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INFINITE SERIES AND THE DERIVED SET OF THE AGGREGATE OF THE FRACTIONAL PARTS OF ITS PARTIAL SUMS

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The aim of this paper is to study the relationship between the nature of an infinite series of real terms in which the general term tends to zero and the derived set (set of limit points) of the aggregate of the fractional parts of its partial sums. For all types of series (in which the nth term tends to zero) we determine the derived set.

We denote by F the set of the fractional parts of the partial sums of the series, and by F' the derived set of F. The principal results of the paper can be stated as follows:

1. the series is convergent if and only if F has at most one limit point in $[0, 1]$ or $F' \subseteq \{0, 1\}$;
2. the series, if non-oscillatory, is divergent if and only if $F' = [0, 1]$;
3. if the series is oscillatory, F' is a closed sub-interval of $[0, 1]$ or $[0, 1] - (a, b)$, where $(a, b) \subsetneq (0, 1)$.

1. Introduction and notation

This paper concerns itself with the study of the relationship between the nature of an infinite series of real terms in which the nth term tends to zero and the derived set of the aggregate of the fractional parts

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of its partial sums. Also, the derived set is determined for all types of series in which the nth term tends to zero.

In this paper, S_n denotes the nth partial sum of the infinite series $\sum_{n=1}^{\infty} u_n$ in which $u_n \to 0$; f_n denotes the fractional part of S_n ; (f_n) , the sequence $f_1, f_2, \ldots, f_n, \ldots$; F , the set of the fractional parts of its partial sums; and F' , the derived set of F . The largest integer not greater than S_n , $[S_n]$, is denoted by I_n .

2. Main theorems

THEOREM 2.1. *The series* $\sum u_n$ *, in which* $u_n \to 0$ *, is convergent if and only if* F *has at most one limit point in* $[0, 1]$ *or* $F' \subseteq \{0, 1\}$.

Closely related to this theorem is

THEOREM 2.1A. *The series* $\sum u_n$ *, in which* $u_n \to 0$ *, is convergent if and only if the sequence* (f_n) *is convergent or oscillates between the two values* 0 *and* 1 .

THEOREM 2.2. *A non-oscillatory series in which the* nth *term tends to zero is divergent if and only if* $F' = [0, 1]$.

Note. That $F' = [0, 1]$ for a divergent series of *positive* terms is known (see [1], Part I, Chapter 3, Problem 101, p. 23). The proof given in this paper is on independent lines.

THEOREM 2.3. *For an oscillatory series in which the* nth *term tends to zero* F' *is a closed sub-interval of* $[0, 1]$ *, or* $[0, 1] - (a, b)$ *, where* $(a, b) \subsetneqq (0, 1)$.

Note. F' may be $[0, 1]$ or may take any of the following forms: $[a, b]$, $[0, a] \cup [b, 1]$, $[0, a] \cup \{1\}$, $\{0\} \cup [b, 1]$, where $0 \le a < b \le 1$.

3. Proof of Theorem 2.1

For convenience, the first half of the theorem is split into two lemmas that follow.

LEMMA 3.1. *If* $\sum u_n$ *converges to a non-integral value,* s *, then the sequence* (f_n) *tends to the fractional part of* s .

The proof is simple.

REMARK. In this case F has at most one limit point, that is, F' is null or a single-element set.

LEMMA 3.2. *If* $\sum u_n$ *converges to an integer, then* F *has no limit point in* $(0, 1)$ *; that is,* $F' \subseteq \{0, 1\}$.

