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THE SYMMETRIC GROUP REPRESENTATION ON COHOMOLOGY OF THE REGULAR ELEMENTS OF A MAXIMAL TORUS OF THE SPECIAL LINEAR GROUP

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Abstract

We give a formula for the character of the representation of the symmetric group S_n on each isotypic component of the cohomology of the set of regular elements of a maximal torus of SL_n, with respect to the action of the centre.

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1. Introduction

Let n be a positive integer. Define the hyperplane complement

$$T(1, n) := \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq 0, \ \forall\, i, z_i \neq z_j, \ \forall\, i \neq j\}.$$

The symmetric group S_n acts on $T(1, n)$ by permuting coordinates; we can identify $T(1, n)$ with the set of regular elements of a maximal torus of $GL_n(\mathbb{C})$, and S_n with the Weyl group of the maximal torus. This action induces representations of S_n on the cohomology groups $H^i(T(1, n))$ (taken with complex coefficients). The characters of these representations are well known (they will be encapsulated in a single 'equivariant generating function' in Theorem 3.1 below). For the purposes of this introduction, we recall only the 'nonequivariant' information, that is, the Betti numbers of $T(1, n)$:

$$\sum_i (-1)^i \dim H^i(T(1, n))\, q^{n-i} = (q-1)\,(q-2)\cdots(q-n). \tag{1.1}$$

These Betti numbers are particularly familiar, since $T(1, n)$ is homotopy equivalent to the configuration space C_{n+1} of $(n + 1)$-tuples of distinct complex numbers.

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Now consider the toral complement

$$ST(1,n) := \{(z_1, z_2, \ldots, z_n) \in T(1,n) \mid z_1 z_2 \cdots z_n = 1\}.$$

This can be identified with the set of regular elements of a maximal torus of $SL_n(\mathbb{C})$. Of course S_n still acts, and there is a commuting action of μ_n (the centre of $SL_n(\mathbb{C})$) by scaling. Thus we have a direct sum decomposition of S_n-representations:

$$H^i(ST(1,n)) \cong \bigoplus_{\chi \in \widehat{\mu_n}} H^i(ST(1,n))_\chi, \tag{1.2}$$

where $H^i(ST(1,n))_\chi$ is the χ-isotypic component of $H^i(ST(1,n))$. In this paper we address the following problem, suggested by Lehrer.

PROBLEM 1.1. Give a formula for the character of the representation of S_n on each $H^i(ST(1,n))_\chi$.

Our solution is given in Section 3 (see especially (3.6)). For now, we state merely the nonequivariant version:

$$\sum_i (-1)^i \dim H^i(ST(1,n))_\chi \, q^{n-1-i}$$
$$= \frac{(-1)^{n-n/r} n!}{r^{n/r}(n/r)!} (q-r-1)\,(q-2r-1)\cdots\left(q-\left(\frac{n}{r}-1\right)r-1\right), \tag{1.3}$$

where r is the order of χ in the character group $\widehat{\mu_n}$.

Summing over χ, we deduce a formula (3.7) for the character of the total representation on $H^i(ST(1,n))$, of which the nonequivariant specialization is

$$\sum_i (-1)^i \dim H^i(ST(1,n)) \, q^{n-1-i}$$
$$= \sum_{r|n} \frac{(-1)^{n-n/r}\phi(r) n!}{r^{n/r}(n/r)!} (q-r-1)\,(q-2r-1)\cdots\left(q-\left(\frac{n}{r}-1\right)r-1\right). \tag{1.4}$$

This total formula was essentially already known. Since $ST(1,n)$ is 'minimally pure' in the sense of [2], (1.4) follows from the fact that the right-hand side counts the number of $\mathbb{F}_q$-points of the variety $ST(1,n)$ for all prime powers q which are congruent to 1 mod n. More generally, (3.7) follows from a count of fixed points of twisted Frobenius maps, which was the result [3, Theorem 5.8] of Fleischmann and Janiszczak. See Remark 3.4 below.

Clearly the quotient of $ST(1,n)$ by μ_n can be identified with $\mathbb{P}T(1,n)$, the image of $T(1,n)$ in $\mathbb{P}^{n-1}(\mathbb{C})$. So, in the case where χ is the trivial character, we are dealing with

$$H^i(ST(1,n))^{\mu_n} \cong H^i(\mathbb{P}T(1,n)). \tag{1.5}$$

In general, as (1.3) suggests, the representation $H^i(ST(1,n))_\chi$ is induced from the wreath product subgroup $W(r, n/r) := \mu_r \wr S_{n/r}$ of S_n. To be precise, define the hyperplane complement

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$$T(r, m) := \{(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m \mid z_i \neq 0,\ \forall\, i,\ z_i^r \neq z_j^r,\ \forall\, i \neq j\},$$

and its image $\mathbb{P}T(r, m)$ in $\mathbb{P}^{m-1}(\mathbb{C})$. For $r \geq 2$, we identify the corresponding reflection group $G(r, 1, m)$ with the wreath product $W(r, m)$ (if $r = 1$, $W(1, m) = S_m$ acts on $T(1, m)$ as seen above). In the following theorem, ε_n denotes the sign character of S_n, and $\det_{n/r}$ the determinant character of $GL_{n/r}(\mathbb{C})$, restricted to $W(r, n/r)$.

