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FREE SUBGROUPS AND THE RESIDUAL NILPOTENCE OF THE GROUP OF UNITS OF MODULAR AND p-ADIC GROUP RINGS

BY

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ABSTRACT. Let G be a group, let RG be the group ring of the group G over the unital commutative ring R and let $U(RG)$ be its group of units. Conditions which imply that $U(RG)$ contains no free noncyclic group are studied, when R is a field of characteristic $p \neq 0$, not algebraic over its prime field, and G is a solvable-by-finite group without p-elements. We also consider the case $R = \mathbb{Z}_p$, the ring of p-adic integers and G torsion-by-nilpotent torsion free group. Finally, the residual nilpotence of $U(\mathbb{Z}_p G)$ is investigated.

1. **Introduction**. Let G be a group, let RG be the group ring of the group G over the unital commutative ring R, and let $U(RG)$ be its group of units. As a sequel to our works [2], [3] and [4], we propose to study conditions which entail that $U(RG)$ contains no free noncyclic subgroup, when R is a field of characteristic $p \neq 0$, not algebraic over its prime field, and G is a solvable-by-finite group without p-elements. We consider too, the case in which $R = \mathbb{Z}_p$, the ring of p-adic integers, and G is an extension of a torsion group by a nilpotent torsion-free group. This is the content of section 2.

In section 3, motivated by a paper of Musson and Weis [7], and pursuing conclusions analogous to [4], Theorems 3.3, 3.4 and 3.5, we study the residual nilpotence of $U(\mathbb{Z}_p G)$.

We are indebted to Professor Sudarshan K. Sehgal, who suggested to us to generalize [7], Theorem VI 4.2 to the modular case, and to the referee for many useful comments, which simplified the proofs of Lemma 2.3 and the sufficiency part of Theorem 3.1.

2. **Free subgroups in the group of units.** We have known, while carrying out this work, that a version of Theorem 2.1 appears in [10]. However, our proof is simpler and our result more general and so, we believe, deserves be stated explicitly.

THEOREM 2.1. *Let K be a field of characteristic $p \neq 0$, not algebraic over its prime field $GF(p)$ such that, if $p = 2$ the degree of transcendence of K over $GF(2)$ is at*

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least 2. Let G be a solvable-by-finite group without p-elements, and suppose that $U(KG)$ contains no noncyclic free subgroups. Then (*) *the set of torsion elements T of G forms an abelian subgroup of G, and every idempotent of KT is central in KG.*

As a partial converse, if G satisfies the condition (*) *above and G/T is nilpotent, then $U(KG)$ contains no noncyclic free subgroups.*

PROOF. This proof is a blend of the proofs of [5], Theorem 1 and [7], Theorem VI.4.2, respectively, and begins copying the proof of Theorem 1 of [5].

Let T be the largest periodic normal subgroup of G, and let H be a solvable normal subgroup of finite index of G. Let M/H' be the torsion subgroup of H/H'. Since each finite set of elements of M lies in some finite extension of H', induction on the derived length of H allows us to assume that the set of elements of finite order of M is a subgroup. The same then holds for H, whence if $L = HT$, then L/T is torsion free. Also, of course $[G:L] < \infty$.

We claim that if a is an element of finite order in G then (1) $[L,a] \leqslant T$.

Suppose not. Then we can choose a of order a power of a prime q such that $[L,a] \nleq T$ but $[L,a^q] \leqq T$. Now, arguing as in [6], proof of Theorem 1, we obtain a-invariant subgroups $T \leq V_1 \leq U_1$ such that U_1/V_1 is free abelian of finite rank and a operates nontrivially on U_1/V_1, while a^q centralizes it.

Let r be a prime different from p and q and write $X = U_1/V_1$. Again, as in [5], we deduce that $[X, a] \nleq X^r$, and applying Maschke's Theorem, that there exists an a-invariant subgroup Y with $X^r \leqq Y \leqq X$, such that a operates irreducibly on Y/X^r and $C_{\langle a\rangle}(Y/X^r) = \langle a^q\rangle$.

Write $Y = U/V_1$, $X^r = V/V_1$. Then we have $V\langle a^q\rangle \lhd U\langle a\rangle$ and $U\langle a\rangle/V\langle a^q\rangle \cong N\langle\bar{a}\rangle$, where $N = U/V$ and $\bar{a}$, the coset containing a, has order q and operates faithfully and irreducibly by conjugation on the elementary abelian r-group N.