Proof. Let the series converge to an integer I . If F is finite, it has no limit points and $F' = \emptyset$. Assume, then, that F is infinite. Let, if possible, α be a limit point of F in $(0, 1)$. Choose a positive $\varepsilon < \min(\alpha, 1-\alpha)$ and also any positive $\delta < \min(\alpha-\varepsilon, 1-\alpha-\varepsilon)$. Now we have

$$0 < \delta < \alpha-\varepsilon < \alpha < \alpha+\varepsilon < 1-\delta < 1 .$$

Since $S_n \to I$, corresponding to this δ there exists a positive integer N such that

$$n > N \text{ implies } I-\delta < S_n < I+\delta .$$

Also, since α is a limit point of F , there exist infinitely many members f_n of F in $(\alpha-\varepsilon, \alpha+\varepsilon)$ and we can choose an f_m in this interval with $m > N$. Since $m > N$,

$$I-\delta < S_m < I+\delta .$$

This implies

$$I-\delta < I_m+f_m < I+\delta ,$$

and so

$$c-\delta < f_m < c+\delta ,$$

where $c = I - I_m$ is an integer. But $0 \le f_m < 1$ and $0 < \delta < 1$ and so we have $-1 < c < 2$, implying that $c = 0$ or 1 . If $c = 0$,

$$0 \le f_m < \delta < \alpha-\varepsilon ;$$

if $c = 1$,

$$\alpha+\varepsilon < 1-\delta < f_m < 1 .$$

In either case we have the contradiction $f_m \notin (\alpha-\varepsilon, \alpha+\varepsilon)$. Hence F has no limit point in $(0, 1)$; that is, its limit point (or limit points) must lie outside $(0, 1)$. Also, F is finite implies $F' = \emptyset$. Thus in all cases, $F' \subseteq \{0, 1\}$.

We will now prove the second half of Theorem 2.1; namely: *if* $u_n \to 0$ *and* F' *is a single-element set, or a subset of* $\{0, 1\}$ *, then* $\sum u_n$ *is convergent.*

CASE (i). Let $F' = \emptyset$. Clearly F is a finite set. If F contains only one element g of (f_n) , then $f_n = g$ for all sufficiently large n and $u_n \to 0$ implies $I_n - I_{n-1} \to 0$. Since I_n and I_{n-1} are integers, we should have $I_n = I_N$ for all n exceeding a sufficiently large positive integer N . The series, therefore, converges to $I_N + g$.

If F contains the elements $g_1, g_2, \ldots, g_k$, let $\delta_1 = \min(|g_i-g_j|)$, $i \neq j$, and $\delta_2 = \max(|g_i-g_j|)$. Since $0 \le g_i < 1$ for $1 \le i \le k$, $0 < \delta_1, \delta_2 < 1$. Choose now any positive $\varepsilon < \min(\delta_1, 1-\delta_2)$. (Note: every f_n is a g_i for some i .)

Since $u_n \to 0$, corresponding to this ε , there exists a positive integer N such that $n > N$ implies $|S_n-S_{n-1}| < \varepsilon$; that is,

$$|I_n-I_{n-1}-(f_{n-1}-f_n)| < \varepsilon ,$$

$$f_{n-1} - f_n - \varepsilon < I_n - I_{n-1} < f_{n-1} - f_n + \varepsilon .$$

But $f_n - f_{n-1} \le |f_n-f_{n-1}| \le \delta_2$ and so the above inequality gives

$$-\delta_2-\varepsilon < I_n-I_{n-1} < \delta_2+\varepsilon ,$$

which results in the inequality $-1 < I_n-I_{n-1} < 1$ since $\varepsilon < 1-\delta_2$. Therefore, $I_n = I_{n-1}$ for all $n > N$. Hence $|f_n-f_{n-1}| < \varepsilon < \delta_1$. But this, by definition of δ_1, implies $f_n = f_{n-1}$ for all $n > N$. Hence $S_n = I_n + f_n = I_N + f_N = S_N$ for $n \geq N$ and the series converges to S_N.

CASE (ii). Let F' contain the lone element ξ, $0 < \xi < 1$. Here F is an infinite set and its only limit point is ξ. Clearly ξ is a limit point of (f_n). We first prove the convergence of (f_n) by proving that its only limit point is ξ. Later we prove that the convergence of (f_n), under the condition that $u_n \to 0$ implies the convergence of the series $\sum u_n$.

