ON THE GANEA STRONG CATEGORY IN PROPER HOMOTOPY

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(Received 23rd April 1996)

Dedicated to the memory of Dr. W. J. Rae Mitchell

This paper contains some basic relations between Ganea strong category and Lusternik Schnirelmann category in proper homotopy theory. We focus our interest on the case of category 2 in order to show that $\mathbb{R}^n$ is the unique open n-manifold with proper Lusternik-Schnirelmann category 2 ($n \neq 3$).

Introduction

The Lusternik-Schnirelmann category (L-S category) $cat(X)$ of a space X is the smallest number k such that there exists an open cover $\{X_1, \ldots X_k\}$ for which each inclusion $X_j \subseteq X$ is nullhomotopic in X. The L-S category turns to be a homotopy invariant of the space X. See [15] for a survey on L-S category.

Associated to the notion of L-S category is the notion of geometric category $gcat(X)$ due to Fox [10], which is the smallest positive integer k such that X can be covered by k open contractible subsets. The geometric category is not an invariant of the homotopy type of X [10]. The homotopy invariant version of $gcat(X)$ was introduced by T. Ganea in [12]. It is called strong category, $Cat(X)$, and it was originally defined as the least integer $k \geq 1$ with the property that X has the homotopy type of a polyhedron which may be covered by k subpolyhedra each of which is contractible in itself. For a comparison of the invariants $cat(-)$ and $Cat(-)$ see [12] or [22].

In this paper we focus our interest on numerical invariants of the Lusternik-Schnirelmann type for non-compact spaces. Ordinary homotopy invariants do not describe what is happening "far out" towards infinity and more subtle invariants are needed to deal with non-compact spaces. To take the "infinity" into account one has to use proper maps instead of just continuous maps and then proper homotopy invariants replace the usual ones for non-compact spaces.

In [2] L-S category invariants are extended to the proper category $\mathcal{P}$ consisting of locally compact σ-compact Hausdorff spaces and proper maps. In this paper we define

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the proper analogue of a Ganea strong category and we relate it to a proper L-S category. As an application, we prove that any open n-manifold with proper L-S category 2 is 1-LC at ∞ and moreover M has the proper homotopy type of $\mathbb{R}^n$ ((3.13) and (3.15)). From this the Euclidean space $\mathbb{R}^n$ ($n \neq 3$) is characterized as the unique open n-manifold with proper L-S category 2 ((3.13)). This result improves previous results in [2].

1. Preliminaries

We shall deal with the category $\mathcal{P}$ of Hausdorff locally compact σ-compact spaces and proper maps and the full subcategory $\mathcal{P}_0 \subseteq \mathcal{P}$ consisting of connected locally path-connected spaces in $\mathcal{P}$. Recall that a *proper map (p-map)* is a continuous map $f : X \to Y$ such that $f^{-1}(K)$ is compact for each compact subset $K \subseteq Y$. Notice that the spaces in $\mathcal{P}$ always are paracompact ([9, 7.6.12]).

Given a space X in $\mathcal{P}$, a *system of ∞-neighbourhoods* of X is a decreasing sequence $\{W_j\}$ of subsets of X where the closures $K_j = \overline{X - W_j}$ form an increasing sequence of compact subsets with $K_j \subseteq \mathrm{int}K_{j+1}$ and $X = \cup \mathrm{int}K_j$. The sequence $\{K_j\}$ will be called an *exhausting sequence* of compact subsets.

A *Freudenthal end* of X is an element of the inverse limit $\mathcal{F}(X) = \varprojlim \mathcal{U}(W_j)$, where $\mathcal{U}(-)$ stands for the family of unbounded connected components. We shall also consider the family $\mathcal{B}(-)$ of bounded components. A subset $A \subseteq X$ is termed *bounded* (*unbounded*, resp.) if the closure $\overline{A}$ is compact (non-compact, respectively). If $\mathcal{F}(X) = \{*\}$ then X is said to be *one-ended*.

The following lemmas are commonplace in the topology of spaces in $\mathcal{P}$, but they are proved here since we do not know precise references for the proofs.

Lemma 1.1. *Let X be a space in $\mathcal{P}_0$. Given a compact connected subset $K \subseteq X$ and a locally finite sequence $\{A_j\}_{j \geq 1}$ of connected compact subsets of X there exists a connected open neighbourhood of K, U, such that $\overline{U}$ is compact, and $FrU \cap A_j = \emptyset$ for all j. Moreover there also exists a compact connected neighbourhood L of K such that U, $\mathcal{B}(X - L) = \emptyset$, and $FrL \cap A_j = \emptyset$ for all j.*

Proof. Since X is normal, locally connected and locally compact and $\{A_j\}$ is locally finite we can find a finite open cover $\{U_1, \ldots, U_m\}$ of the compact connected subset $K' = K \cup \{A_j; A_j \cap K \neq \emptyset\}$ with $\overline{U_i}$ compact and $\overline{U_i} \subseteq X - \{A_j; A_j \cap K = \emptyset\}$. Let U be the connected component of $\cup\{U_i; 1 \leq i \leq m\}$ containing K'. Clearly $FrU \cap A_j = \emptyset$ and $\overline{U}$ is compact. Let $L = \overline{U} \cup \{D; D \in \mathcal{B}(X - \overline{U})\}$. It is easy to check that L is connected and $\mathcal{U}(X - L) = \mathcal{U}(X - \overline{U})$. Hence $\mathcal{B}(X - L) = \emptyset$. Moreover, $FrL \subseteq FrU$ implies $FrL \cap A_j = \emptyset$ for all j. Finally L is compact. For this let $M \subseteq X$ be a compact set with $\overline{U} \subseteq \mathrm{int}M$. Then the compactness of FrM yields that the family $\mathcal{A}_1$ of components of $X - \overline{U}$ which meet FrM is finite. Then $L - \overline{U}$ is contained in the union of the families $\mathcal{A}_1$ and $\mathcal{A}_2 = \{D \in \mathcal{B}(X - \overline{U}); D \subseteq M\}$. As $\mathcal{A}_1 = \overline{\cup\{D; D \in \mathcal{A}_1\}}$ and $\mathcal{A}_2 = \overline{\cup\{D; D \in \mathcal{A}_2\}} \subseteq M$ are both compact sets it follows that $L \subseteq \overline{U} \cup \mathcal{A}_1 \cup \mathcal{A}_2$ is also compact.

Lemma 1.2. *Let X be a space in $\mathcal{P}_0$. Then there exists a sequence $U_1 \subseteq L_1 \subseteq U_2 \subseteq L_2 \ldots$ such that U_i is a connected open set and L_i is compact with $\mathcal{B}(X - L_i) = \emptyset$. Moreover, if $\{A_j\}_{j \geq 1}$ is a locally finite sequence of compact connected subsets of X the U_i's and L_i's can be chosen in such a way that each A_j is contained in a difference $U_i - L_{i-1}$ $(i \geq 1)$ with $L_0 = \emptyset$.*

Proof. Let $K_1 \subseteq K_2 \ldots$ be an exhausting sequence of compact subsets. Since $K_1 \subseteq intK_2$ we can find a finite open cover $\{W_k\}$ with $W_k \subseteq intK_2$ connected and $\overline{W}_k$ compact. Moreover, we can join the connected components of $W = \cup\{W_k\}$ by finite chains of connected open subsets of compact closures in some $intK_{i_1}$. Let V_1 denote the union of all the open subsets which form these chains. Then V_1 is connected and open and $\overline{V}_1 \subseteq intK_{i_1}$ is compact. Hence the union $H_1 = \overline{V}_1 \cup \{A_j; A_j \cap \overline{V}_1 \neq \emptyset\}$ is compact and connected and contained in some $intK_{j_1}$ with $j_1 \geq i_1$. Since X is normal and $\{A_j\}$ is locally finite we can find an open set Z_1 with $H_1 \subseteq Z_1 \subseteq \overline{Z}_1 \subseteq intK_{j_1} - \cup\{A_j; A_j \cap \overline{V}_1 = \emptyset\}$. Let U_1' denote the connected component of Z_1 which contains H_1. Now we apply (1.1) to $\overline{U_1'}$ and we get an open connected subset U_1 and a compact connected set L_1 with $\overline{U_1'} \subseteq U_1 \subseteq L_1$ and $\mathcal{B}(X - L_1) = \emptyset$ and moreover $A_j \cap FrU_1 = \emptyset$ for all j. Assume $L_1 \subseteq intK_{s_1}$. If we replace K_1 and K_2 by L_1 and K_{s_1} respectively, an inductive process can now be started to construct the required sequence.