Note that $\langle V\langle a^q\rangle, a\rangle$ is not normal in $U\langle a\rangle$, since otherwise $\langle\bar{a}\rangle$ is normal in $N\langle\bar{a}\rangle$ and $N\langle\bar{a}\rangle = N \times \langle\bar{a}\rangle$, a contradiction.

By Wedderburn's Theorem

$$K(N\langle\bar{a}\rangle) \cong \bigoplus_{i=1}^{m} M(n_i, D_i),$$

a direct sum of full matrix rings over division rings. Let $|\langle a\rangle| = q^n$. Since $\bar{e}$, for $e = 1/q^n \sum_{i=1}^{q^n} a^i$ is a noncentral idempotent of $K(N\langle\bar{a}\rangle)$, its projection in at least one component is noncentral. Thus, we have a homomorphism

$$\pi: K(U\langle a\rangle) \to K(N\langle\bar{a}\rangle) \to M(l,D),\ l > 1.$$

After conjugation by an element of $GL(l,D)$ we may assume that

$$\pi(e) = \left[\begin{array}{c|c} I_S & \\ \hline & \end{array}\right]$$

where I_S is the identity matrix of dimension $s < l$.

Now, let us denote by E_{ij} the $l \times l$ matrix with 1 in its (i,j)-entry and zero elsewhere. Choose $X_{1l}, X_{l1} \in K(U\langle a\rangle)$ such that

$$\pi(X_{1l}) = E_{1l}, \qquad \pi(X_{l1}) = E_{l1}$$

Hence

$$\pi((1-e)X_{l1}e) = E_{l1}, \qquad \pi(eX_{1l}(1-e)) = E_{1l}$$

We consider two cases: (i) $p \neq 2$. Let λ be an element of K transcendental over $GF(p)$ and put

$$\alpha = \lambda(1-e)X_{l1}e, \qquad \beta = \lambda e X_{1l}(1-e)$$

Then $\alpha^2 = \beta^2 = 0$, $1+\alpha$ and $1+\beta$ are units of $K(U\langle a\rangle)$ and $\pi(1+\alpha) = 1 + \lambda E_{l1}$, $\pi(1+\beta) = 1 + \lambda E_{1l}$.

As is well known, see [11], Lemma 2.8,

$$\langle \pi(1+\alpha), \quad \pi(1+\beta)\rangle = \langle \pi(1+\alpha)\rangle * \langle \pi(1+\beta)\rangle,$$

which contains a free noncyclic group, a contradiction.

(ii) $p = 2$. Let λ and μ be elements of K algebraically independent over $GF(2)$.

Then, using the same notation as in the case $p \neq 2$, we consider the elements

$$\alpha = \lambda e + \lambda^{-1}(1-e) + \mu(1-e)X_{l1}e$$

and

$$\beta = \lambda e + \lambda^{-1}(1-e) + \mu e X_{1l}(1-e)$$

belonging to $K(U\langle a\rangle)$, whose inverses are

$$\alpha^{-1} = \lambda^{-1}e + \lambda(1-e) - \mu(1-e)X_{l1}e$$

and

$$\beta^{-1} = \lambda^{-1}e + \lambda(1-e) - \mu e X_{1l}(1-e).$$

Therefore

$$\pi(\alpha) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\lambda I_s} & 0 \\ \hline 0 & 0 & \lambda^{-1} I_{l-s} \\ \mu & 0 & \end{array}\right]$$

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and

$$\pi(\beta) = \left[\begin{array}{c|c} \lambda I_s & \begin{matrix} 0 & \mu \\ 0 & 0 \end{matrix} \\ \hline 0 & \lambda^{-1} I_{l-s} \end{array}\right]$$

and we have a homomorphism $\tau : \langle \pi(\alpha), \pi(\beta) \rangle \to SL(2, K)$ such that

$$\tau(\pi(\alpha)) = \begin{bmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{bmatrix}, \qquad \tau(\pi(\beta)) = \begin{bmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{bmatrix}$$

By [11], Exercise 2.2 $\tau(\pi(\alpha))$ and $\tau(\pi(\beta))$ freely generate a free group, which is a contradiction.

By (1), we now have

$$[G/T : C_{G/T}(aT)] < \infty$$

Hence, by Dietzmann's Lemma, aT lies in a finite normal subgroup of G/T.

This gives $a \in T$ and hence T is a subgroup.

