}{r_i^{2\beta_i}} + l\Big) + r_i^{2\beta_i-1} s + \frac{2r_i^{2\beta_i-1} - 1}{3}.$$

Hence, (3.2) follows immediately from (2.2) and (2.11). This completes the proof. □

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REMARK. For fixed $a \geq 1$, let $R = 5^{2a}$ and $t = \frac{1}{3}(5^{2a-2} - 1) + 5^{2a-2}k$. When $k = 3, 13, 18$ or 23, it is easy to see that $\mathrm{ord}_5(3t + 1) = 2a - 1$. According to the second part of Theorem 3.1, the first claim of Conjecture 1.2 holds.

The following corollary is an immediate consequence of Theorem 3.1.

COROLLARY 3.2. *Let $p \equiv 2 \pmod 3$ be a prime. For $a \geq 1$, $0 \leq b < a$ and $n \geq 0$,*

$$b_9\Big(p^{2a}n + \frac{c_p \cdot p^{2b+1} - 1}{3}\Big) \equiv 0 \pmod 3, \tag{3.4}$$

whenever $c_p \equiv 2 \pmod 3$ and is not divisible by p.

REMARK. In fact Theorem 1.1 is a special case of (3.4). Letting $p = 2$, $b = a - 1$ and $c_2 = 5$, Theorem 1.1 follows immediately.

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