The usual notions of homotopy, deformation, homotopy equivalence, etc . . . can be defined in the natural way in $\mathcal{P}$. Actually, the ordinary cylinder functor $IX = X \times I$ endows the category $\mathcal{P}$ of a structure of I-category and hence $\mathcal{P}$ is a cofibration category in the sense of Baues (see [5] and [1]). The symbols $\simeq, \simeq_p$, and $\cong$ stand for homotopy equivalence, p-homotopy equivalence, and homeomorphism respectively. The set of p-homotopy classes of p-maps $f : X \to Y$ will be denoted by $[X, Y]_p$. A *ray* is a proper map $\alpha : J \to X$ where J denotes the *half-line* $J = [0, \infty)$. A *strong end* of X is the p-homotopy class of a ray in $F(X) = [J, X]_p$. Clearly each strong end defines a Freudenthal end. When $F(X) = \{*\}$ we say that X is *strongly one-ended*.

In the category $\mathcal{P}$ the constant map $X \to \{*\}$ is not defined if X is not compact. However the role of the point is played in $\mathcal{P}$ by the half-line J. For this we recall that for any space X in $\mathcal{P}$, there always exists a proper map $r : X \to J$ ([8; 6.3.5]). Moreover the map r is unique up to p-homotopy. We shall briefly describe the construction of such a map r. If $\{U_j\}_{j \geq 0}$ is a system of ∞-neighbourhoods of X with $U_0 = X$, the Tietze Extension Theorem yields continuous maps $r_j = \overline{U_j - U_{j+1}} \to [j, j+1]$ with $r_j(FrU_{j+1}) = j + 1$, $r_j(FrU_j) = j$. It is now clear that the maps r_j define a proper map $r : J \to X$. By using r, we can define (up to p-homotopy) the proper cone C_pX as the push-out in $\mathcal{P}$ of the diagram $X \times I \xleftarrow{i_0} X \xrightarrow{r} J$. Also if $f : X \to Y$ is a proper map, the proper cone of f, C_pf is naturally defined in $\mathcal{P}$. When $f = r$, the proper cone $C_pr = \Sigma_pX$ is called the proper suspension. Since J is contractible it is not hard to check that the proper homotopical constructions in the I-category $\mathcal{P}$ have the same ordinary homotopy type as the ordinary ones.

The following lemma leads us to the proper analogue of Borsuk's homotopy extension theorem ([19, p. 41]). Recall that a space X is said to be an *absolute*

neighbourhood retract (ANR-space) if for any metrizable space Y and any continuous map $f : A \to X$ with $A \subseteq Y$ closed f admits a continuous extension $\tilde{f} : U \to X$ for some neighbourhood U of A.

Lemma 1.3 ([4, 3.5]).   *Let X be a locally compact metrizable space and Y a locally compact ANR in $\mathcal{P}$. Let $C \subseteq X$ be a closed subset and $f, g : C \to Y$ p-homotopic maps. If $W \subseteq X$ is open with $C \subseteq W$ and $\tilde{f}, \tilde{g} : W \to Y$ are continuous extensions of f and g respectively then there exists a closed neighbourhood U of C in W such that $\tilde{f}|_U$ and $\tilde{g}|_U$ are p-homotopic.*

Proposition 1.4.   *Every ANR in $\mathcal{P}$ has the proper homotopy extension property (PEHP) with respect to all pairs (X, A) in $\mathcal{P}$ with A closed and X metrizable.*

Proof.   It is similar to the classical proof in ([19, p. 41]). If V is any closed neighbourhood of A in X and $C = X \times \{0\} \cup A \times I$, let $\phi : X \to I$ be a Urysohn map with $\phi(A) = 1$ and $\phi(X - int V) = \emptyset$. Then $r : X \times I \to X \times I$ given by $r(x, y) = (x, \phi(x)t)$ is a p-map such that $r|_C = id_C$ and $r(X \times I) \subseteq V \times I$.

Now let $H : C \to Y$ a p-map with Y an ANR in $\mathcal{P}$. Since Y is an ANR we can extend H to a continuous map $\tilde{H} : W \to Y$ where W is an open neighbourhood of C in $X \times I$. Then there exists a closed neighbourhood $U \subseteq W$ of C such that $\tilde{H}|_C$ is proper. Here we use (1.3) with $f = g = H$. By using the compactness of I we can find a closed neighbourhood V of A such that $V \times I \subseteq U$. Then $\tilde{H}r : X \times I \to Y$ is a proper extension of H.

We also derive from (1.3) the following lemma which will be used later.

Lemma 1.5.   *Let X be a metrizable ANR-space in $\mathcal{P}$ and let C be closed in X. Assume that the inclusion $i_c : C \subseteq X$ is p-homotopic to a composition of p-maps $C \overset{r}{\to} J \overset{\alpha}{\to} X$. Then there exists a closed neighbourhood U of C and a proper extension $r' : U \to J$ such that the composition $\alpha r'$ is p-homotopic to the inclusion $U \subseteq X$.*

Proof.   Let $r_0 : X \to J$ be any continuous extension of r. Here we use the Tietze Extension Theorem. If $H : i_c \simeq_p \alpha r$ is a p-homotopy we consider the map

$$h = id_X \cup H \cup \alpha r_0 : X \times \{0, 1\} \cup C \times I \to X.$$

As X is an ANR-space we can extend h to $\tilde{h} : W \to X$ for some neighbourhood W of $X \times \{0, 1\} \cup C \times I$ in $X \times I$. By using the compactness of I one finds a neighbourhood V of C in X with $V \times I \subseteq W$. Let $r_1 = r_0|V$. Then $\tilde{h}|V \times I$ yields a homotopy $i_v \simeq \alpha r_1$. As an immediate consequence of (1.3) we get a closed neighbourhood $U \subseteq V$ and a p-homotopy $i_U \simeq_p \alpha r'$ where $r' = r_1|U$.

2. Numerical proper homotopy invariants

In [2] the proper L-S category of a space in $\mathcal{P}$ was defined. Namely, a closed subset $A \subseteq X$ is said to be *properly deformable to J* in X if there exists a diagram in $\mathcal{P}$

$$
\begin{array}{ccc}
 & \xrightarrow{\ r\ } & J \\
A & & \downarrow{\scriptstyle \alpha} \\
 & \searrow_{i_A} & \\
 & & X
\end{array}
$$
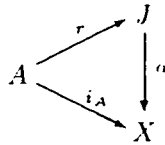
which is commutative up to p-homotopy. That is, there exists a p-homotopy $H : i_A \simeq_p \alpha r$. Notice that we can use the restriction to A of any proper map $r : X \to J$ but the p-homotopy class of $\alpha : J \to X$ can depend on A. An open cover of X $\{V_i\}$ is said to be properly categorical (p-categorical) if each closure $\overline{V}_i$ is p-deformable to J in X. The *proper Lusternik Schnirelmann category* (p-category) of X, $p - cat(X)$, is the least number n such that X admits a p-categorical open cover with n elements. It is not hard to check that $p - cat(-)$ is a proper homotopy invariant. Moreover, the inequality $p - cat(X) \geq max\{Card(\mathcal{F}(X)), cat(X)\}$ always holds. Here cat denotes the ordinary L-S category. Also it is clear that $\mathbb{R}^n$ verifies that $p - cat(\mathbb{R}^n) = 2$. As in ordinary homotopy closed covers can be used in the definition of proper L-S category for ANR in $\mathcal{P}$. This is an immediate consequence of (1.5).