If ξ is not the only limit point of (f_n), let g_1 be another limit point of it. Choose a positive number $\delta < \min(\xi, 1-\xi, |\xi-g_1|)$, so that the interval $(\xi-\delta, \xi+\delta)$ lies completely within $[0, 1]$ and does not contain g_1. Outside the interval $(\xi-\delta, \xi+\delta)$, there exist only finitely many members of the *aggregate* F since ξ is its only limit point; and it is possible that some (or all) of these are the limit points of the sequence (f_n). Our assumption that $F' = \{\xi\}$ implies that the only limit point of (f_n), around which an infinity of its *distinct* members cluster, is ξ. Hence the other limit points of the sequence (f_n) are members of it, each occurring in it an infinite number of times.

Let $g_1, \ldots, g_k$ be the limit points of (f_n) lying outside the closed interval $[\xi-\delta, \xi+\delta]$. (Note that one of $\xi - \delta$ and $\xi + \delta$ or both can be limit points of (f_n).) Since outside $(\xi-\delta, \xi+\delta)$ there are at most finitely many members of (f_n) which are *not* limit points of it, there must be a stage in the sequence beyond which f_n is one of $g_1, \ldots, g_k$ or $f_n \in [\xi-\delta, \xi+\delta]$, that is, there exists a positive integer N_1 such that $n > N_1$ implies

(1) $$f_n \in \{g_1, \ldots, g_k\} \cup [\xi-\delta, \xi+\delta] .$$

Let δ_1 be the least of the positive differences between any two

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members of the set

$$\{g_1, \ldots, g_k, \xi-\delta, \xi+\delta\}$$

and δ_2 the greatest of the same. Since $0 < \delta_1, \delta_2 < 1$, we can choose a positive $\varepsilon < \min(\delta_1, 1-\delta_2)$. As in Case (i), the condition $u_n \to 0$ will result in the inequality

$$(2) \qquad f_{n-1} - f_n - \varepsilon < I_n - I_{n-1} < f_{n-1} - f_n + \varepsilon ,$$

which holds for all n greater than a certain positive integer N_2. If $N > \max(N_1, N_2)$, $n > N$ implies (1) and (2). But (1) clearly implies $|f_n-f_{n-1}| < \delta_2$, and since $\varepsilon < 1-\delta_2$, (2) should give $-1 < I_n-I_{n-1} < 1$. Hence, again as in Case (i), $I_n = I_{n-1}$ and $|f_n-f_{n-1}| < \varepsilon$ for $n > N$.

Since g_1 repeats itself an infinite number of times, there exists a positive integer $m > N$ such that $f_m = g_1$. Hence

$$|g_1-f_{m-1}| = |f_m-f_{m-1}| < \varepsilon < \delta_1 .$$

But, if $g_1 \neq f_{m-1}$, we should have $|g_1-f_{m-1}| \geq \delta_1$, by definition of δ_1. Therefore, $g_1 = f_m = f_{m-1}$. In a similar way we have, also, $f_m = f_{m+1}$. So we should have

$$g_1 = f_N = f_{N+1} = f_{N+2} = \ldots .$$

But then F is a finite set, that is $F' = \emptyset$, which is a contradiction. Hence (f_n) is convergent to ξ.

The convergence of (f_n) implies $f_n - f_{n-1} \to 0$, which, together with the condition $u_n \to 0$, implies $I_n - I_{n-1} \to 0$: and this leads to the conclusion that (I_n) converges to an integer I. Therefore, the sequence (S_n) is convergent, that is, $\sum u_n$ is convergent.

CASE (iii). Let F' be a non-empty subset of $\{0, 1\}$. Clearly, F is an infinite set.

Let $0 < \alpha < 1$. Consider the series $\sum v_n$, where $v_1 = u_1 + \alpha$ and $v_n = u_n$ for $n > 1$. For the v-series, let g_n denote the fractional part of its nth partial sum and G, the range of the sequence (g_n). Clearly, g_n equals the fractional part of $f_n + \alpha$.