THEOREM 1.2. *Let r be the order of $\chi \in \widehat{\mu_n}$. For every i, we have an isomorphism of representations of S_n:*

$$H^i(ST(1, n))_\chi \cong \varepsilon_n \otimes \mathrm{Ind}_{W(r,n/r)}^{S_n}(\det_{n/r} \otimes H^{i-n+n/r}(\mathbb{P}T(r, n/r))).$$

The proof of this theorem given in Section 4 merely equates the characters of both sides; a more conceptual understanding of the isomorphism (or rather the related isomorphism (4.2)), involving Orlik–Solomon-style bases for the cohomology groups, is given in [6].

When χ is faithful (that is, $r = n$), Theorem 1.2 says that

$$H^i(ST(1, n))_\chi \cong \begin{cases} \varepsilon_n \otimes \mathrm{Ind}_{\mu_n}^{S_n}(\psi) & \text{if } i = n-1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

where μ_n is embedded in S_n as the subgroup generated by an n-cycle, and $\psi \in \widehat{\mu_n}$ is any faithful character (it does not matter which—note also that tensoring with ε_n makes no difference unless $n \equiv 2 \pmod 4$). This result for prime n was proved in [2, Section 4.4].

2. Equivariant weight polynomials

Suppose that X is an irreducible complex variety which is *minimally pure* in the sense that $H_c^i(X)$ is a pure Hodge structure of weight $2i - 2\dim X$ for all i (see [2]). Let Γ be a finite group acting on X. We define the *equivariant weight polynomials* of this action by

$$P(\gamma, X, q) := \sum_i (-1)^i\, \mathrm{tr}(\gamma, H_c^i(X))\, q^{i-\dim X},$$

for all $\gamma \in \Gamma$, where q is an indeterminate ($= t^2$ in the notation of [2]). We also define

$$P^\Gamma(X, q) := \sum_i (-1)^i [H_c^i(X)] q^{i-\dim X} \in R(\Gamma)\,[q],$$

where $R(\Gamma)$ is the complexified representation ring of Γ. If Δ is an abelian finite group acting on X whose action commutes with that of Γ, and χ is a character of Δ, we define

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$$\begin{aligned} P(\gamma, \chi, X, q) &:= \sum_i (-1)^i \operatorname{tr}(\gamma, H_c^i(X)_\chi)\, q^{i-\dim X} \\ &= \sum_i (-1)^i \frac{1}{|\Delta|} \sum_{\delta\in\Delta} \chi(\delta)^{-1} \operatorname{tr}((\gamma, \delta), H_c^i(X)) q^{i-\dim X} \\ &= \frac{1}{|\Delta|} \sum_{\delta\in\Delta} \chi(\delta)^{-1} P((\gamma, \delta), X, q), \end{aligned}$$

and similarly

$$P^\Gamma(\chi, X, q) := \sum_i (-1)^i [H_c^i(X)_\chi] q^{i-\dim X} \in R(\Gamma)\,[q].$$

If X is nonsingular, we can translate knowledge of $H_c^i(X)$ and $H_c^i(X)_\chi$ into knowledge of $H^i(X)$ and $H^i(X)_\chi$ by Poincaré duality.

Now for any positive integers r and n, $T(r, n)$ (respectively, $\mathbb{P}T(r, n)$) is a nonsingular irreducible minimally pure variety of dimension n (respectively, $n-1$); minimal purity is a standard property of hyperplane complements [2, Example 3.3]. Also, $ST(1, n)$ is clearly a nonsingular irreducible variety of dimension $n-1$. To show that it is minimally pure, one can use [2, Corollary 4.2], or else observe that it is the quotient of $\mathbb{P}T(n, n)$ by the free action of a finite group, as follows.