The proper analogue of Borsuk's homotopy extension theorem (1.4) yields

Proposition 2.1. *Let X be a metrizable ANR in $\mathcal{P}$ and let $P \subseteq X$ be a closed subspace p-deformable to J in X. If $\{C_i\}_{i\geq 1}$ is a locally finite sequence of pairwise disjoint compact subspaces of P and $x_i \in C_i$ there exists a p-deformation of $\cup_{i\geq 1} C_i$ to $A = \{x_i\}_{i\geq 1}$ relative to A. Moreover, this p-deformation can be extended to a p-deformation of P to J in X.*

Proof. The result is a consequence of the proper homotopy extension property (PHEP) in (1.4). We shall give some details. Assume we have a diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\ r\ } & J \\
P & & \downarrow{\scriptstyle \alpha} \\
 & \searrow_{i_P} & \\
 & & X
\end{array}
$$
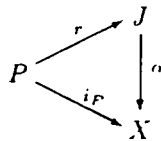
and a p-homotopy $H : i_P \simeq_p \alpha r$. Since any two p-maps in J are p-homotopic we can use the PHEP of $(P, \cup C_i)$ to change r for a new r_1 with $r_1(C_i) = i \in \mathbb{N}$ for all $i \geq 1$. We have a new p-homotopy $H^1 : i_P \simeq_p \alpha r_1$. By using the PHEP of $(J, \mathbb{N})$ we replace α by $\alpha_1 \simeq_p \alpha$ with $\alpha_1(i) = x_i$. Here we use the paths $H^1|_{\{x_i\}\times I}$ between $\alpha(i)$ and x_i given by the p-homotopy H^1. Let $H^2 : i_P \simeq_p \alpha_1 r_1$ be a p-homotopy.

By using again the (PEHP) of $(J, \mathbb{N})$ we find α_2 and a p-homotopy $F : \alpha_1 \simeq \alpha_2$ such that $F(i, t) = H^2(x_i, 1 - t)$ for all $i \in \mathbb{N}$. Then by putting together the homotopies H^2 and $F(r \times id)$ we get a new p-homotopy $H^3 : i_P \simeq \alpha_2 r_1$ such that $H^3|\{x_i\} \times I$ is a loop γ_i homotopic to the constant path c_{x_i}. Moreover the homotopy $G_i : \gamma_i \simeq c_{x_i}$ rel $\{x_i\} \times \{0, 1\}$ can be chosen with $G_i(\{x_i\} \times I \times I) \subseteq H^2(\{x_i\} \times I)$ and so we have a p-homotopy $G = \cup G_i : A \times I \times I \to X$ rel $A \times \{0, 1\}$. Finally we apply the (PEHP) to the pair.

$$(P \times I \times I, P \times I \times \{0\} \cup P \times \{0, 1\} \times I \cup A \times I \times I)$$

to extend $H^3 \cup i_P \times \{0\} \times I \cup \alpha_2 r_1 \times \{1\} \times I \cup G$ to a p-homotopy $H^4 : P \times I \times I \to X$. Then $H^4|P \times I \times \{1\} : i_P \simeq \alpha_2 r_1$ is the required p-deformation of P extending a p-deformation of C onto A relatively to A.

A subspace $A \subseteq X$ in $\mathcal{P}$ will be called *properly admissible* if the family of connected components of A is locally finite in X (and hence countable).

Lemma 2.2. *Let $\{U_1, \ldots, U_n\}$ be an open p-categorical cover of a space X in $\mathcal{P}_0$. Then there exists an open p-categorical cover $\{U'_1, \ldots, U'_n\}$ such that each $\overline{U'_i}$ is properly admissible.*

Proof. Let $K_1 \subseteq K_2 \ldots$ be an exhausting sequence of compact subsets of X. Let $\mathcal{C}_m(U_i)$ denote the family of path components of $U_i - K_m$. Here $m \geq 0$ with $K_0 = \emptyset$. Since X is locally path connected each $C \in \mathcal{C}_m(U_i)$ is an open subset. By compactness there exist finite families $\mathcal{C}'_m(U_i) \subseteq \mathcal{C}_m(U_i)$ such that $\overline{K_{m+1} - K_m} \subseteq \cup\{C; C \in \mathcal{C}'_{m-1}(U_i)\}$ $(m \geq 0)$. Here $\mathcal{C}'_{-1}(U_i) = \mathcal{C}_0(U_i) = \mathcal{C}(U_i)$. For each i we define $U'_i = \cup\{C; C \in \mathcal{C}'_m(U_i); m \geq 0\}$. It is clear that $\{U'_i\}_{1 \leq i \leq n}$ is an open cover and it is p-categorical since $U'_i \subseteq U_i$. Moreover the family $\mathcal{C}(U'_i)$ of path components of U'_i is locally finite since any point in K_m only meets path components of $\mathcal{C}'_j(U_i)$ with $j \leq m - 1$. Furthermore, if $D \subseteq \overline{U'_i}$ is a connected component then $D = \cup\{\overline{C}; C \in \mathcal{C}(U'_i)$ and $C \subseteq D\}$. Indeed, since D is closed D contains the union of closures. Conversely, if $x \in D$ we have that $x \in \overline{C}$ with $C \in \mathcal{C}(U'_i)$ since $\mathcal{C}(U'_i)$ is locally finite. As $\overline{C} \subseteq \overline{U'_i}$ is connected it follows $\overline{C} \subseteq D$. Finally the family of connected components of $\overline{U'_i}$ is locally finite since given an open G which only meets the path components $C_1 \ldots C_k$ of U'_i then G only meets the components $D \subseteq \overline{U'_i}$ which contain some C_s $(1 \leq s \leq k)$.

Proposition 2.3. *Assume X is a one-ended metrizable ANR-space in $\mathcal{P}_0$ and $H_1(X; \mathbb{Z}_2) = 0$. If P is a properly admissible closed subset p-deformable to J with $\mathcal{U}(P) \neq \emptyset$ there exists a closed subset $P' \subseteq X$ with $P \subseteq P'$, $\mathcal{B}(P') = \emptyset$, and such that P' is p-deformable to J in X.*

Proof. Let $\emptyset = L_0 \subseteq U_1 \subseteq L_1 \ldots$ be an increasing sequence with the properties of (1.2). Moreover we can assume that each bounded component $B \in \mathcal{B}(P)$ is contained in some difference $D_k = U_k - L_{k-1}$ $(k \geq 1)$. We shall also use the differences $D^j_k = U_k - L_{j-1}$ for $j \leq k$. Note $D^k_k = D_k$. Since both $X - L_j$ and U_k are open connected

and hence path connected sets, the reduced Mayer-Vietoris exact sequence yields the isomorphism

$$0 = H_1(X; \mathbb{Z}_2) \to \tilde{H}_0(D_k^j; \mathbb{Z}_2) \to \tilde{H}_0(X - L_{j-1}; \mathbb{Z}_2) \oplus \tilde{H}_0(U_k; \mathbb{Z}_2) \to \tilde{H}_0(X; \mathbb{Z}_2) = 0$$

and so each D_k^j is path connected.

