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A CHARACTERIZATION OF STARSHAPED SETS

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1. Introduction. In 1946 Krasnoselskii proved that if every $n + 1$ points of a compact connected set S in a Euclidean space E_n can see at least one point of S via S then S is starshaped [**1**]. This result was expanded by Robkin in 1965 [**2**]. In this paper, we characterize starshaped sets in E_n by a local arcwise convexity property relative to a point p. We also show that a closed connected set $S \subset E_n$ is arcwise convex relative to a point p in S if and only if it is locally arcwise convex relative to p. This result extends the scope of Tietze's theorem which asserts that for a closed connected set $S \subset E_n$ to be convex it is necessary and sufficient that S be locally convex [**4**].

2. Preliminaries. The terminology follows Valentine [**5**]. For example, the closure, complement, interior and convex hull of a set S are denoted by cl S, compl S, int S, conv S, respectively. If x and y are distinct points, then $L(x, y)$ = line determined by x and y, xy = closed line segment joining x and y, intv xy = relative interior of segment xy, and $R_x(y)$ = ray having x as end-point and containing y. The expression $\Delta(p, x, y)$ denotes the simplex with vertices p, x and y. All points and sets are taken to be in E_n for $n \geqq 2$.

A set S is *starshaped* relative to a point p if for each point $x \in S$, it is true that $px \subset S$. Thus starshapedness is a weak convexity assumption. Moreover, a starshaped set is polygonally connected and arcwise connected. The latter notion was studied by Valentine [**3**]. The following concept is related to arcwise connectedness and starshapedness.

Definition. Let p be a fixed point. A set S is *p-arcwise convex* if each pair of points $x \in S$, $y \in S$ can be joined by a convex arc $C(x, y)$ lying in $S \cap \Delta(p, x, y)$. A convex arc $C(x, y)$ joining x and y with $C(x, y)$ contained in $\Delta(p, x, y)$ is called a *p-arc*. A set S is *locally p-arcwise convex at $z \in S$* is there exists a neighborhood N of z such that $N \cap S$ is p-arcwise convex. A set S is *locally p-arcwise convex* if it is locally p-arcwise convex at each of its points.

We note that the p-arc $C(p, x)$ joining p and x is the line segment px itself Thus, if $p \in S$ and if S is locally p-arcwise convex, then there exists a neighborhood N of p such that $N \cap S$ is starshaped relative to p. Loosely speaking, these assumptions on S give localized starshapedness at p. In the following, we will establish that if $p \in S$ and S is locally p-arcwise convex, then S is starshaped relative to p.

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The following diagram indicates the interrelation of the various properties of a closed connected set S containing a point p:

Local convexity $\Leftrightarrow$ convexity $\Rightarrow$ starshapedness relative to $p \Leftrightarrow$ local p-arcwise convexity $\Leftrightarrow$ p-arcwise convexity.

The first equivalence is Tietze's theorem and the second implication follows from definitions. We will prove the last two equivalences in this paper.

3. The main results.

THEOREM 1. *Let $S \subset E_n$ be a closed connected set containing a point p. The set S is starshaped relative to p if and only if S is locally p-arcwise convex.*

THEOREM 2. *A necessary and sufficient condition for a closed connected set $S \subset E_n$ containing a point p to be p-arcwise convex is that it be locally p-arcwise convex.*

In order to prove Theorem 1 we need the following two lemmas.

LEMMA 1. *Let $S \subset E_2$ be a closed set and p a fixed point in E_2. If S is a locally p-arcwise convex, then the complement of S has no nonempty bounded components.*

Proof. Suppose there exists a nonempty bounded component H in compl S. The set H, being a component of the open set compl S in E_2, is open. Choose a point $q \in H$ with $q \neq p$. Let $K(p, q)$ denote the line through p and q and let $M(q)$ be a line through q perpendicular to $K(p, q)$. Let A^+ and A^- denote the closed half-planes bounded by $M(q)$ such that $p \in A^-$ and $p \notin A^+$. The closed convex set cl (conv H) $\cap A^+$, being a closed subset of the compact set conv (cl H), is compact. It is a convex body whose boundary is not contained in cl (conv H) $\cap$ $M(q)$. Thus, there exists an exposed point z of cl (conv H) such that $z \notin M(q)$ [**5**]. Since $z \in$ cl (conv H), every neighborhood of z contains a point of H. Because $H \subset$ compl S and $z \notin H$, the point z lies in bd S. Since S is closed, $z \in S$. Since $z \in S$ and S is locally p-arcwise convex, there is a neighborhood G of z such that $G \cap S$ is p-arcwise convex. We may choose G to be convex. There are the following two cases: (a) $z \notin K(p, q)$; and (b) $z \in K(p, q)$.