Recall that $W(r, m) = S_m \ltimes \mu_r^m$ acts on $T(r, m)$ and $\mathbb{P}T(r, m)$; the S_m factor acts by permuting the coordinates, and μ_r^m acts by scaling them. Define a surjective map $\varphi : \mathbb{P}T(n, n) \to ST(1, n)$ by

$$\varphi([x_1 : x_2 : \cdots : x_n]) = \frac{1}{x_1 x_2 \cdots x_n}(x_1^n, x_2^n, \ldots, x_n^n).$$

The fibres of φ are clearly the orbits of the normal subgroup $S\mu_n^n \subset W(n, n)$, where

$$S\mu_n^m := \{(\zeta_1, \ldots, \zeta_m) \in \mu_n^m \mid \zeta_1\zeta_2 \cdots \zeta_m = 1\}.$$

The action of $S\mu_n^n$ on $\mathbb{P}T(n, n)$ becomes free once one factors out the subgroup $\{(\zeta, \zeta, \ldots, \zeta)\}$, which acts trivially. Thus $ST(1, n)$ is minimally pure, and solving Problem 1.1 amounts to computing the polynomials $P(w, \chi, ST(1, n), q)$, for all $w \in S_n$ and $\chi \in \widehat{\mu_n}$.

Consider the quotient of $T(n, m)$ by $S\mu_n^m$ for arbitrary $m \geq 1$. This can be identified with

$$T^{(n)}(1, m) := \{((z_i), z) \in T(1, m) \times \mathbb{C}^\times \mid z^n = z_1 \cdots z_m\}.$$

The quotient map $\psi : T(n, m) \to T^{(n)}(1, m)$ is given by

$$\psi(x_1, x_2, \ldots, x_m) = ((x_1^n, x_2^n, \ldots, x_m^n), x_1x_2 \cdots x_m).$$

The group $S_m \times \mu_n \cong W(n, m)/S\mu_n^m$ acts on the quotient $T^{(n)}(1, m)$ in the obvious way: S_m acts on the $T(1, m)$ component, and μ_n acts by scaling z.

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When $m = n$, we have an isomorphism

$$T^{(n)}(1, n) \xrightarrow{\sim} ST(1, n) \times \mathbb{C}^\times,$$
$$((z_1, \ldots, z_n), z) \mapsto ((z_1 z^{-1}, \ldots, z_n z^{-1}), z),$$

which respects the S_n-action, and transforms the μ_n-action on $T^{(n)}(1, n)$ into the inverse of the μ_n-action on $ST(1, n)$, and a scaling action on $\mathbb{C}^\times$. Since the latter has no effect on cohomology,

$$P(w, \chi, ST(1, n), q) = \frac{1}{q-1} P(w, \chi^{-1}, T^{(n)}(1, n), q). \tag{2.1}$$

So we aim to compute $P(w, \chi^{-1}, T^{(n)}(1, n), q)$; it turns out to be convenient to compute the polynomials $P(w, \chi^{-1}, T^{(n)}(1, m), q)$ for all $m \geq 1$ and $w \in S_m$ together.

REMARK 2.1. One can see *a priori* that allowing $m \neq n$ incurs no extra work, thanks to the following neat argument, pointed out by Lehrer. If $d = \gcd(m, n)$, the action of $\mu_{n/d} \subset \mu_n$ on $T^{(n)}(1, m)$ is part of the action of the connected group $\mathbb{C}^\times$ defined by

$$t \cdot ((z_i), z) = ((t^{n/d} z_i), t^{m/d} z).$$

Hence $\mu_{n/d}$ acts trivially on cohomology, so $P(w, \chi^{-1}, T^{(n)}(1, m), q) = 0$ unless $\chi|_{\mu_{n/d}} = 1$, that is, $\chi^d = 1$, or $r|m$, where r is the order of χ. Moreover, if $r|m$, then writing χ° for the character of μ_r such that $\chi(\zeta) = \chi^\circ(\zeta^{n/r})$ for all $\zeta \in \mu_n$, and χ' for the character of μ_m defined by $\chi'(\zeta) = \chi^\circ(\zeta^{m/r})$ for all $\zeta \in \mu_m$,

$$\begin{aligned} P(w, \chi^{-1}, T^{(n)}(1, m), q) &= P(w, (\chi^\circ)^{-1}, T^{(r)}(1, m), q) \\ &= P(w, (\chi')^{-1}, T^{(m)}(1, m), q). \end{aligned}$$

We will not actually use this observation.

The identification of $T^{(n)}(1, m)$ with the quotient of $T(n, m)$ by $S\mu_n^m$ has the following consequence for equivariant weight polynomials.

PROPOSITION 2.2. *For any $w \in S_m$ and $\chi \in \widehat{\mu_n}$,*

$$\begin{aligned} &P(w, \chi^{-1}, T^{(n)}(1, m), q) \\ &\quad = \frac{1}{n^m} \sum_{(\zeta_i) \in \mu_n^m} \chi(\zeta_1 \cdots \zeta_m) P(w(\zeta_1, \ldots, \zeta_m), T(n, m), q). \end{aligned}$$