In order to connect all the bounded components of P to unbounded components we now proceed in each difference D_k in the same way as in ([10, 16.2]). Namely, we construct for each $k \geq 1$ a union ("forest") $F_k = \cup T_j^k$ of pairwise disjoint directed "trees" $T_j^k \subseteq D_k$ whose "vertices" are all bounded components $B \in \mathcal{B}(P)$ with $B \subseteq D_k$ together with some points in D_k. Furthermore we shall construct each F_k in such a way that the source vertices are bounded components and the sink vertices are points of unbounded components which meet D_k. Hence for each bounded component $B \subseteq T_j^k$ there will be only one directed path γ_B from a point $x_B \in B$ to some unbounded component $U_B \in \mathcal{U}(P)$. The forest F_k is constructed inductively as follows. Assume we have already constructed a forest F_k'. Let m_k' denote the number of bounded components in $D_k - F_k'$. Then we consider an arc γ from a bounded component $B \subseteq D_k - F_k'$ to some unbounded component U with $U \cap D_k \neq \emptyset$. Let p be the first point in $\gamma \cap (F_k' \cap U)$ and let q be the last point in the subarc from B to p belonging to some component $B' \subseteq D_k - F_k'$. Let $\gamma' \subseteq \gamma$ be the subarc between q and p, and let $F_k'' = F_k' \cup \gamma' \cup B'$. Then $m_k'' = m_k' - 1$. Hence any forest F_k' can be extended to a forest F_k with $m_k = 0$. The result follows by starting with $F_k' = \emptyset$.

Once we have constructed a complete forest F_k in each D_k, we follow the above argument to extend $F_k \cup F_{k-1}$ to a new forest F_k^{k-1} in the difference D_k^{k-1} containing all components in D_k^{k-1}. Then $F = \cup\{F_k^{k-1}; k \geq 2\}$ is a locally finite forest in X containing all components in $\mathcal{B}(P)$. Now we use (2.1) to choose a p-deformation of P relatively to each point x_B, $B \in \mathcal{B}(P)$. The construction of the (directed) forest yields an order to push each tree $T \subseteq F$ onto $\Omega_P = \cup\{U; U \in \mathcal{U}(P)\}$. Note that this deformation is proper since any x_B is pushed along a finite tree inside D_k^{k-1} if $B \subseteq D_k^{k-1}$. Hence $P' = P \cup F$ is p-deformable onto P in X and so P' is p-deformable to J in X. This finishes the proof.

Proposition 2.4. *Let X be a one-ended ANR-space in $\mathcal{P}_0$ with $H_1(X; \mathbb{Z}_2) = 0$. Assume that $\{P, Q\}$ is a properly admissible p-categorical cover of X. Then there exists a new closed cover $\{P', Q'\}$ with the same properties and such that the intersection $P' \cap Q'$ also is properly admissible. Moreover, if $\mathcal{B}(P) = \mathcal{B}(Q) = \emptyset$ then $\mathcal{B}(P' \cap Q') = \emptyset$.*

Proof. Let $\{C_m\}$ be the connected components of P. If $\{K_i\}$ is an exhausting sequence of compact subsets of X there exists an increasing sequence of positive integers $m_1 < m_2 \dots$ such that for each $i \geq 1$ $C_m \cap K_i = \emptyset$ if $m \geq m_i$ since $\{C_m\}$ is locally finite. As X is normal and locally connected we can inductively construct connected open subsets U_m with $C_m \subseteq U_m \subseteq \overline{U}_m \subseteq X - (\cup_{j<m}\overline{U}_j) \cup (\cup_{k>m} C_k)$, $U_m \cap U_{m'} = \emptyset$ if $m \neq m'$ and $U_m \subseteq X - K_i$ if $m \geq m_i$. Hence $\{\overline{U}_m\}$ is a locally finite family of pairwise disjoint closed

sets and we define $P' = \cup\{\overline{U}_m\}$. Notice that P' can be assumed p-deformable to J in X by (1.5), and $\mathcal{B}(P') = \emptyset$ if $\mathcal{B}(P) = \emptyset$ since the components of P' are $\overline{U}_m$. Similarly we can construct $Q' = \cup\{\overline{W}_n\}$ with W_n connected and open. On the other hand, since $H_1(X; \mathbb{Z}_2) = 0$ the Mayer-Vietoris exact sequence yields the short exact sequence

$$0 = H_1(X; \mathbb{Z}_2) \xrightarrow{\Delta} H_0(U \cap W; \mathbb{Z}_2) \to H_0(U; \mathbb{Z}_2) \oplus H_0(W; \mathbb{Z}_2) \to H_0(X; \mathbb{Z}_2) = \mathbb{Z}_2 \to 0$$

where $U = \cup\{U_m\}$ and $W = \cup\{W_n\}$. This implies that the (path) components of $U \cap W$ are of the form $U_m \cap W_n$. Moreover we next show that the components of $P' \cap Q'$ are of the form $\overline{U_m \cap W_n}$. Indeed, let $C \subseteq P' \cap Q'$ be a component. Then there exist m and n such that $C \subseteq \overline{U}_m \cap \overline{W}_n$. But $\overline{U_m \cap W_n} = \overline{U}_m \cap \overline{W}_n$ since any $x \in \overline{U}_m \cap \overline{W}_m$ with $x \notin U_m$ necessarily belongs to W_n because $X = U \cup W$. Hence, for any open neighbourhood Ω of x, $\Omega \cap W_n \cap U_m \neq \emptyset$ and so $x \in \overline{U_m \cap W_n}$. Similarly if $x \notin W_n$. As $U_m \cap W_n$ is a (path) component of $U \cap W$ then $\overline{U_m \cap W_n} \subseteq P' \cap Q'$ is connected and hence $C = \overline{U_n \cap W_m}$.

Finally, if $\mathcal{B}(P) = \mathcal{B}(Q) = \emptyset$ the U_n and W_n are unbounded for all m, n. Assume that for some intersection $U_m \cap W_n$ is bounded. Then there exists a compact subset $K \subseteq X$ with $U_m \cap W_n \subseteq K$. As X is one-ended we can find another compact subset L with $K \subseteq L$ and such that any two points in $X - L$ can be joined by an arc in $X - K$. Since U_m and W_n are unbounded we can find points $x \in U_m - L$ and $y \in W_n - L$. Let $\gamma \subseteq X - K$ be a path joining x to y. In addition paths $\alpha \subseteq U_m$ and $\beta \subseteq W_n$ can be found joining $x_0 \in U_m \cap W_n$ to x and y respectively. In this way we have constructed a loop $\xi = \gamma \cup \alpha \cup \beta$ which represents an element $w_\xi \in H_1(X; \mathbb{Z}_2) = 0$, and its image $\Delta(w_\xi) \in H_0(U \cap W; \mathbb{Z}_2)$ is the sum of components of $U \cap W$ which are crossed by ξ when ξ passed from U to W. As γ joins U_m to W_n outside $U_m \cap W_n$, it follows that $\Delta(w_\xi) \neq 0$ which is a contradiction. Therefore any intersection $U_m \cap W_n$ is unbounded and hence any component $C \subseteq P' \cap Q'$ is unbounded.