We prove that if $F' = \{0\}$ or $\{1\}$ or $\{0, 1\}$, then $G' = \{\alpha\}$, which, by Case (ii) implies convergence of the v-series and hence of $\sum u_n$.

(a) First, let $F' = \{0\}$. Choose a positive $\varepsilon < \min(\alpha, 1-\alpha)$, so that

$$0 < \varepsilon , \quad \alpha-\varepsilon < \alpha < \alpha+\varepsilon < 1 .$$

Since 0 is the only limit point of F, at most finitely many members of F lie in the interval $(\varepsilon, 1)$. We can easily verify that $g_i \notin (\alpha, \alpha+\varepsilon)$ implies $f_i \in (\varepsilon, 1)$. Hence, at most a finite number of members of G lie outside $(\alpha, \alpha+\varepsilon)$. Thus α is the only limit point of G; that is, $G' = \{\alpha\}$.

(b) If $F' = \{1\}$, only a finite number of members of F lie in the interval $(0, 1-\varepsilon)$ and $g_i \notin (\alpha-\varepsilon, \alpha)$ implies $f_i \in (0, 1-\varepsilon)$. Hence, as in (a), $G' = \{\alpha\}$.

(c) If $F' = \{0, 1\}$, there exist only finitely many members in the interval $(\varepsilon, 1-\varepsilon)$ and $g_i \notin (\alpha-\varepsilon, \alpha+\varepsilon)$ implies $f_i \in (\varepsilon, 1-\varepsilon)$, so that, again, α is the only limit point of G.

Thus, $G' = \{\alpha\}$, if $\emptyset \neq F' \subseteq \{0, 1\}$, and Case (iii) is now disposed of.

If 0 and 1 are the only limit points of the sequence (f_n), then we could prove, practically on the same lines as above, that α is the only limit point of (g_n). Thus (g_n) is convergent and this leads to the convergence of $\sum u_n$.

Theorem 2.1 is completely proved. We have, by the way, proved Theorem 2.1A also.

REMARK 3.3. If $F' = \{0\}$ or $\{1\}$ or $\{0, 1\}$, then, the series must converge to an integer; for, otherwise, by Lemma 3.1, F' must contain a non-zero real number less than 1. The different implications of the three cases may be stated as follows:

(a) $F' = \{0\}$ if and only if

(i) there exists a subsequence of (S_n) which tends to the integer, say I, through values greater than I,

(ii) $S_n < I$ for at most finitely many n;

(b) $F' = \{1\}$ if and only if

(i) there exists a subsequence of (S_n) which tends to I through values less than I,

(ii) $S_n > I$ for at most finitely many n;

(c) $F' = \{0, 1\}$ if and only if there exist subsequences of (S_n) tending to I from either side of it.

The above results can be proved easily and we omit their proof.

4. Proof of Theorem 2.2

Suppose, first, that the series $\sum u_n$ diverges to $+\infty$. For any $\alpha \in (0, 1)$ we shall prove that there exists a member of F in any neighbourhood of α.

Consider $m + \alpha$, where m is any positive integer. Since $S_n \to \infty$, there exists a positive integer N such that $n > N$ implies $m + \alpha < S_n$. Let m' be the least positive integer such that $m + \alpha < S_{m'}$. Clearly, m uniquely determines m' and $m \to \infty$ implies $m' \to \infty$. Further we have

$$S_{m'-1} \leq m+\alpha < S_{m'} .$$

(We take S_0 to be 0 here.) Since $u_{m'} = S_{m'} - S_{m'-1} \to 0$ as $m \to \infty$, the length of the interval $[S_{m'-1}, S_{m'})$ can be made as small as desired by making m sufficiently large. Now choose a positive $\varepsilon < \min(\alpha, 1-\alpha)$, so that