PROOF. It is well known that if V is a representation of the finite group G and V^H is the subspace invariant under the normal subgroup H, the character of G/H on V^H is given by

$$\mathrm{tr}(gH, V^H) = \frac{1}{|H|} \sum_{h \in H} \mathrm{tr}(gh, V).$$

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Now apply this with $V = H_c^i(T(n, m))$, $G = W(n, m)$, and $H = S\mu_n^m$, so that $V^H \cong H_c^i(T^{(n)}(1, m))$ and $G/H \cong S_m \times \mu_n$. We find that, for all $\zeta \in \mu_n$,

$$P((w, \zeta), T^{(n)}(1, m), q) = \frac{1}{n^{m-1}} \sum_{\substack{(\zeta_i) \in \mu_n^m \\ \zeta_1 \cdots \zeta_m = \zeta}} P(w(\zeta_1, \dots, \zeta_m), T(n, m), q).$$

Combining this with the fact that

$$P(w, \chi^{-1}, T^{(n)}(1, m), q) = \frac{1}{n} \sum_{\zeta \in \mu_n} \chi(\zeta) P((w, \zeta), T^{(n)}(1, m), q)$$

gives the desired result. □

3. Generating functions

In this section we will compute the sum in Proposition 2.2 using the known formula for the equivariant weight polynomials of $T(r, m)$. As is usual in dealing with characters of symmetric groups and wreath products, the computations become easier if one uses suitable 'equivariant generating functions'.

For any $r \geq 1$, let $\Lambda(r)$ denote the polynomial ring $\mathbb{C}[p_i(\zeta)]$ in countably many independent variables $p_i(\zeta)$ where i is a positive integer and $\zeta \in \mu_r$. Define an $\mathbb{N}$-grading on $\Lambda(r)$ by $\deg(p_i(\zeta)) = i$. Also let $\Lambda(r)[q] := \Lambda(r) \otimes_{\mathbb{C}} \mathbb{C}[q]$, with the $\mathbb{N}$-grading given by the first factor (so $\deg(q) = 0$). Let $\mathbb{A}(r) = \mathbb{C}[\![p_i(\zeta)]\!]$ be the completion of $\Lambda(r)$, and set $\mathbb{A}(r)[q] = \mathbb{A}(r) \otimes \mathbb{C}[q]$.

As in [10, Ch. I, Appendix B], we define an isomorphism $\mathrm{ch}_{W(r,m)} : R(W(r, m)) \xrightarrow{\sim} \Lambda(r)_m$ by

$$\mathrm{ch}_{W(r,m)}([M]) = \frac{1}{r^m m!} \sum_{y \in W(r,m)} \mathrm{tr}(y, M) p_y,$$

where $p_y = \prod_{i,\zeta} p_i(\zeta)^{a_i(\zeta)}$ if y has $a_i(\zeta)$ cycles of length i and type ζ. Note that to recover $\mathrm{tr}(y, M)$ from $\mathrm{ch}_{W(r,m)}([M])$ one must multiply the coefficient of $\prod_{i,\zeta} p_i(\zeta)^{a_i(\zeta)}$ by the order of the centralizer of y, which is $\prod_{i,\zeta} a_i(\zeta)!(ri)^{a_i(\zeta)}$. Write $\mathrm{ch}_{W(r,m)}$ also for the induced isomorphism $R(W(r, m))[q] \xrightarrow{\sim} \Lambda(r)[q]_m$.

The result we need on the equivariant weight polynomials for $T(r, m)$ can be conveniently stated in terms of the equivariant generating function $P(r, q) \in \mathbb{A}(r)[q]$ defined by

$$\begin{aligned} P(r, q) &:= 1 + \sum_{m \geq 1} \mathrm{ch}_{W(r,m)}(P^{W(r,m)}(T(r, m), q)) \\ &= 1 + \sum_{m \geq 1} \frac{1}{r^m m!} \sum_{y \in W(r,m)} P(y, T(r, m), q) p_y. \end{aligned}$$

In the following result $\mu(d)$ denotes the Möbius function.

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THEOREM 3.1. *If* $R_{r,i,\theta} := \sum_{d|i} |\{\zeta \in \mu_r \ : \ \zeta^d = \theta\}| \mu(d)\ (q^{i/d} - 1) \in \mathbb{C}[q]$,

$$P(r, q) = \prod_{\substack{i \geq 1 \\ \theta \in \mu_r}} (1 + p_i(\theta))^{R_{r,i,\theta}/ri}.$$

PROOF. This follows from Hanlon's result [4, Corollary 2.3] on the Möbius functions of Dowling lattices. Within the reflection group context, it follows from results of Lehrer in [8] ($r = 1$), [7] ($r = 2$) and [1, 9] (general r). A short proof for all $r \geq 2$, based on an 'equivariant inclusion–exclusion' argument of Getzler, is given in [5, Theorem 8.4] ($T(r, m)$ is the same as what is there called $M(r, m)$). The $r = 1$ case can be proved by the same method (note that $T(1, m)$ is different from the variety $M(1, m)$ considered in [5, Theorem 8.2], since it has the extra condition of nonzero coordinates). □