In addition to the L-S category, another numerical homotopy invariant was introduced by T. Ganea ([12]) with the name of strong category. This notation admits the following generalization to proper homotopy. Given a space X in $\mathcal{P}$, the *proper strong category* of X is the smallest number $p - Cat(X) = k$ such that there exists a space Y in $\mathcal{P}$ with $X \simeq_p Y$ and such that Y is covered with the interiors of k closed subspaces $Y_i \simeq_p J$ $(1 \leq i \leq k)$. Many results concerning strong category have their corresponding analogues in proper homotopy. We shall prove below two of them (cf. ([6], 5.4]) and ([6], 5.5.])) which will be used later on. The key point in the proofs is the fact that $\mathcal{P}$ is an I-category in the sense of [5]. In particular, categorical wedge and cone constructions are defined in $\mathcal{P}$ up to p-homotopy equivalence. Moreover, if $i : A \rightarrowtail X$ is a p-cofibration in $\mathcal{P}$, the cofibre of i is defined up to p-homotopy by the push-out diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle r}\downarrow & & \downarrow \\ J & \longrightarrow & X/^p A \end{array}$$

It can be shown that $X/{}^p A$ is p-homotopically equivalent to the proper mapping cone of i $C_p i$. In general any space X p-homotopically equivalent to $X/{}^p A$ will be called the homotopy cofibre of i.

Proposition 2.5.   *Let X be a space in $\mathcal{P}$ such that $p - cat(X) \leq n$. Then X is a p-retract of a space $\tilde{X}$ in $\mathcal{P}$ with $p - Cat(\tilde{X}) \leq n$ and conversely.*

Proof.   Let $\{W_k\}_{1 \leq k \leq n}$ be a p-categorical open cover of X. Then we have diagrams in $\mathcal{P}$

$$\begin{array}{ccc} & \xrightarrow{\ r_k\ } & J \\ \overline{W}_k & & \Big\downarrow a_k \\ & \searrow{\scriptstyle i_k} & \\ & & X \end{array}$$
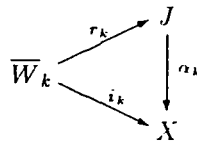
which are commutative up to p-homotopies H^k. Let $C_p \overline{W}_k$ denote the proper cone constructed by using r_k. The homotopies H^k and the push-out properties of $C_p \overline{W}_k$ yield p-maps $\tilde{H}^k : C_p \overline{W}_k \to X$ such that $\tilde{H}^k|\overline{W}_k$ is the inclusion i_k. Hence the union of the p-maps $\tilde{H}^k$ give a p-retract $r : \tilde{X} = \cup\{C_p \overline{W}_k\} \to X$. Clearly each cone $C_p \overline{W}_k$ is p-homotopically trivial, and so $p - Cat(\tilde{X}) \leq n$. The converse is easy, since the existence of a p-retraction as above implies $p - cat(X) \leq p - Cat(\tilde{X})$.

Proposition 2.6.   *If X is a space in $\mathcal{P}$ then $p - Cat(X) \leq n + k$ if and only if X has the p-homotopy type of the push-out, $Z_A(X_0, X_1)$, of a diagram $X_0 \sqcup X_1 \leftarrow A \sqcup A \overset{(i_0, i_1)}{\rightarrowtail} A \times I$ (double mapping cylinder) where $p - Cat(X_0) \leq n$ and $p - cat(X_1) \leq k$.*

Proof.   The sufficiency is clear. Let now $Y \simeq_p X$ with $Y = \cup_{1 \leq i \leq n+k} int\, Y_i$, Y_i closed and $Y_i \simeq_p J$. Then $X_0 = \cup_{1 \leq i \leq n} Y_i = \cup_{1 \leq i \leq n} int_{X_0} Y_i$ and $X_1 = \cup_{n+1 \leq i \leq n+k} Y_i = \cup_{n+1 \leq i \leq n+k} int_{X_1} Y_i$ verify $p - Cat(X_0) \leq n$ and $p - Cat(X_1) \leq k$. Moreover, we claim that Y has the same p-homotopy type as the double mapping cylinder $Z_A(X_0, X_1)$ where $A = X_0 \cap X_1$. Indeed, we consider the natural closed embedding $Z_A(X_0, X_1) \subseteq Y \times I$ with $X_0 \subseteq Y \times \{0\}$ and $X_1 \subseteq Y \times \{1\}$. As Y is paracompact let $\{\phi_i\}$ be a partition of unity subordinate to the open cover $\{int\, Y_i\}$. We now define the p-map $\psi : Y \to Y \times I$ by $\psi(y) = (y, \phi_{X_1}(y))$ where $\phi_{X_1} = \Sigma\{\phi_i; i \geq n + 1\}$. Clearly, $\phi_{X_1}(y) = 0$ and $\phi_{X_1}(y) = 1$ if $y \notin X_1$ and $y \notin X_0$ respectively. Therefore $\psi(Y) \subseteq Z_A(X_0, X_1)$.

Let $\psi' : Y \to Z_A(X_0, X_1)$ and $q : Z_A(X_0, X_1) \to Y$ denote the restrictions of ψ and the projection $\pi_1 : Y \times I \to Y$ respectively. Clearly $q\psi' = id_Y$. Moreover the formula $H(w, s) = (\pi_1(w), (1 - s)\phi'_{X_1}(w) + s\pi_2(w))$ where $\phi'_{X_1}(w) = \phi_{X_1}\pi_1(w)$ yields a well-defined p-map which gives us a p-homotopy $H : \psi'q \simeq_p id_{Z_A(X_0, X_1)}$.

Corollary 2.7.   (cf. [12, 2.1])) *If X is a connected space in $\mathcal{P}$ $p = Cat(X) \leq n$ if and only if there exists a cofibration $i : A \rightarrowtail Y$ in $\mathcal{P}$ such that $p - Cat(Y) \leq n - 1$ and X is the homotopy cofibre of i.*

Proof. According to (2.6) we have $X \simeq_p Z_A(X_0, X_1)$ for a certain double mapping cylinder with $X_0 \simeq_p J$ and $p - Cat(X_1) \le n - 1$. By general properties of cofibration categories ([**5**, I.1.1]) the natural p-map $X_0 \rightarrowtail Z_A(X_0, X_1)$ is a cofibration in $\mathcal{P}$ and the p-homotopy equivalence $X_0 \simeq_p J$ yields a p-homotopy equivalence $Z_A(X_0, X_1) \simeq_p C = Z_A(X_0, X_1)/^p X_0$. As X is connected we have $A \ne \emptyset$ and clearly C is homeomorphic to the mapping cone of $i : A \hookrightarrow X_1$ which is the push-out of the diagram $J \leftarrow A \rightarrowtail M_i$ where M_i denotes the mapping cylinder of i in $\mathcal{P}$. As $M_i \simeq_p X_1$, $p - Cat(M_i) \le n - 1$ and the result follows.

Remark 2.8. Corollary (2.7) can be restated in the following way. The space X verifies $p - Cat(X) \le n$ if and only if there exist cofibration sequences $A_i \rightarrowtail X_i \rightarrow X_{i+1}$ $(1 \le i \le n - 1)$ in $\mathcal{P}$ where $X_1 \simeq_p J$ and $X_n \simeq_p X$. Similarly, the Cornea cone-length invariant ([**7**]) can also be defined in $\mathcal{P}$. Namely, the proper cone-length of X is $\le n$ if there exist cofibration sequences as above where $A_i = \Sigma_p^i Z_i$ for some space Z_i in $\mathcal{P}$. These definitions suggest natural abstract notions of Ganea strong category and Cornea cone-length in cofibration categories. The abstract notions of both L-S category and Cornea cone-length have been already developed for J-categories in [**14**] and [**7**] respectively. Since the proper category $\mathcal{P}$ provides an example of cofibration category which does not have a natural structure of J-category, it seems to be interesting to develop the basic properties of abstract numerical homotopy invariants in the setting of cofibration categories.