As a torsion solvable-by-finite group T is locally finite. Since KT has no free noncyclic group, by [2], Theorem 3.1, T is abelian. Now, arguing as in [5], Lemma 4 and [7], Lemma VI.3.12, we prove that every subgroup of T is normal in G and that every idempotent of KT is central in KG, respectively.

For the proof of the partial coverse argue as in [7], Theorem VI.4.12.

Now, we turn our attention to the group of units of p-adic group rings.

LEMMA 2.2. *Let p be a rational prime, let $\mathbb{Z}_p$ be the ring of p-adic integers with field of fractions $\mathbb{Q}_p$, and let G be a finite group whose order is relatively prime to p. Then*

$$\mathbb{Z}_p G = \bigoplus_{i=1}^{r} M(n_i, E),$$

a direct sum of full matrix rings over E_i, where E_i is the integral closure of $\mathbb{Z}_p$ in a finite extension of $\mathbb{Q}_p$.

PROOF. See [8], Lemma 2.

Again let p be a rational prime, let $\mathbb{Z}_{(p)}$ be localization of $\mathbb{Z}$ at the prime ideal (p), let $\mathbb{H}$ be the usual quaternion algebra over the rationals, i.e.,

$$\mathbb{H} = \{\alpha + \beta i + \gamma j + \delta k \mid i^2 = j^2 = -1, \quad ij = -ji = k, \quad \alpha, \beta, \gamma, \delta \in \mathbb{Q}\}$$

and let

$$\mathbb{H}_{(p)} = \{\alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{Z}_{(p)}\}$$

We will denote by $\mathbb{H}^*_{(p)}$ the group of units of $\mathbb{H}_{(p)}$.

LEMMA 2.3. *$\mathbb{H}^*_{(p)}$ is not solvable-by-finite.*

PROOF. Suppose not. Then the infinite group $G = \mathbb{H}^*_{(p)}/\mathbb{Z}^*_{(p)}$ is solvable-by-finite, and hence contains a nontrivial abelian normal subgroup. To see this take $H \lhd G$ with G/H finite and H soluble, and consider the last nontrivial term of the derived series of H.

Thus, let $A \lhd \mathbb{H}^*_{(p)}/\mathbb{Z}^*_{(p)}$ be an abelian subgroup. Let $\alpha \in A$ be such that $\bar{\alpha} \neq \bar{1}$, and let $\gamma \in \mathbb{H}^*_{(p)}$. Then denoting $\gamma^{-1}\alpha\gamma$ by α^γ we have $\bar{\alpha}\,\bar{\alpha}^\gamma = \bar{\alpha}^\gamma\,\bar{\alpha}$, i.e., $\alpha\,\alpha^\gamma = \alpha^\gamma\,\alpha\delta$, where $\delta \in \mathbb{Z}^*_{(p)}$.

Applying the norm function N defined on the real quaternions to both sides of the last equality, we conclude that $N(\delta) = \delta^2 = 1$, and so $\delta = \pm 1$.

Therefore we have shown that if $\gamma \in \mathbb{H}^*_{(p)}$, either (a) $\alpha\,\alpha^\gamma = \alpha^\gamma\,\alpha$ or (b) $\alpha\,\alpha^\gamma = -\alpha^\gamma\,\alpha$.

Let $\alpha = x + yi + zj + wk$. Taking $\gamma = i, j, k, 1 + pi, 1 + pj, 1 + pk$, and considering the possibilities (a) and (b) above, we are able to show that the elements appearing in the support of α are all zero. This is a contradiction and proves the Lemma.

THEOREM 2.4. *Let G be a finite group. Then $U(\mathbb{Z}_pG)$ does not contain a free noncyclic group if, and only if, G is abelian.*

PROOF. Only necessity deserves a proof.

Suppose that G is not abelian. Since $\mathbb{Z}_p G$ contains $\mathbb{Z}G$, by [5], Theorem 2, G is a Hamiltonian 2-group. i.e., G is the direct product of an elementary abelian 2-group by the quaternion group of order 8, K_8.

Assume first, that $p \neq 2$. Hence, by Lemma 2.2 $U(\mathbb{Z}_pK_8)$ contains $GL(2, \mathbb{Z})$ which, as in well known, contains the free group of rank two

$$\left\langle \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle$$

Now, let us assume that $p = 2$.