Recovering the traces of individual elements by the above rule, we get an equivalent statement, closer to Hanlon's and Lehrer's: if y in $W(r, m)$ has $a_i(\zeta)$ cycles of length i and type ζ,

$$P(y, T(r, m), q) = \prod_{\substack{i \geq 1 \\ \zeta \in \mu_r}} R_{r,i,\zeta}(R_{r,i,\zeta} - ri) \cdots (R_{r,i,\zeta} - (a_i(\zeta) - 1)ri). \quad (3.1)$$

There is an alternative description of the polynomials $R_{r,i,\theta}$. Define

$$R_i^{(r)} := \sum_{\substack{d|i \\ \gcd(d,r)=1}} \mu(d)\ (q^{i/d} - 1) \in \mathbb{C}[q].$$

LEMMA 3.2. *If* $t(\theta)$ *denotes the order of* θ,

$$R_{r,i,\theta} = \sum_{s|\gcd(r/t(\theta),i)} s\mu(s) R_{i/s}^{(r)}.$$

PROOF. Since μ_r is cyclic of order r,

$$|\{\zeta \in \mu_r \ : \ \zeta^d = \theta\}| = \begin{cases} \gcd(d, r) & \text{if } \gcd(d, r) \mid (r/t(\theta)), \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$R_{r,i,\theta} = \sum_{s|(r/t(\theta))} s \sum_{\substack{d|i \\ \gcd(d,r)=s}} \mu(d)\ (q^{i/d} - 1).$$

The sum over d has no terms unless $s|i$, in which case it equals

$$\sum_{\substack{d|(i/s) \\ \gcd(d,r)=1}} \mu(ds)\ (q^{i/ds} - 1).$$

Since $\gcd(d, r) = 1$ implies $\mu(ds) = \mu(d)\mu(s)$, this is $\mu(s) R_{i/s}^{(r)}$. □

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This lemma makes it clear that the $r = 2$ case of (3.1) is indeed equivalent to [7, Theorem 5.6].

As for $\mathbb{P}T(r, m)$, we have that, for all $y \in W(r, m)$,

$$P(y, \mathbb{P}T(r, m), q) = \frac{1}{q-1} P(y, T(r, m), q). \tag{3.2}$$

For this, one need only show that the isomorphism

$$\varphi : T(r, m) \to \mathbb{P}T(r, m) \times \mathbb{C}^\times : (z_1, \ldots, z_m) \mapsto ([z_1 : \cdots : z_m], z_1)$$

induces a $W(r, m)$-equivariant map on cohomology. It is enough to check that $w \circ \varphi$ and $\varphi \circ w$ are homotopic for all w in a set of generators for $W(r, m)$, which is straightforward.

Now, for any $\chi \in \widehat{\mu_n}$, define the generating function

$$\begin{aligned} P(\chi, q) &:= 1 + \sum_{m \geq 1} \mathrm{ch}_{S_m}(P^{S_m}(\chi^{-1}, T^{(n)}(1, m), q)) \\ &= 1 + \sum_{m \geq 1} \frac{1}{m!} \sum_{w \in S_m} P(w, \chi^{-1}, T^{(n)}(1, m), q) p_w \in \mathbb{A}(1)[q]. \end{aligned}$$

We want a formula for this similar to Theorem 3.1. Define

$$P_i^{(r)} := \prod_{s \mid \gcd(r, i)} (1 - (-p_i)^{r/s})^{s\mu(s) R_{i/s}^{(r)}/ri} \in \mathbb{A}(1)[q].$$

THEOREM 3.3. *If $\chi \in \widehat{\mu_n}$ has order r, $P(\chi, q) = \prod_{i \geq 1} P_i^{(r)}$.*

PROOF. Using Proposition 2.2, we see that $P(\chi, q)$ equals

$$1 + \sum_{m \geq 1} \frac{1}{n^m m!} \sum_{\substack{w \in S_m \\ (\zeta_i) \in \mu_n^m}} \chi(\zeta_1 \cdots \zeta_m) P(w(\zeta_1, \ldots, \zeta_m), T(n, m), q) p_w,$$

which is precisely the result of applying to $P(n, q)$ the specialization $p_i(\theta) \to \chi(\theta) p_i$.