Incidentally the proper analogue of the Takens bound $Cat \le cat + 1$ ([**24**]) still holds for strongly one-ended spaces (cf. [**6**]). In fact, the short proof of this result due to Ganea ([**24**, § 5]) is based on results available in any cofibration category. Hence one can prove

Proposition 2.9. *If X is a strongly one-ended connected space in $\mathcal{P}$ then $p - Cat(X) \le p - cat(X) + 1$.*

Proof. Let $\{U_1, \ldots, U_k\}$ be a family of p-categorical closed subsets with $X = \cup int U_i$. If $\tilde{X} = \cup C_p U_i$ is the space in the proof of (2.5), we clearly have $p - Cat(\tilde{X}) \le k$. Moreover, the homotopy invariance of the push-out construction in any cofibration category ([**5**, II.1.2]) yields $\tilde{X} \simeq_p X \vee (\Sigma_p U_1) \vee \ldots \vee (\Sigma_p U_k)$ where "$\vee$" denotes the wedge in $\mathcal{P}$. As X is strongly one-ended we can assume that all the proper wedges are constructed along the same ray $\alpha : J \to X$. Now we get $X \simeq_p X \vee C_p Z$ where $Z = \Sigma_p U_1 \vee \ldots \vee \Sigma_p U_k$, and so $p - Cat(X) \le k + 1$ by (2.7).

Remark 2.10. If K is the Berstein-Hilton space with $cat(K) = 2 < 3 = Cat(X)$ ([**22**]), it is clear that $X = K \times \mathbb{R}$ verifies $p - cat(X) = 4 < 6 = p - Cat(X)$. However, we do not know whether Proposition 2.9 is true for spaces with only one Freudenthal end.

We now apply (2.7) above to show (cf. ([**22**, 2.2])).

Corollary 2.11. *Let X be a connected space in $\mathcal{P}$. Then $p - Cat(X) \leq 2$ if and only if X has the p-homotopy type of a proper suspension in $\mathcal{P}$. Moreover, if $X = \operatorname{int} P \cup \operatorname{int} Q$ with P, Q p-homotopically trivial closed subspaces then $X \simeq_p \Sigma_p(P \cap Q)$.*

Proof. According to the proof of (2.7) one gets that X has the same p-homotopy type as the cofibre of the inclusion $i : P \cap Q \rightarrowtail P$. Hence $X \simeq_p P/{}^p P \cap Q \simeq_p C_p i$. As $P \simeq_p J$, it follows that $X \simeq_p \Sigma_p(P \cap Q)$.

As an immediate consequence of (2.5) and (2.11) we get

Corollary 2.12. *If X is a connected space in $\mathcal{P}$ with $p - cat(X) \leq 2$ then X is a p-retract of a proper suspension. Furthermore, if $X = \operatorname{int} P \cup \operatorname{int} Q$ with P and Q p-deformable closed subspaces then $X \simeq_p \Sigma_p(P \cap Q)$.*

3. Proper category and the fundamental pro-group

It is known that if a space has L-S category 2 then its fundamental group is free ([10, 23.1]). In this section we state some results concerning spaces with proper category 2. In particular we prove that the proper homotopy type of an open manifold with proper category 2 is $\mathbb{R}^n$. For this we recall the basic algebraic pro-invariants in proper homotopy theory.

Given a category $\mathcal{C}$, the *category of towers* of $\mathcal{C}$, $tow - \mathcal{C}$, is the category of inverse sequences $\mathbf{A} = \{A_1 \leftarrow A_2 \leftarrow \ldots\}$ in $\mathcal{C}$ and pro-morphisms. See [23] for a complete reference about pro-categories. We are interested in the full subcategory of $\mathcal{M}or(tow - \mathcal{C})$ whose objects are arrows $\underline{f} : \underline{X} \to A$ where $\underline{X}$ is a $(tow - \mathcal{C})$-object and A is a $\mathcal{C}$-object regarded as a constant tower whose bonding maps are the identity. This category is denoted $(tow - \mathcal{C}, \mathcal{C})$. A $(tow - \mathcal{C}, \mathcal{C})$-morphism from $\underline{f} : \underline{X} \to A$ to $g : \underline{Y} \to B$ can be regarded as a $\mathcal{C}$-morphism between A and B and a $(tow - \mathcal{C})$-morphism from $\underline{X}$ to $\underline{Y}$ such that both morphisms are compatible via the bonding maps. We shall use the above constructions for $\mathcal{C} = \mathcal{G}r, \mathcal{A}b$, the categories of groups and abelian groups respectively.

Given a ray $\alpha : J \to X$, the *n-th homotopy pro-group* of (X, α) is the inverse sequence in $(tow - \mathcal{G}r, \mathcal{G}r)$

$$pro - \pi_n(X, \alpha) = \{\pi_n(X, x_0) \leftarrow \pi_n(U_1, x_1) \leftarrow \pi_n(U_2, x_2) \leftarrow \ldots\}$$

where $\{U_j\}$ is a system of ∞-neighbourhoods, $x_j = \alpha(t_j)$ with $\alpha([t_j, \infty)) \subseteq U_j$, and the bonding morphisms are induced by inclusions and base-point change isomorphisms. When $n = 1$ $pro - \pi_1(X, \alpha)$ will be called the *fundamental pro-group* of (X, α). It is known that the set of strong ends of X, $F(X)$ is in $1 - 1$ correspondence with the (pointed) set $\varprojlim^1 pro - \pi_1(X, \alpha)$ (see [21]).

The *n-th homology pro-group* of X is defined as the inverse sequence in $(tow - \mathcal{A}b, \mathcal{A}b)$

$$pro - H_n(X) = \{H_n(X) \leftarrow H_n(U_1) \leftarrow H_n(U_2) \leftarrow \ldots\}.$$

Next we describe the fundamental pro-group of proper suspensions.

Lemma 3.1. *If $\Sigma_p X$ is the proper suspension of X in $\mathcal{P}$, $pro - \pi_1(\Sigma_p X, \alpha)$ is pro-isomorphic to an inverse sequence of free groups.*

Proof. Let $r : X \to J$ be any p-map. We can use the ∞-neighbourhood system $\{U_j\}$ of X with $U_j = r^{-1}([j, \infty))$. In this way, it is easily checked that $\{\Sigma_p U_j\}$ is a system of ∞-neighbourhoods of the proper suspension $\Sigma_p X$, where $\Sigma_p U_j$ is constructed by using $r|_{U_j} : U_j \to [j, \infty)$. As the proper suspension has the same ordinary homotopy type as the ordinary suspension, one gets that $pro - \pi_1(\Sigma_p X, \alpha)$ is pro-isomorphic to the inverse sequence of free groups $\pi_1(\Sigma_p U_j, x_j)$.

Remark 3.2. An explicit description of the pro-group $pro - \pi_1(\Sigma_p X, \alpha)$ can be given as follows. With the notation of the proof of (3.1) we fix a component $c_0^i \in H_0(U_i; \mathbb{Z}_2)$ for each i. Then each component $c^i \in H_0(U_i; \mathbb{Z}_2)$ $c^i \neq c_0^i$, defines a generator $x(c_i, c_i^0) \in \pi_1(\Sigma_p U_i, x_j)$ represented by a loop in $\Sigma_p U_i$ which only meets U_i in two points, one in c^i and the other in c_0^i. Then $pro - \pi_1(\Sigma_p X, \alpha)$ is pro-isomorphic to the inverse sequence

$$\underline{\mathbf{S}} = \{\sqcup_{(c, c_0)} \mathbb{Z} \leftarrow \sqcup_{(c^1, c_0^1)} \mathbb{Z} \leftarrow \ldots\}$$

where $\alpha : J \to \Sigma_p X$ is the canonical ray, "$\sqcup$" denotes free product and the bonding morphisms of $\underline{\mathbf{S}}$ carry $x(c^i, c_0^i)$ to $x(c^{i-1}, c_0^{i-1})$ if $c^i + c_0^i = c^{i-1} + c_0^{i-1}$ in $H_0(U_{i-1}; \mathbb{Z}_2)$.