Case (a). $z \notin K(p, q)$. Since $z \notin M(q)$, we may assume that G is contained in int A^+ and in one of the open half-planes bounded by $K(p, q)$. In particular, we note that $q \notin G$. Let L_1 be a line of support to cl (conv H) through z for which $L_1 \cap$ cl (conv H) $= \{z\}$. Choose a point $t \in H =$ int H sufficiently close to z so that there exists a line L_2 through t parallel to L_1 with $L_2 \neq L_1$ such that $L_2 \cap$ cl (conv H) $\subset G$. From the choice of z, G, L_1 and L_2, it is clear that q and z lie in distinct open half-planes bounded by L_2. See Figure 1.

Let ab denote the intersection of L_2 and cl (conv H) where $t \in$ intv ab. The points a and b are boundary points of cl (conv H) and do not lie in H. Since $t \in$ (intv ab) $\cap H$, $H \cap ab \neq \emptyset$. Let $cd \subset ab$ be such that $cd \cap H =$ intv cd and that $t \in$ intv cd. The points c and d are bounadry points of S and are

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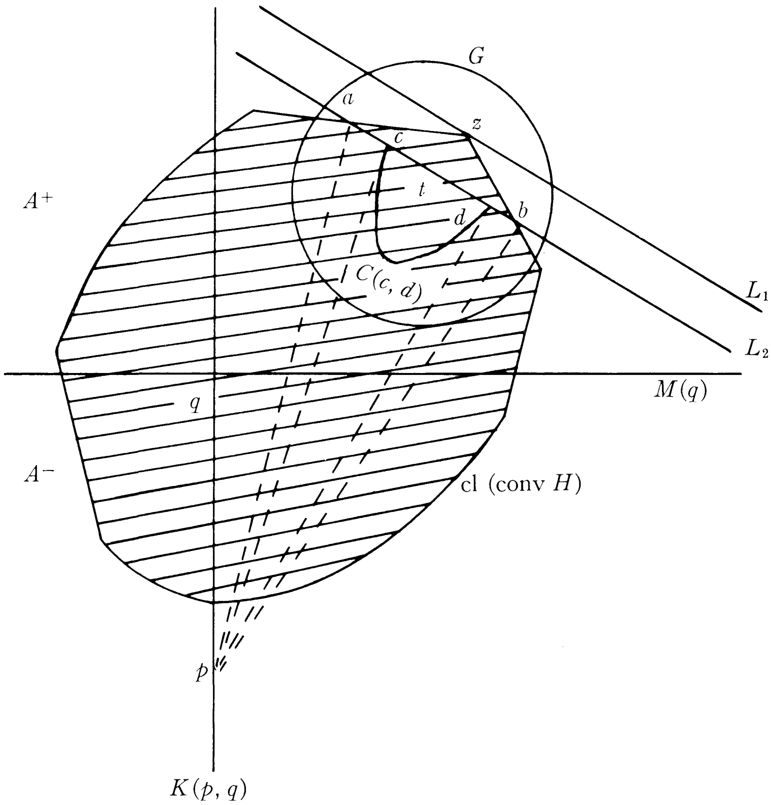

FIGURE 1

boundary points of $L_2 \cap H$. Hence, c and d are in S. With c, d in $G \cap S$, we can find a p-arc $C(c, d)$ joining c and d in S. From the choice of z and G, and since c and d lie in G, clearly $q \notin \Delta(p, c, d)$. Let R_c be the ray on L_2 with endpoint c not containing t and let R_d be the ray on L_2 with endpoint d not containing t. It follows from the choice of the points c and d that $H \cap (R_c \cup R_d) = \emptyset$. Let $C = R_c \cup C(c, d) \cup R_d$. Since $(R_c \cup R_d) \cap H = \emptyset$ and $C(c, d) \subset S$, we have $C \cap H = \emptyset$. The curve C separates the set H, with points of H lying arbitrarily close to z and the point $q \in H$ on distinct sides of the curve C. This contradicts the connectedness of H.