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So, by Theorem 3.1,

$$
\begin{aligned}
P(\chi, q) &= \exp \sum_{\substack{i\geq 1\\ \theta\in\mu_n}} \frac{R_{n,i,\theta}}{ni} \log(1+\chi(\theta)p_i)\\
&= \exp \sum_{\substack{i\geq 1\\ \theta\in\mu_n\\ m\geq 1}} \frac{-R_{n,i,\theta}}{nmi} \chi(\theta)^m(-p_i)^m\\
&= \exp \sum_{\substack{i\geq 1\\ d|i\\ \zeta\in\mu_n\\ m\geq 1}} \frac{-\mu(d)}{nmi} \chi(\zeta)^{dm}(q^{i/d}-1)\,(-p_i)^m\\
&= \exp \sum_{\substack{i\geq 1\\ d|i\\ m\geq 1,\ r|dm}} \frac{-\mu(d)}{mi}(q^{i/d}-1)\,(-p_i)^m,
\end{aligned}
$$

since $\sum_{\zeta\in\mu_n}\chi(\zeta)^{dm}=n$ if χ^{dm} is trivial, and vanishes otherwise. Thus we need only show that

$$P_i^{(r)} = \exp \sum_{\substack{d|i\\ m\geq 1,\ r|dm}} \frac{-\mu(d)}{mi}(q^{i/d}-1)\,(-p_i)^m. \tag{3.3}$$

The condition $r|dm$ is equivalent to $(r/\gcd(d,r))|m$. Writing s for $\gcd(d,r)$, the right-hand side becomes

$$
\begin{aligned}
&\exp \sum_{\substack{s|r\\ d|i,\gcd(d,r)=s\\ m\geq 1,\ (r/s)|m}} \frac{-\mu(d)}{mi}(q^{i/d}-1)\,(-p_i)^m\\
&\quad= \exp \sum_{s|\gcd(r,i)} \frac{s}{ri}\log(1-(-p_i)^{r/s}) \sum_{\substack{d|i\\ \gcd(d,r)=s}} \mu(d)\,(q^{i/d}-1).
\end{aligned}
$$

By the same argument as in the proof of Lemma 3.2, the sum over d equals $\mu(s)R_{i/s}^{(r)}$ as required. □

Note that $P(\chi, q)$ depends only on r, not on n or χ, and that, as predicted in Remark 2.1, its nonzero homogeneous components all have degree divisible by r.

There is no formula as neat as (3.1) for the individual polynomials $P(w, \chi^{-1}, T^{(n)}(1,m), q)$. However, if $w\in S_m$ has a_i cycles of length i, we know that

$$P(w, \chi^{-1}, T^{(n)}(1,m), q) = \prod_{i\geq 1} a_i!\, i^{a_i}\,(\text{coefficient of } p_i^{a_i} \text{ in } P_i^{(r)}). \tag{3.4}$$

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Note that, for the right-hand side to be nonzero, a_i must be divisible by $(r/\gcd(r, i))$ for all i. In the special case that $\gcd(r, i) = 1$,

$$P_i^{(r)} = (1 - (-p_i)^r)^{R_i^{(r)}/ri},$$

and the coefficient of $p_i^{a_i}$, where a_i is divisible by r, is

$$(-1)^{a_i - a_i/r} \frac{R_i^{(r)}(R_i^{(r)} - ri)\,(R_i^{(r)} - 2ri) \cdots (R_i^{(r)} - (a_i - r)i)}{(ri)^{a_i/r}(a_i/r)!}. \tag{3.5}$$

Now consider some further special cases. The $r = 1$ case of Theorem 3.3 says that

$$P(\text{triv}, q) = \prod_{i \geq 1} (1 + p_i)^{R_i^{(1)}/i} = P(1, q),$$

reflecting the fact that the quotient of $T^{(n)}(1, m)$ by μ_n is $T(1, m)$. Slightly more interesting is the $r = 2$ case. We have

$$P_i^{(2)} = \begin{cases} (1 - p_i^2)^{R_i^{(2)}/2i} & \text{if } i \text{ is odd}, \\ (1 - p_i^2)^{R_i^{(2)}/2i}(1 + p_i)^{-R_{i/2}^{(2)}/i} & \text{if } i \text{ is even}. \end{cases}$$

Hence if i is even, the coefficient of $p_i^{a_i}$ is

$$\sum_{j=0}^{\lfloor a_i/2 \rfloor} (-1)^j \binom{R_i^{(2)}/2i}{j} \binom{-R_{i/2}^{(2)}/i}{a_i - 2j}.$$

Returning to Problem 1.1, (2.1) and (3.4) tell us that, if $w \in S_n$ has a_i cycles of length i,

$$P(w, \chi, ST(1, n), q) = \frac{1}{q - 1} \prod_{i \geq 1} a_i!\, i^{a_i} (\text{coefficient of } p_i^{a_i} \text{ in } P_i^{(r)}). \tag{3.6}$$

Since there are $\phi(r)$ characters $\chi \in \widehat{\mu_n}$ of order r, we deduce that

$$P(w, ST(1, n), q) = \frac{1}{q - 1} \sum_{r|n} \phi(r) \prod_{i \geq 1} a_i!\, i^{a_i} (\text{coefficient of } p_i^{a_i} \text{ in } P_i^{(r)}). \tag{3.7}$$