Lemma 3.3. *If X is a properly admissible space in $\mathcal{P}$ and $\mathcal{B}(X) = \emptyset$ the bonding morphisms of $\underline{\mathbf{S}}$ are onto.*

Proof. If $\{C_j\}_{j \geq 1}$ are the components, we can assume that $C_j \subseteq U_{j-1}$ and the result follows from the definition of $\underline{\mathbf{S}}$.

Lemma 3.4. *If X is a properly admissible space in $\mathcal{P}$ and $\mathcal{U}(X) = \emptyset$, the bonding morphisms of $\underline{\mathbf{S}}$ are induced by basis inclusions.*

Proof. Since $\mathcal{U}(X) = \emptyset$ all the components $\{C_j\}_{j \geq 1}$ of X are compact, and we can choose $U_i = \cup_{j \geq i} C_j$. Now the result follows easily from the definition of $\underline{\mathbf{S}}$.

Corollary 3.5. *If X is a connected space in $\mathcal{P}$ with $p - cat(X) \leq 2$ then $pro - \pi_1(X)$ is pro-isomorphic to an inverse sequence of free groups.*

Proof. The inverse sequence $pro - \pi_1(X, \alpha)$ is pro-isomorphic to its image in $pro - \pi_1(\Sigma_p Y, \alpha)$ for some Y in $\mathcal{P}$ according to (2.12). Now the result follows from (3.1).

Corollary 3.6.   *Let X be a one-ended connected space in $\mathcal{P}$ such that $X = \operatorname{int}P \cup \operatorname{int}Q$ with P, Q properly admissible closed subspaces p-deformable to J in X. Assume that $\mathcal{B}(P \cap Q) = \emptyset$. Then X is strongly one-ended.*

Proof.   According to (3.3) all the bonding morphisms of the inverse sequence $pro - \pi_1(\Sigma_p(P \cap Q), \alpha)$ are onto, and so $\varprojlim{}^1 pro - \pi_1(\Sigma_p(P \cap Q), \alpha) = \{[\alpha]\}$ (see ([**19**, II.6.2])). Hence $\Sigma_p(P \cap Q)$ has only one strong end. Finally, by (2.12) we know that there exists a p-retraction $r : Z \to X$ where $Z \simeq_p \Sigma_p(P \cap Q)$. This retraction implies that X is strongly one-ended.

Corollary 3.7.   *Let X be an one-ended metrizable ANR in $\mathcal{P}_0$ such that $pro - H_1(X)$ is trivial in $(tow - \mathcal{A}b, \mathcal{A}b)$. Assume that $p - cat(X) = 2$. Then $pro - \pi_1(X, \alpha)$ is trivial in $(tow - \mathcal{G}r, \mathcal{G}r)$.*

Recall that a space X is 1-LC at ∞ if for any ∞-neighbourhood $U \subseteq X$ there exists an ∞-neighbourhood $V \subseteq U$ such that any loop in V is homotopically trivial in U. Hence the triviality of $pro - \pi_1(X, \alpha)$ is equivalent to say that X is simply connected and 1-LC at ∞.

Proof.   Let P, Q be two properly admissible closed subspaces p-deformable to J with $X = \operatorname{int}P \cup \operatorname{int}Q$ (see (2.2)). If $\mathcal{U}(P)$ or $\mathcal{U}(Q)$ is empty then $\mathcal{U}(P \cap Q) = \emptyset$. Otherwise we can apply (2.3) and (2.4) to assume that $\mathcal{B}(P \cap Q) = \emptyset$. Now the result follows from (2.12) and (3.2) and the following result.

Proposition 3.8.   *Let $\underline{A} = \{A_0 \leftarrow A_1 \ldots \leftarrow \ldots\}$ be an inverse sequence of groups which is a retract in $(tow - \mathcal{G}r, \mathcal{G}r)$ of one of the following inverse sequences*

$$(a) \quad \underline{L} = \{L_0 \overset{k_1}{\leftarrow} L_1 \overset{k_2}{\leftarrow} L_2 \ldots\}$$

$$(b) \quad \underline{P} = \{P_0 \overset{p_1}{\leftarrow} P_2 \overset{p_2}{\leftarrow} P_2 \ldots\}$$

where P_i and L_i are free groups, k_i are basis inclusions, and p_i are epimorphisms. Then $\underline{A}$ is trivial in $(tow - \mathcal{G}r, \mathcal{G}r)$ if and only if its abelianization $\underline{A}^{ab} = \{A_0^{ab} \leftarrow A_1^{ab} \ldots \leftarrow \ldots\}$ is trivial in $(tow - \mathcal{A}b, \mathcal{A}b)$.

In the proof of (3.8) we shall use Lemma 3.9 below on commutator subgroups. Given a group G, the subgroup $[G, G]^n$ ($n \geq 1$) is defined inductively as $[G, G]^1 = [G, G]$ (the commutator subgroup of G) and $[G, G]^n = [[G, G]^{n-1}, [G, G]^{n-1}]$. Moreover if F is a free group the subgroup $G \subseteq F$ is called a *free factor* if a basis of G can be extended to a basis of the whole of F.

Lemma 3.9.   *Let $G \hookrightarrow F$ a free factor. Then $[G, G]^n$ is a free factor of $[F, F]^n$ and $[F, F]^n \cap G = [G, G]^n$ for all $i \geq 1$.*

Proof. The commutator $[F, F]$ consists of all words $w \in F$ such that the sum of coefficients of each generator in w is trivial. This directly implies $[G, G] = [F, F] \cap G$ since G is a free factor. In particular, this equality yields $[F, F]^2 \cap G = [F, F]^2 \cap [G, G]$. Now we apply ([18, Ex. 32, p. 117]) to $[F, F] \subseteq F$ to show that $[G, G]$ is a free factor in $[F, F]$, and then $[G, G]^2 = [F, F]^2 \cap [G, G] = [F, F]^2 \cap G$ by the same result. We can now proceed inductively.

Proof of (3.8). Assume $\underline{A}^{ab}$ is trivial in $(tow - \mathcal{A}b, \mathcal{A}b)$. If $\underline{A} \xrightarrow{k} \underline{P} \xrightarrow{r} \underline{A}$ is a retraction in $(tow - \mathcal{G}r, \mathcal{G}r)$, then $\mathbf{k}$ is a monomorphism in $(tow - \mathcal{G}r, \mathcal{G}r)$ and there exists a pro-isomorhism $\underline{A} \stackrel{\phi}{\cong} \underline{B} = \{B_i\}$ with $B_i = Imk_i$ for a representative $\{k_i : A_{m_i} \to P_i\}$ of $\mathbf{k}$. Hence we have a retract $\underline{B} \xrightarrow{k'} \underline{P} \xrightarrow{r'} \underline{B}$ by using ϕ.

On the other hand $\underline{B}$ is also pro-isomorphic to the inverse sequence $\underline{C} = \{C_i\}$ with $C_i = Imr'_i \subseteq B_i$ for a representative $\{r'_i : P_{n_i} \to B_i\}$ of $\mathbf{r}'$. In particular, $\underline{C}^{ab} = 0$. Moreover, since the bonding morphisms of $\underline{P}$ are onto, it follows that the bonding morphisms of $\underline{C}$ are also onto. As the abelianization functor preserves epimorphisms, it follows $C_i^{ab} = 0$ for all i, and hence $C_i = 0$ since $C_i \subseteq B_i$ is a free group. Therefore $\underline{A} = 0$.