Let $\mathbb{Q}_2$ be the 2-adic completion of $\mathbb{Q}$, let

$$K_8 = \langle a, b \mid a^4 = 1, \quad a^2 = b^2, \quad bab^{-1} = a^3 \rangle$$

and let

$$\mathbb{H}_2 = \mathbb{Q}_2 \underset{\mathbb{Q}}{\otimes} \mathbb{H}$$

The function

$$\psi : \mathbb{Q}_2 K_8 \to \mathbb{Q}_2 \oplus \mathbb{Q}_2 \oplus \mathbb{Q}_2 \oplus \mathbb{Q}_2 \oplus \mathbb{H}_2$$

$$\psi(a) = (1, 1, -1, -1, i)$$

$$\psi(b) = (1, -1, 1, -1, j)$$

defines an isomophism of $\mathbb{Q}_2$-algebras. $\mathbb{Q}_2 K_8$ has a central primitive idempotent $e = 1/2\,(1 - a^2)$ and the image of $\mathbb{Z}_2 K_8 e$ by ψ is $R = \{\alpha + \beta i + \gamma j + \delta k \in \mathbb{H}_2 \mid \alpha, \beta, \gamma, \delta \in \mathbb{Z}_2\}$. Arguing as in [5], Lemma 1, we prove that $U(\mathbb{Z}_2 K_8)e$ has finite index in $U(\mathbb{Z}_2 K_8 e)$. So, we get a contradiction once we can show that $U(R)$ contains a free noncyclic group. And this is ensured by [9], Theorem 1, which gives as only possible alternative that $\mathbb{H}_2^*$ be solvable-by-finite, and this is precluded by Lemma 2.3.

COROLLARY 2.5. *Let G be the extension of a locally finite group T by a nilpotent torsion free group G/T, and suppose that $U(\mathbb{Z}_p G)$ has no free noncyclic subgroups. Then T is abelian, with every subgroup normal in G, and moreover, if $T = P \times H$ is the decomposition of T as a direct product of a p-subgroup P by a p'-subgroup H, then every idempotent of $\mathbb{Z}_p H$ is central in $\mathbb{Z}_p G$.*

As a partial converse, if either $T = P$ or $T = H$ and the conditions above are satisfied, then $U(\mathbb{Z}_p G)$ has no noncyclic free subgroups.

PROOF. **Necessity**: By Theorem 2.4 it follows that T is abelian. So G is solvable and by [5], Theorem 1, every subgroup of T is normal in G.

Let now $T = P \times H$ be as above, let e be an idempotent of $\mathbb{Z}_p H$, and suppose that some $x \in G \backslash T$ does not centralize e. We can consider $G = \langle x, H_1 \rangle$, where $H_1 = \langle supp\, e \rangle$ is the finite abelian subgroup generated by the support of e.

By Lemma 2.2, and keeping the same notation

$$\mathbb{Z}_p H_1 = \bigoplus_{i=1}^{r} E_i$$

Therefore some idempotent of $\mathbb{Z}_p H_1$ that appears in its decomposition is noncentral in $\mathbb{Z}_p G$, and so by [7] Lemma VI 3.12 $U(\mathbb{Z}_p G)$ contains $GL(m, \mathbb{Z}_p)$, the $m \times m$ general linear group over $\mathbb{Z}_p$, for some $m > 1$. This is a contradiction.

Sufficiency: We claim that $U(\mathbb{Z}_p G)$ is solvable. We consider two cases. (i) $T = P$. Then apply [1], Theorem A. (ii) $T = H$. Then same proof of [7], Theorem VI 4.12 works, if we observe that supposing G finitely generated implies that T is finite. So, by Lemma 2.2

$$\mathbb{Z}_p T = \bigoplus_{i=1}^{r} E_i,$$

and by [7], Lemma VI.3.22, every $u \in U(\mathbb{Z}_p G)$ can be written as $u = \Sigma \alpha_i g_i$, with $\alpha_i \in U(E_i)$ and $g_i \in G$.

3. **Residual nilpotence of the group of units of *p*-adic group rings**. Now, with almost no additional extra effort we can determine when $U(\mathbb{Z}_p G)$ is residually nilpotent.

First we recall that a group G is p-abelian if G', the commutator subgroup of G, is a p-group.

THEOREM 3.1. *Let G be a finite nonabelian group. Then $U(\mathbb{Z}_p G)$ is residually nilpotent if and only if G is nilpotent and p-abelian.*

PROOF. **Necessity**: Since $U(\mathbb{Z}_pG)$ is residually nilpotent and $U(\mathbb{Z}G) \subseteq U(\mathbb{Z}_pG)$, by [6], Corollary 2.3, G is nilpotent and q-abelian for some prime q.