REMARK 3.4. As mentioned in the introduction, if q is specialized to a prime power congruent to 1 mod n, the right-hand side of (3.7) equals the formula given in [3, Theorem 5.8] for the number of $\mathbb{F}_q$-points of the regular set of a maximal torus of $SL_n(\mathbb{F}_q)$ obtained from a maximally split one by twisting with w. (To see this, use the expression (3.3) for $P_i^{(r)}$; the coefficient of $p_i^{a_i}$ is called $R^i_{a_i,n}(q)$ in [3].) This is no surprise: general principles imply that for all but finitely many primes,

$$P(w, ST(1, n), q) = |ST(1, n)\,(\overline{\mathbb{F}_q})^{wF}|.$$

See [2, Section 5.3, Example 5.6] for the details.

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4. Induction

We now aim to prove Theorem 1.2 by interpreting the generating function $P(\chi, q)$ in terms of induced characters. Recall that $W(r, m)$ can be embedded in S_{rm} as the centralizer of the product of m disjoint r-cycles. For any $\theta \in \mu_r$, let $t(\theta)$ denote the order of θ.

LEMMA 4.1. *For any $W(r, m)$-module M,*

$$\mathrm{ch}_{S_{rm}}([\mathrm{Ind}_{W(r,m)}^{S_{rm}}(M)]) = \mathrm{ch}_{W(r,m)}([M])|_{p_i(\theta)\to p_{it(\theta)}^{r/t(\theta)}}.$$

PROOF. This is a direct consequence of Frobenius' formula for induced characters, once one observes that a cycle of length i and type θ in $W(r, m)$ becomes the product of $r/t(\theta)$ disjoint $it(\theta)$-cycles when regarded as an element of S_{rm}. □

LEMMA 4.2. *For any $W(r, m)$-module M,*

$$\begin{aligned}\mathrm{ch}_{S_{rm}}&([\varepsilon_{rm} \otimes \mathrm{Ind}_{W(r,m)}^{S_{rm}}(\det_m^{-1} \otimes M)])\\ &= (-1)^{rm-m}\mathrm{ch}_{W(r,m)}([M])|_{p_i(\theta)\to -\theta^{-1}(-p_{it(\theta)})^{r/t(\theta)}}.\end{aligned}$$

PROOF. If $y \in W(r, i)$ is a cycle of length i and type θ,

$$\varepsilon_{ri}(y)\det_i(y)^{-1} = (-1)^{i-1+(it(\theta)-1)r/t(\theta)}\theta^{-1} = (-1)^{i-1+ri-r/t(\theta)}\theta^{-1}.$$

Also, if $y \in W(r, m)$,

$$p_y|_{p_i(\theta)\to(-1)^{ri+i}p_i(\theta)} = (-1)^{rm-m}p_y.$$

Hence

$$\begin{aligned}\mathrm{ch}_{W(r,m)}&([\varepsilon_{rm} \otimes \det_m^{-1} \otimes M])\\ &= (-1)^{rm-m}\mathrm{ch}_{W(r,m)}([M])|_{p_i(\theta)\to -\theta^{-1}(-1)^{r/t(\theta)}p_i(\theta)},\end{aligned}$$

and the result follows by applying the previous lemma. □

Now define an element $P'(r, q) \in \mathbb{A}(1)[q]$ by

$$\begin{aligned}P'(r, q) &:= P(r, q)|_{p_i(\theta)\to -\theta^{-1}(-p_{it(\theta)})^{r/t(\theta)}}\\ &= 1 + \sum_{m\geq 1}\mathrm{ch}_{W(r,m)}(P^{W(r,m)}(T(r, m), q))|_{p_i(\theta)\to -\theta^{-1}(-p_{it(\theta)})^{r/t(\theta)}}\\ &= 1 + \sum_{m\geq 1}(-1)^{rm-m}\mathrm{ch}_{S_{rm}}(\varepsilon_{rm} \otimes \mathrm{Ind}_{W(r,m)}^{S_{rm}}(\det_m^{-1} \otimes P^{W(r,m)}(T(r, m), q))).\end{aligned}$$

PROPOSITION 4.3. $P'(r, q) = \prod_{i\geq 1} P_i^{(r)}$.

