BULL. AUSTRAL. MATH. SOC. 16A76
VOL. 19 (1978), 1-4.

A construction of monogenic near-ring groups, and some applications

S.D. Scott

Let V be a group generated by elements v_1 and v_2 of finite coprime order, and let N be the near-ring generated by the inner automorphism induced by v_1. It is proved that V is a monogenic N-group. Certain consequences of this result are discussed. There exist finite near-rings N with identity generated by a single distributive element μ, such that $\mu^2 = 1$ and where the radical $J_2(N)$ (see Günter Pilz, *Near-rings. The theory and its applications*, 1977, p. 136) is non-nilpotent.

Throughout this paper all groups will be written additively. This does not imply commutativity. All near-rings considered will be left distributive and zero-symmetric.

Let N be a near-ring and V an N-group. We call V *monogenic by* v, if v is an element of V such that $vN = V$ [1, p. 75].

Let V be a group, v an element of V, and k a positive integer. The sum $v + v + \ldots + v$ of v taken k times will be denoted by vk.

PROPOSITION. *Let V be a group and v_1 an element of V. Let α be the inner automorphism of V induced by v_1 and N the near-ring of all maps of V into V generated by α. If v_1 has finite order, then*

Received 17 April 1978.

N has an identity and V is a unitary N-group.

Proof. It is easily seen that V is an N-group. Also if $|v_1| = n$, then $\alpha^n = 1$ is the identity of N and $v1 = v$ for all v in V .

We now prove the following theorem.

THEOREM. *Let V be a group generated by two elements v_1 and v_2 of finite order. Let α be the inner automorphism of V induced by v_1 and N the near-ring of all maps of V into V generated by α . If $|v_1|$ and $|v_2|$ are coprime, then V is monogenic by $v_1 + v_2$.*

Proof. By the above proposition, V is unitary. Thus $(v_1+v_2)1 = v_1 + v_2$ is in $(v_1+v_2)N$. We shall prove by induction that $v_2m + v_1m$ is in $(v_1+v_2)N$ for all positive integers m . Since $v_1 + v_2$ is in $(v_1+v_2)N$, $(v_1+v_2)\alpha = v_2 + v_1$ is in $(v_1+v_2)N$, and the statement holds for $m = 1$. Assume $v_2(m-1) + v_1(m-1)$ is in $(v_1+v_2)N$. It follows that

$$v_1 + v_2 + v_2(m-1) + v_1(m-1) = v_1 + v_2m + v_1(m-1)$$

is in $(v_1+v_2)N$. Now

$$\begin{aligned}(v_1+v_2m+v_1(m-1))\alpha &= -v_1 + v_1 + v_2m + v_1(m-1) + v_1 \\ &= v_2m + v_1m\end{aligned}$$

is in $(v_1+v_2)N$ and the statement holds for all positive integers.

Since $|v_1|$ and $|v_2|$ are coprime, there exists a positive integer r such that $|v_1|r \equiv 1 \bmod |v_2|$. Hence $v_2|v_1|r + v_1|v_1|r = v_2$ is in $(v_1+v_2)N$ and similarly v_1 is in $(v_1+v_2)N$. Thus $(v_1+v_2)N = V$ and the theorem is proved.

In what follows all near-rings have an identity and all N-groups are unitary.

A non-zero N-group V is of *type* 0 (see [1, p. 77]), if there

https://doi.org/10.1017/S0004972700008388 Published online by Cambridge University Press

exists v in V such that $vN = V$ and V contains no proper submodules.

A non-zero N-group is of *type* 2 if it contains no proper N-subgroups. A near-ring with a faithful N-group of type 0 or type 2 will be called 0-*primitive* or 2-*primitive*, respectively (see [*1*, p. 103]).

Many finite non-abelian simple groups have two generators of coprime order. Let V be such a group and v_1, v_2, and N be as in the statement of the above theorem. Now V has no proper submodules as it has no proper normal subgroups. By the theorem proved above, V is a type 0 N-group. The cyclic subgroup of V generated by v_1 is a proper N-subgroup and therefore V is not of type 2. It is not difficult to show that N is 0-primitive but not 2-primitive, and that the radical $J_2(N)$ (see [*1*, p. 136]) of such a near-ring is non-nilpotent. These near-rings are clearly distributively generated. What is more they are generated by a single distributive element.

The projective unimodular groups $P(K, 2)$ of dimension 2 over a finite prime field K where $|K| > 3$ are finite, non-abelian, and simple (see [*2*, p. 122]). Any such group V is generated by an element v_1 of order two and an element v_2 of order three [*2*, p. 166]. Our theorem allows us to construct an infinite family of finite 0-primitive near-rings that are not 2-primitive. Furthermore, any member N of this family is generated by a single distributive element μ such that $\mu^2 = 1$. Thus, even near-rings N generated in this very simple manner may have a non-nilpotent radical $\left(J_2(N)\right)$.

References

[*1*] Günter Pilz, *Near-rings. The theory and its applications* (North-Holland Mathematics Studies, **23**. North-Holland, Amsterdam, New York, Oxford, 1977).

https://doi.org/10.1017/S0004972700008388 Published online by Cambridge University Press

[2] Eugene Schenkman, *Group theory* (Van Nostrand, Princeton, New Jersey; Toronto; New York; London; 1965).

Department of Mathematics,
University of Auckland,
Auckland,
New Zealand.