(3) $$0 < \alpha-\varepsilon < \alpha < \alpha+\varepsilon < 1 ,$$

and allow the interval to diminish to a length less than ε by taking m sufficiently large. Since $m + \alpha \in [S_{m'-1}, S_{m'})$, $m + \alpha \pm \varepsilon$ cannot lie inside the interval. Hence

(4) $$m+\alpha-\varepsilon < S_{m'-1} \le m+\alpha < S_{m'} \le m+\alpha+\varepsilon ,$$

and by inequality (3) we have

$$m < S_{m'-1} \le m+\alpha < S_{m'} < m+1 ,$$

which implies

$$I_{m'-1} = I_{m'} = m .$$

Therefore (4) gives

$$\alpha-\varepsilon < f_{m'-1} \le \alpha < f_{m'} \le \alpha+\varepsilon ,$$

showing that α is a limit point of F . Also 0 and 1 , which are limit points of $(0, 1)$ are limit points of F also. Hence F is everywhere dense in $[0, 1]$; that is, $F' = [0, 1]$.

If $\sum u_n$ diverges to $-\infty$, then $\sum (-u_n)$ diverges to $+\infty$. The fractional part of the nth partial sum, $-S_n$, of $\sum (-u_n)$ is $1 - f_n$ if $f_n \ne 0$, and 0 if $f_n = 0$. Clearly the set of the fractional parts of the partial sums $-S_n$ is everywhere dense in $[0, 1]$ if and only if F is everywhere dense in $[0, 1]$. Thus, in this case also, $F' = [0, 1]$, and this completes the proof of the first half of the theorem.

If $F' = [0, 1]$, certainly the series $\sum u_n$ cannot converge by Theorem 2.1: and since the series is assumed to be non-oscillatory, the series is divergent. Theorem 2.2 is now completely proved.

REMARK 4.1. If $u_n \to 0$ and $\sum u_n$ oscillates between finite and infinite limits or between $-\infty$ and $+\infty$, we should still have $F' = [0, 1]$ since (S_n) contains a subsequence that diverges to $+\infty$, or $-\infty$.

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5. Proof of Theorem 2.3

We now prove a lemma that is used in the proof of Theorem 2.3.

LEMMA 5.1. *Let* Λ, λ *be the limit superior and limit inferior of the set of values within which the bounded series* $\sum u_n$ *in which* $u_n \to 0$ *oscillates. Let* A *denote the closed interval* $[\lambda, \Lambda]$ *and* F_A *, the set of the fractional parts of the members of* A *. Then* $F_A = F'$ *, if* A *does not contain any integer; if* A *contains an integer, then*

$$F_A \cup \{1\} = F' \cup \{0, 1\} .$$

Proof. It is known (see [1], Part I, Chapter 3, Problem 100, p. 23) that for an oscillatory series in which the nth term tends to zero, the sequence of partial sums is everywhere dense between its limit superior and limit inferior. Thus, if $s \in [\lambda, \Lambda]$, there exists a subsequence of (S_n) that tends to s . Hence it follows that the set of values within which the series $\sum u_n$ oscillates is $[\lambda, \Lambda] = A$.

Suppose that A does not contain any integer. If $\alpha \in F_A$ $(\alpha \neq 0)$, there exists an integer I , such that $I + \alpha \in A$. Hence a subsequence of (S_n) converges to $I + \alpha$. If F_1 is the set of the fractional parts of its members, we have, by Lemma 3.1, $F_1' = \{\alpha\}$. Clearly $F_1 \subset F$ and so $F_1' \subset F'$ implying that $\alpha \in F'$. Thus $F_A \subset F'$.