Let us suppose that $q \neq p$. Since G is nilpotent and q-abelian, G can be written as the direct product $G = H \times P$, of the abelian q'-group H by the nonabelian q-subgroup P.

By Lemma 2.2, with the same notation,

$$\mathbb{Z}_pP = \bigoplus_{i=1}^{r} M(n_i, E_i).$$

But P is nonabelian and hence, for some k, $1 \leq k \leq r$, we have that $n_k > 1$, and therefore it follows that $GL(n_k, E_k)$ is residually nilpotent, a contradiction.

Sufficiency: Since G is nilpotent and p-abelian, as above, we have $G = H \times P$. Hence

$$\mathbb{Z}_pG = \mathbb{Z}_p(H \times P) = (\mathbb{Z}_pH)P$$

By Lemma 2.2

$$\mathbb{Z}_pH = \bigoplus_{i=1}^{r} E_i$$

and therefore

$$\mathbb{Z}_pG = \left(\bigoplus_{i=1}^{r} E_i\right)P = \bigoplus_{i=1}^{r} E_iP.$$

We claim that for every i, $1 \leq i \leq r$, $U(E_iP)$ is residually nilpotent.

It is well known that there is a unique extension of the p-adic valuation of $\mathbb{Q}_p$ to F_i, and that E_i is the discrete valuation ring of F_i. Let π_iE_i be the maximal ideal of E_i and $pE_i = \pi_i^m E_i$. Then

$$\bigcap_n p^n E_i = \bigcap_n \pi_i^{mn} E_i = \{0\}$$

Let $R_n = E_i/p^n E_i$. Then the group of units of R_nP is nilpotent, for the augmentation ideal of R_nP is nilpotent. Since $\bigcap_{n=1}^{\infty} p^n E_i = \{0\}$ we find that $U(E_iP)$ is residually nilpotent.

COROLLARY 3.2. *Let p be a rational prime and let G be a nilpotent group such that every p'-element has prime order. Then, if $U(\mathbb{Z}_pG)$ is residually nilpotent the torsion subgroup T of G can be written as $T = P \times H$, the direct product of a p-subgroup P by a central p'-subgroup H.*

PROOF. Since only local properties are involved, we can assume that G is finitely generated and so T is finite.

By Theorem 3.1, $T = P \times H$, the direct product of a finite p-subgroup P, by a finite abelian p'-subgroup H, with both P and H normal in G.

We claim, first, that every subgroup of T is normal in G. Suppose not. Then there exist a subgroup L of T and an element $x \in G$ which does not normalize L. Arguing as in [5], Lemma 4, we conclude that $U(\mathbb{Z}_pG)$ contains $GL(2, \mathbb{Z}_p)$, a contradiction.

Suppose now that T is not central. We may assume that $G = \langle T, x \rangle$ and x, $|\langle x \rangle| = \infty$, does not centralize T.

By Lemma 2.2

$$\mathbb{Z}_p T = \bigoplus_{i=1}^{r} E_i.$$

Observe that every idempotent of $\mathbb{Z}_p T$ that appears in the decomposition above is central in $\mathbb{Z}_p G$. If this is not the case, by [7], Lemma VI.3.12, $U(\mathbb{Z}_p G)$ contains $GL(m, \mathbb{Z}_p)$ for some $m > 1$, a contradiction.

Hence

$$\mathbb{Z}_p G = (\mathbb{Z}_p T)_\sigma \langle x \rangle = \left(\bigoplus_{i=1}^{r} E_i \right)_\sigma \langle x \rangle = \bigoplus_{i=1}^{r} (E_i)_\sigma \langle x \rangle,$$

where $\alpha \mapsto \sigma(\alpha) = x\,\alpha\,x^{-1}$ is the automorphism of E_i induced by conjugation by x, and $(E_i)_\sigma \langle x \rangle$ denotes the skew group ring of $\langle x \rangle$ over E_i, with automorphism σ.

Thus $U(\mathbb{Z}_p G)$ contains the nonabelian subgroup $L = \langle \theta, x \,|\, \theta^q = 1, x\theta x^{-1} = \theta^j \rangle$, where q is a rational prime greater than 2, and q and j are relatively prime.

Finally, as in [4], Theorem 3.5, L is not residually nilpotent.

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