Case (b). $z \in K(p, q)$. We use the same notations and arguments as in (a). In this case, we choose a convex neighborhood G of such that $G \cap S$ is p-arcwise convex with $G \subset \text{int } A^+$ and $G \subset \text{int conv } [L(p, q_1), L(p, q_2)]$ where q_1 and q_2 are distinct points in $H \cap M(q)$ with $q \in \text{intv } q_1q_2$. The latter condition can be fulfilled since $q \in \text{int } H$ and the points p, q and z are collinear. As in (a), we obtain a curve C disjoint from H separating the set H with points of

H arbitrarily close to z on one side of C and the points q_1 and q_2 of H on the other side of C, contradicting the connectedness of H. Hence, H is empty and the complement of S has no nonempty bounded components.

LEMMA 2. *Let p be a fixed point in E_2 and let $S \subset E_2$ be a closed connected set which is locally p-arcwise convex. If x and y are two distinct points in S that can be joined by a convex arc $J(x, y)$ in S such that $J(x, y) \subset \operatorname{conv}[R_p(x), R_p(y)]$ and such that $J(x, y) \cap \operatorname{int} \Delta(p, x, y) = \emptyset$, then $xy \subset S$.*

Proof. Let $J(x, y)$ be as described in Lemma 2. Since compl S has no nonempty bounded components by Lemma 1 and since the set S is closed, we may take $J(x, y)$ to be the minimal such arc in S in the sense that if $J_1(x, y)$ is any other convex arc joining x and y in S with $J_1(x, y) \subset \operatorname{conv}[R_p(x), R_p(y)]$ and such that $J_1(x, y) \cap \operatorname{int} \Delta(p, x, y) = \emptyset$, it is true that $\operatorname{conv} J(x, y) \subset \operatorname{conv} J_1(x, y)$.

Suppose $J(x, y) \neq xy$. Consider the case where the points p, x, and y are not collinear. There exists an exposed point z of $\operatorname{conv} J(x, y)$ with $z \in J(x, y) - \{x, y\}$. Let N be a convex neighborhood such that $N \cap S$ is p-arcwise convex with $N \cap xy = \emptyset$. Let L be a line of support to $\operatorname{conv} J(x, y)$ through z for which $L \cap \operatorname{conv} J(x, y) = \{z\}$. Since $J(x, y)$ is compact, so is $\operatorname{conv} J(x, y)$. Because z is a boundary point of the convex body conv $(J(x, y)$, there exists an interior point of $\operatorname{conv} J(x, y)$ arbitrarily close to z. Thus, we can choose a line L_0, distinct from L, through a point $t \in \operatorname{int} \operatorname{conv} J(x, y)$ with L_0 parallel and sufficiently close to L such that $L_0 \cap \operatorname{conv} J(x, y) \subset N$. Let $ab = L_0 \cap \operatorname{conv} J(x, y)$. We note that the points a and b are in $J(x, y) - \{x, y\}$ and $t \in \operatorname{intv} ab$. Since a and b are in $N \cap [J(x, y) - \{x, y\}] \subset N \cap S$, a p-arc $C(a, b)$ joining a and b exists in S. Since $C(a, b) \cup J(a, b) \subset S$ and since compl S has no nonempty bounded components, the compact set bounded by $C(a, b)$ and $J(a, b)$ belongs to S, where $J(a, b)$ is the subarc of $J(x, y)$ joining a and b with $z \in J(a, b)$. In particular, $ab \subset S$ and ab is a nondegenerate line segment since $t \in \operatorname{intv} ab$. Since z is an exposed point on $J(a, b)$, clearly $J(a, b) \neq ab$. Let $J_0(x, y) = J(x, a) \cup ab \cup J(b, y)$ where $J(x, a)$ and $J(b, y)$ are the subarcs of $J(x, y)$ joining the indicated pairs of points. It is obvious that $J_0(x, y)$ is a convex arc joining x and y in S such that $J_0(x, y) \subset \operatorname{conv}[R_p(x), R_p(y)]$ and such that $J_0(x, y) \cap \operatorname{int} \Delta(p, x, y) = \emptyset$. The existence of $J_0(x, y)$ in S contradicts the minimality of $J(x, y)$. Hence $J(x, y) = xy$ holds and $xy \subset S$. See Figure 2.