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PROOF. By Theorem 3.1 and Lemma 3.2,

$$P'(r,q) = \prod_{\substack{i\geq 1\\ \theta\in\mu_r}} (1-\theta^{-1}(-p_{it(\theta)})^{r/t(\theta)})^{R_{r,i,\theta}/ri}$$
$$= \prod_{\substack{i\geq 1\\ \theta\in\mu_r\\ s|\gcd(r/t(\theta),i)}} (1-\theta^{-1}(-p_{it(\theta)})^{r/t(\theta)})^{s\mu(s)R^{(r)}_{i/s}/ri}.$$

Applying to this the Möbius inversion formula for cyclotomic polynomials, in the form

$$\prod_{\substack{\theta\in\mu_r\\ t(\theta)=t}} (1-\theta^{-1}X) = \prod_{u|t} (1-X^{t/u})^{\mu(u)},$$

we obtain

$$P'(r,q) = \prod_{\substack{i\geq 1\\ t|r\\ s|\gcd(r/t,i)\\ u|t}} (1-(-p_{it})^{r/u})^{s\mu(s)\mu(u)R^{(r)}_{i/s}/ri}.$$

Write this as $\prod_{i\geq 1} Q_i^{(r)}$, where $Q_i^{(r)}$ is the product of all factors involving the variable p_i. Thus

$$Q_i^{(r)} = \exp \sum_{\substack{t|\gcd(r,i)\\ s|\gcd(r/t,i/t)\\ u|t}} \frac{st\mu(s)\mu(u)}{ri} R^{(r)}_{i/st} \log(1-(-p_i)^{r/u})$$
$$= \exp \sum_{\substack{v|\gcd(r,i)\\ u|v\\ s|(v/u)}} \frac{v\mu(s)\mu(u)}{ri} R^{(r)}_{i/v} \log(1-(-p_i)^{r/u}),$$

where we have set $v = st$. Since $\sum_{s|(v/u)} \mu(s)$ is nonzero if and only if $u = v$, we find that $Q_i^{(r)} = P_i^{(r)}$ as required. □

COROLLARY 4.4. *If $\chi \in \widehat{\mu_n}$ has order r, $P(\chi, q) = P'(r, q)$.*

PROOF. Combine Theorem 3.3 and Proposition 4.3. □

COROLLARY 4.5. *If $\chi \in \widehat{\mu_n}$ has order r, and $r|m$, we have the following equality in $R(S_m)[q]$:*

$$P^{S_m}(\chi^{-1}, T^{(n)}(1,m), q)$$
$$= (-1)^{m-m/r}\varepsilon_m \otimes \mathrm{Ind}_{W(r,m/r)}^{S_m}(\det{}_{m/r}^{-1} \otimes P^{W(r,m/r)}(T(r,m/r),q)).$$

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PROOF. Under the isomorphism ch_{S_m}, the left-hand side corresponds to the degree-m term of $P(\chi, q)$, and the right-hand side corresponds to the degree-m term of $P'(r, q)$. □

To translate Corollary 4.5 into an isomorphism of S_m-modules, we take coefficients of q^{i-m} on both sides and multiply by $(-1)^i$ to obtain

$$H_c^i(T^{(n)}(1, m))_{\chi^{-1}} \cong \varepsilon_m \otimes \mathrm{Ind}_{W(r,m/r)}^{S_m}(\det_{m/r}^{-1} \otimes H_c^{i-m+m/r}(T(r, m/r))). \quad (4.1)$$

By Poincaré duality, this is equivalent to

$$H^{2m-i}(T^{(n)}(1, m))_\chi \cong \varepsilon_m \otimes \mathrm{Ind}_{W(r,m/r)}^{S_m}(\det_{m/r} \otimes H^{m/r+m-i}(T(r, m/r))),$$

which after replacing $2m - i$ by i gives

$$H^i(T^{(n)}(1, m))_\chi \cong \varepsilon_m \otimes \mathrm{Ind}_{W(r,m/r)}^{S_m}(\det_{m/r} \otimes H^{i-m+m/r}(T(r, m/r))). \quad (4.2)$$

Finally, we prove Theorem 1.2. Equations (2.1), (3.2), and Corollary 4.5 together imply

$$\begin{aligned}
&P^{S_n}(\chi, ST(1, n), q)\\
&\quad= \frac{(-1)^{n-n/r}}{q-1}\varepsilon_n \otimes \mathrm{Ind}_{W(r,n/r)}^{S_n}(\det_{n/r}^{-1} \otimes P^{W(r,n/r)}(T(r, n/r), q))\\
&\quad= (-1)^{n-n/r}\varepsilon_n \otimes \mathrm{Ind}_{W(r,n/r)}^{S_n}(\det_{n/r}^{-1} \otimes P^{W(r,n/r)}(\mathbb{P}T(r, n/r), q)).
\end{aligned}$$

Taking coefficients of q^{i-n+1} on both sides and multiplying by $(-1)^i$, we get an isomorphism of S_n-modules:

$$H_c^i(ST(1, n))_\chi \cong \varepsilon_n \otimes \mathrm{Ind}_{W(r,n/r)}^{S_n}(\det_{n/r}^{-1} \otimes H_c^{i-n+n/r}(\mathbb{P}T(r, n/r))). \quad (4.3)$$

Since the right-hand side depends only on n and the order of χ, this remains true if χ is replaced by χ^{-1}. Then Theorem 1.2 follows by Poincaré duality.

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