Assume now that $\underline{A} \xrightarrow{k} \underline{L} \xrightarrow{r} \underline{A}$ is a retraction in $(tow - \mathcal{G}r, \mathcal{G}r)$. As above we can replace this retract by a new retract $\underline{B} \xrightarrow{k'} \underline{L} \xrightarrow{r'} \underline{B}$ where $\underline{B}$ is an inverse sequence of free groups. Moreover, as the bonding morphisms of $\underline{L}$ are injective the bonding morphisms of $\underline{B}$ are also injective. More explicitly, we have an increasing sequence of positive integers $n_1 < n_2 < \dots$ and commutative diagrams
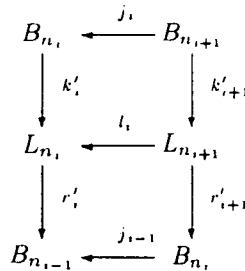
where all morphisms are injective except the r_i's, and $r'_{i+1}k'_{i+1} = j_i$ for all i.

As $\underline{A}^{ab} = \underline{B}^{ab} = 0$ we can assume without loss of generality that $j_i^{ab} = 0$ and hence $B_{n_{i+1}}$ is contained in the commutator subgroup $[B_{n_i}, B_{n_i}] \subseteq [L_{n_i}, L_{n_i}]$. Therefore $B_{n_{i+2}} \subseteq L_{n_{i+2}} \cap [L_{n_i}, L_{n_i}]^2$, and so $B_{n_{i+2}} \subseteq [L_{n_{i+2}}, L_{n_{i+2}}]^2$ by (3.9). In fact the same argument shows $B_{n_i} \subseteq [L_{n_i}, L_{n_i}]^2$ for all $i \geq 3$. Hence $B_{n_{i+2}} \subseteq [B_{n_{i+1}}, B_{n_{i+1}}] \subseteq [L_{n_{i+1}}, L_{n_{i+1}}]^3 \subseteq [L_{n_i}, L_{n_i}]^3$. Then (3.9) yields $B_{n_{i+2}} \subseteq [L_{n_{i+2}}, L_{n_{i+2}}]^3$; that is, the inclusion $B_{n_i} \subseteq [L_{n_i}, L_{n_i}]^3$ holds for all $i \geq 3$ and we now apply inductively (3.9) to get all $i \geq 3$ $B_{n_i} \subseteq \cap_{m \geq 1}[L_{n_i}, L_{n_i}]^m = 0$ (see [17, I.3.4]). Therefore $\underline{A} \cong \underline{B} = 0$ in $(tow - \mathcal{G}r, \mathcal{G}r)$.

We devote the rest of the paper to characterize the manifolds whose proper L-S category is 2. The proof of the next result is part of the proof of ([**2**, 3.6]).

Proposition 3.10.   *If M is a connected open n-manifold with $p - cat(M) = 2$ then M is homologically trivial and simply connected. In particular M is contractible.*

We briefly recall the proof. As $cat(M) \leq p - cat(M) = 2$ the fundamental group $\pi_1(M)$ is free by ([**10**, 23.1]). Also $p - cat(M) = 2$ implies that $cup - length(M) = 1$ for the cup-product $H_c^*(M; \mathbb{Z}_p) \oplus H^*(M; \mathbb{Z}_p) \to H_c^*(M; \mathbb{Z}_p)$ for all prime p. Here H_c^* is the cohomolgy of compact supports. This fact together with the Poincaré Duality yield that M is homologically trivial and as $H_1(M) = abelianized(\pi_1(M))$ is free with the same rank as $\pi_1(M)$ it follows that M is also simply connected. The result follows from the Whitehead Theorem ([**26**, IV.7.13]).

The following result gives us the pro-homology at ∞ of homologically trivial manifolds.

Proposition 3.11.   *Any homologically trivial connected open n manifold is one-ended and $pro - H_q(M) = 0$ if $q \neq 0$, $n - 1$ and $pro - H_p(M)$ is pro-isomorphic to the constant inverse sequence $\{\mathbb{Z} = \mathbb{Z} = \ldots\}$ for $p = 0, n - 1$.*

Proof.   The manifold M is one-ended by ([**13**, I.7]). Let H_*^∞ and H_*^e denote the homology of infinite cycles and the homology at infinity respectively. Then the exact sequence ([**20**, p. 306])

$$\ldots \to H_q(M) \to H_q^\infty(M) \to H_q^e(M) \to H_{q-1}(M) \to \ldots$$

yields isomorphisms $H_q^\infty(M) \cong H_q^e(M)$ for $q \geq 2$ and $H_1^e(M) \cong H_0(M) \cong \mathbb{Z}$ since $H_0^\infty(M) = 0$. Now the Poincaré Duality $H_q^\infty(M) \cong H^{n-q}(M)$ yields that $H_q^\infty(M) = 0$ if $q \neq n$. If $\{W_j\}$ denotes a system of ∞-neighbourhoods of M we have the Milnor exact sequence ([**3**, 2.1]).

$$0 \to \lim_{\leftarrow}{}^1\{H_{q+1}(W_j) \to H_q^e(M) \to \lim_{\leftarrow}\{H_q(W_j)\} \to 0$$

and the groups $\lim_{\leftarrow}{}^1\{H_p(W_j)\}$ and $\lim_{\leftarrow}\{H_q(W_j)\}$ are trivial if $p \geq 0$ and $q \neq 0$, $n - 1$ and $\lim_{\leftarrow}\{H_q(W_j)\} \cong \mathbb{Z}$ if $q = 0$, $n - 1$. Hence the inverse sequence $\{H_q(W_j)\}$ is trivial if $q \neq 0$, $n - 1$ by ([**19**, 6.2 Lemma 2]) and $\{H_q(W_j)\}$ is pro-isomorphic to the constant sequence $\{\mathbb{Z} = \mathbb{Z} = \ldots\}$ if $q = 0, n - 1$ by ([**19**, 6.2 Corollary 8]).

As an immediate consequence of (3.7), (3.10), and (3.11) one gets

Corollary 3.12.   *Any connected open n-manifold M $(n \geq 3)$ with $p - cat(M) = 2$ is $1 - LC$ at ∞.*

As a consequence of (3.10) and (3.12), the homotopical characterizations of Euclidean spaces due to Siebenmann ([23]) and Freedman ([11]) can be restated in the following way

Theorem 3.13. (cf. [2, 3.6]) *A connected open n-manifold M ($n \neq 3$) is homeomorphic to $\mathbb{R}^n$ if and only if $p - cat(M) = 2$.*

Remark 3.14. The characterization in dimension 3 depends on the Poincaré Conjecture and (3.13) can be only stated for irreducible open 3-manifolds as a consequence of [25]. We recall that a 3-manifold is irreducible if every embedded 2-sphere bounds a 3-cell.

For any dimension we can use basic results of proper homotopy to show

Theorem 3.15. (cf. [15, p. 336]) *Let M be a connected (triangulated) open n-manifold. Then $p - cat(M) = 2$ if and only if M has the same proper homotopy type as $\mathbb{R}^n$.*

Proof. Assume $p - cat(M) = 2$. Then M is contractible by (3.10) and so $M = \mathbb{R}^n$ for $n \leq 2$. If $n \geq 3$ we know (see the proof of (3.11)) that $H_q^{\infty}(M) = 0$ if $q \neq n$. Since $pro - \pi_1(M) = 0$ by (3.7), the proper Hurewicz Theorem for CW-complexes in ([3, 3.4]) yields $H_n^{\infty}(M) \cong [\mathbb{R}^n, M]_p$, and the fundamental class $\mu_M \in H^{\infty}(M)$ can be represented by a p-map $f : \mathbb{R}^n \to M$. Moreover $f_* : H_n^{\infty}(\mathbb{R}^n) \to H_n^{\infty}(M)$ is an isomorphism and by ([3, 3.8]) we conclude that f is a p-homotopy equivalence.

Remark 3.16. Since any open n-manifold has the p-homotopy type of a finite dimensional locally finite polyhedron ([16, p. 123]), Theorem (3.15) actually holds for any topological manifold.

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