If p, x and y are collinear and $p \in xy$, the proof is similar. If p, x and y are collinear and $x \in \operatorname{intv} py$ or $y \in \operatorname{intv} px$, then $J(x, y)$ as described in Lemma 2 reduces to xy. Hence, in all cases, $xy \subset S$.

We now give the proof of Theorem 1.

Proof. The necessity is clear. To prove the sufficiency, let y be an arbitrary point of S distinct from p. Claim that p and y can be joined by a union of a finite number of p-arcs $C(x_i, y_i)$, $1 \leqq i \leqq m$, where $y_i = x_{i+1}$, $1 \leqq i \leqq m - 1$, with $x_1 = p$ and $y_m = y$. To see this, let S_p denote the points of S which can be

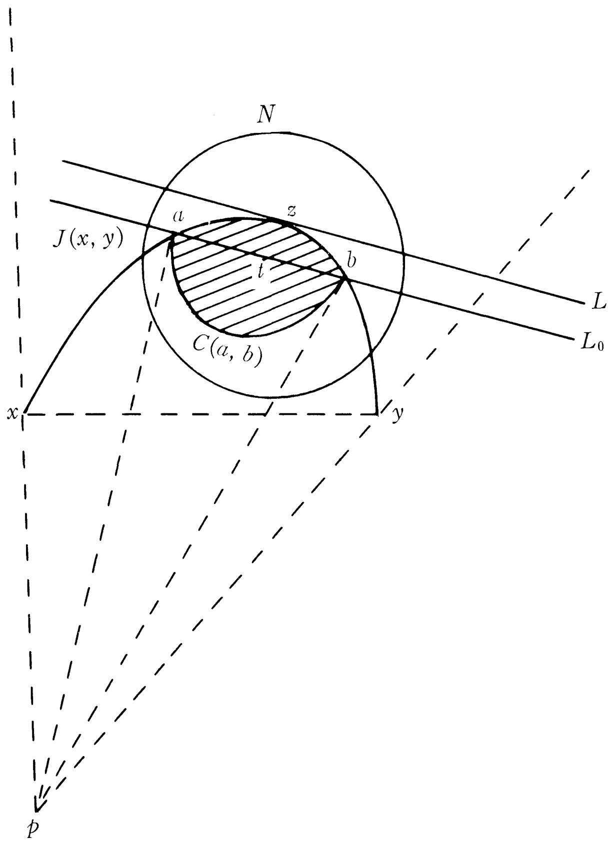

FIGURE 2

joined to p by a union of a finite number of p-arcs in S as described above. Clearly $S_p \neq \emptyset$ since $p \in S_p$. Since S is locally p-arcwise convex, it follows that the set S_p is both open and closed in S. Since S is connected, we have $S_p = S$. Thus, $y \in S$ can be joined to p by such a union of a finite number of p-arcs in S. Hence, it suffices to prove that $py \subset S$ if p and y are connected by a union C of two p-arcs in S, $C(p, x)$ and $C(x, y)$, joining p to x and x and y for some $x \in S$. Since the p-arc $C(p, x)$ is equal to px, $C = px \cup C(x, y)$. If the points p, x and y are collinear, then we have $py \subset S$. Thus we assume that p, x and y are noncollinear. Let F denote the two-dimensional flat in E_n determined by p, x and y. Clearly F contains the path $C = px \cup C(x, y)$ and the component K of $S \cap F$ containing C is a closed connected subset of F. Moreover, for each point $k \in K$, there exists a neighborhood N of k in F such that each pair of points $a \in N \cap K$, $b \in N \cap K$ can be joined by a p-arc lying in $N \cap K$. Thus, it follows from Lemma 1 that the complement of K in F has no nonempty bounded components in F.