If, on the other hand, $\alpha \in F'$, there exists a subsequence $(f_{m_i})_{i=1}^{\infty}$ of (f_n) tending to α . Let the corresponding subsequence of (S_n) be $(S_{m_i})_{i=1}^{\infty}$. Since the series is bounded, (S_{m_i}) either converges or oscillates within finite limits. (We do not know if the difference between consecutive members of this sequence tends to zero.) If it converges to b , then, by Lemma 3.1, the sequence (f_{m_i}) tends to b_f , the fractional part of b . Hence $\alpha = b_f$. Clearly, $b \in A$ and so $\alpha \in F_A$. If the sequence (S_{m_i}) oscillates, it has a convergent subsequence; and since its corresponding subsequence of fractional parts (being a subsequence of (f_{m_i})) also tends to α , it follows, as in the previous case, that $\alpha \in F_A$. Thus $F' \subset F_A$. Hence $F_A = F'$.

If A contains an integer, we have to alter the result as $F_A \cup \{1\} = F' \cup \{0, 1\}$, which can be proved easily.

Proof of Theorem 2.3. Case (i). Let $\Lambda-\lambda < 1$.

(a) Let λ and Λ lie between two consecutive integers. Also let none of them be an integer. If Λ_f and λ_f denote the fractional parts of Λ and λ , then $F_A = [\lambda_f, \Lambda_f]$. So, by Lemma 5.1, $F' = [\lambda_f, \Lambda_f]$.

(b) If λ is an integer, Λ is not an integer and $F_A = [0, \Lambda_f]$ since $\lambda_f = 0$. Clearly there exists a subsequence of (S_n) which tends to λ through values greater than λ . By Remark 3.3 (a), $0 \in F'$. If there is *no* subsequence of (S_n) tending to λ through values less than it, then, again by (b) of the remark, $1 \notin F'$. In this case, the result $F_A \cup \{1\} = F' \cup \{0, 1\}$ implies $F_A \cup \{1\} = F' \cup \{1\}$. The result reduces to

$$F' = F_A = [0, \Lambda_f]$$

since 1 is not a member of F_A or F' .

If there *is* a subsequence of (S_n) which tends to λ through values less than it, then there exists a subsequence of (f_n) tending to 1 and $1 \in F'$. Already, $0 \in F'$ and we have $F' \cup \{0, 1\} = F'$. Thus, in this case,

$$F' = [0, \Lambda_f] \cup \{1\} = [0, 1] - (\Lambda_f, 1) .$$

(c) If Λ is an integer, λ is not an integer and $\Lambda_f = 0$. Here $F_A = \{0\} \cup [\lambda_f, 1)$. In this case $1 \in F'$. If there exists a subsequence of (S_n) which tends to Λ through values greater than it, then $0 \in F'$; otherwise $0 \notin F'$. Hence

$$F' \cup \{0, 1\} = F_A \cup \{1\}$$

implies

$$F' \cup \{0\} = [\lambda_f, 1] \cup \{0\} ,$$

which implies

$$F' = [\lambda_f, 1]$$

if $0 \notin F'$, and

$$F' = [\lambda_f, 1] \cup \{0\} = [0, 1] - (0, \lambda_f) ,$$

if $0 \in F'$.

(d) If none of λ and Λ is an integer, then $[\Lambda]$ lies between λ and Λ , and $F_A = [\lambda_f, 1) \cup [0, \Lambda_f]$. Since $\Lambda-\lambda < 1$, $\Lambda_f < \lambda_f$ and hence

$$F_A \cup \{1\} = [0, \Lambda_f] \cup [\lambda_f, 1] .$$

Since $\lambda < [\Lambda] < \Lambda$, there exists subsequences of (s_n) tending to $[\Lambda]$ from either side of it (by Remark 3.3 (c)) and so 0 and $1 \in F'$. Therefore, in this case,

$$F' = [0, \Lambda_f] \cup [\lambda_f, 1] = [0, 1] - (\Lambda_f, \lambda_f) .$$

Case (ii). If $\Lambda - \lambda \geq 1$, we can prove, on the lines of Case (i), that $F' = [0, 1]$. If the series has infinite oscillation, then also, by Remark 4.1, $F' = [0, 1]$.

The proof of Theorem 2.3 is now complete.

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