The path $C = px \cup C(x, y)$ is a union of two p-arcs joining p to y in K. Claim there exists such a union of two p-arcs in K which has minimal length. To prove this, let T be the family of all paths in K joining p to y of the form $pw \cup A$ where $w \in K$ and A is a p-arc in K joining w to y such that

$$f(pw \cup A) = f(pw) + f(A) \leqq f(px) + f[C(x, y)]$$

where f is the length function. The family T is uniformly bounded. We will show that T is a compact family in the sense that every sequence in T has a convergent subsequence whose limit belongs to T. Suppose $\{C_i\}$ is a sequence of infinitely many paths in T where $C_i = pw_i \cup A_i$. The sequence of points $\{w_i\}$ is bounded; it contains a subsequence $\{w_{ij}\}$ converging to a point w_0. Since the sequence of p-arcs $\{A_{ij}\}$ associated with the subsequence $\{C_{ij}\}$ of $\{C_i\}$ is uniformly bounded, it follows from the Blaschke convergence theorem [**5**] that it converges to a limit which is a p-arc or a single point. If the limit is a p-arc, it must join w_0 to y. Let this p-arc be denoted by $A(w_0, y)$. If the limit of $\{A_{ij}\}$ is a single point, then $w_0 = y$. Similarly, the sequence of line segments $\{pw_{ij}\}$ converges to the limit pw_0 which may reduce to the point $p = w_0$. Since we have assumed $p \neq y$, only one of the sequences $\{pw_{ij}\}$ and $\{A_{ij}\}$ may converge to a point. Let $A_0 = pw_0 \cup A(w_0, y)$. The path A_0 connecting p to y lies in K since K is closed. In the event that one of the sequences $\{pw_{ij}\}$ and $\{A_{ij}\}$ converges to a point, we have $A_0 = py \subset K \subset S \cap F \subset S$ and hence S is starshaped relative to p. Assume that this trivial case does not hold. Since f is continuous, we have $f(A_0) \leqq f(px) + f[C(x, y)]$. Therefore $A_0 \in T$ and T is a compact family. Hence, f takes on its minimum in T. For this reason, we may take the path $C = px \cup C(x, y)$ to be the shortest such path joining p to y by a union of two p-arcs in K. With the assumption that p, x, and y are non-collinear, we proceed to obtain a contradiction to the minimality of C by the following steps.

(a) We have $C(x, y) \cap px = \{x\}$; otherwise the minimality of C is contradicted.

(b) The p-arc $C(x, y)$ contains no nondegenerate line segment at x. For if $C(x, y) = xz \cup C(z, y)$ where xz is a nondegenerate line segment in $C(x, y)$ and where $C(z, y)$ is a p-arc joining z and y such that $C(z, y) \subset C(x, y)$, then $px \cup xz$ is a convex arc joining p and z as described in Lemma 2. Hence, we have $pz \subset K$ by Lemma 2. The path $C_0 = pz \cup C(z, y)$ is of shorter length than $C = px \cup C(x, y)$, contradicting the minimality of C.

(c) Thus, we assume $C = px \cup C(x, y)$ is such that $C(x, y) \cap px = \{x\}$ and $C(x, y)$ has no nondegenerate line segment at x. In particular, $C(x, y) \neq xy$. Let N be a neighborhood of x in F such that $N \cap K$ is p-arcwise convex, and let Q denote the compact region in F bounded by $px \cup C(x, y) \cup py$. Since the complement of K in F has no nonempty bounded components in F and $N \cap K$ is p-arcwise convex, there exists points u and v in $N \cap K$ with $u \in px - \{p, x\}$ and $v \in C(x, y) - \{x, y\}$ that can be connected by a p-arc $D(u, v)$ in $N \cap K$ such that $D(u, v) - \{u, v\}$ is contained in the interior of Q

in F. The compact region D_x in F bounded by $ux \cup D(u, v) \cup C(x, v) \subset K$ is nonempty, where $C(x, v)$ is the p-arc joining x and v such that $C(x, v) \subset C(x, y)$. Therefore, $D_x \subset K$. Since $C(x, y) \subset \Delta(p, x, y)$ and $C(x, y)$ has no nondegenerate line segment at x, there exists an exposed point $w \in C(x, v) - \{x\}$ and a line of support L to $C(x, y)$ through w for which $L \cap C(x, y) = \{w\}$ such that L intersects xu in a point t with $t \neq x$, $t \neq w$ and $tw - \{t, w\}$ is contained in the interior of D_x in F. See Figure 3.

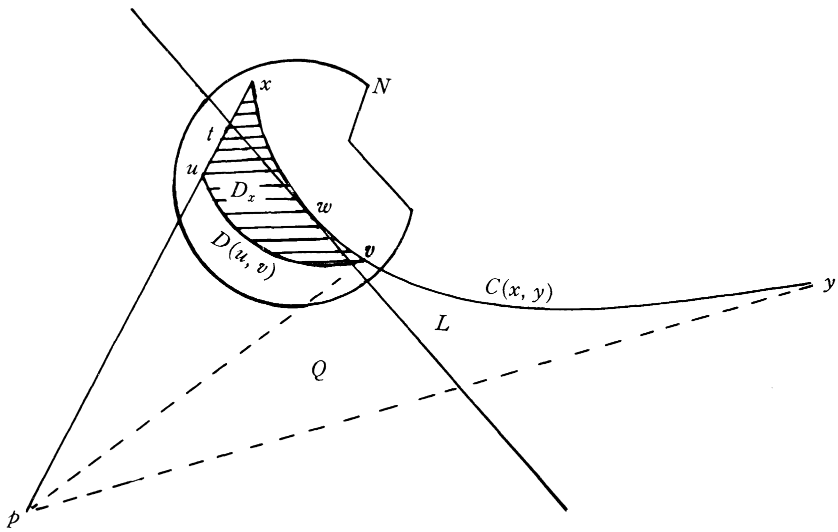

FIGURE 3

Let $C_0 = pt \cup tw \cup C(w, y)$ where $C(w, y)$ is the p-arc joining w and y such that $C(w, y) \subset C(x, y)$. We note that $tw \cup C(w, y)$ is a p-arc in K. Hence $C_0 = pt \cup [tw \cup C(w, y]$ is a union of two p-arcs joining p to y in K. Since $x \notin tw$, tw is of shorter length than $tx \cup C(x, w)$ where $C(x, w)$ is the p-arc joining x and w such that $C(x, w) \subset C(x, y)$. Hence, C_0 is of shorter length than C, contradicting the minimality of C. Therefore, we must have p, x and y collinear and $C = py \subset K \subset S \cap F \subset S$. Since y is an arbitrary point in S, the set S is starshaped relative to p. This completes the proof of Theorem 1.

Theorem 2, a Tietze-type globalization result on p-arcwise convex sets, follows from Theorem 1. The necessity is clear. To prove the sufficiency, suppose that S is locally p-arcwise convex. Let x and y be any pair of points of S. By Theorem 1, S is starshaped relative to p. Thus $px \subset S$ and $py \subset S$. Let $C(x, y) = px \cup py$. Since $C(x, y)$ is a convex arc joining x and y and $C(x, y) \subset \Delta(p, x, y)$, $C(x, y)$ is a p-arc in S joining x and y. Hence S is p-arcwise convex.

Theorems 1 and 2 do not hold if the set S is not closed. For example, let $S = \Delta(p, x, y) - \{a\}$ where p, x and y are noncollinear and $a \in \operatorname{intv} px$. The connected set S is locally p-arcwise convex; but S is neither starshaped relative to p nor p-arcwise convex.

Theorems 1 and 2 also hold in any topological linear space. Moreover, these results can still be proved if we weaken the local p-arcwise convexity condition in S as follows: For each $z \in S$, there exists a neighborhood N of z such that any pair of points $x \in N \cap S$, $y \in N \cap S$ can be joined by a p-arc in S.

It is natural to ask whether a Krasnoselskii-type theorem involving p-arcs can be proved. The author is presently investigating the following conjecture. Let p be a fixed point in E_2 and let $S \subset E_2$ be a nonempty simply connected compact set. If every three points in S can see at least one point of S via p-arcs in S, then there exists a point $k \in S$ such that every point in S can see k via a p-arc in S.

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