HOMOTOPY CLASSIFICATION OF FILTERED COMPLEXES

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The homology functor from the category of free abelian chain complexes and homotopy classes of maps to that of graded abelian groups is full and replete (surjective on objects up to isomorphism) and reflects isomorphisms. Thus such a complex is determined to within homotopy equivalence (although not a unique homotopy equivalence) by its homology. The homotopy classes of maps between two such complexes should therefore be expressible in terms of the homology groups, and such an expression is in fact provided by the Künneth formula for Hom, sometimes called 'the homotopy classification theorem'.

In [4] Kelly showed that the functor assigning to a short exact sequence of free abelian chain complexes its long exact homology sequence is again full and replete and reflects isomorphisms. Partial information about the kernel of this functor was found in [5]: but not enough to provide a homotopy classification theorem for this case.

Since a short exact sequence of free abelian chain complexes may be considered as a free abelian chain complex with a filtration of length 2 the question arises whether the above results admit appropriate generalizations for complexes with a filtration of finite length $N-1$.

The main purpose of this paper is to exhibit for such filtered complexes a functor which is full and replete and reflects isomorphisms, and to provide a a homotopy classification theorem for this case. We do not entirely restrict ourselves to free abelian complexes, but then we must content ourselves with an analogue of the Künneth spectral sequence instead of the short exact sequence.

The image category $\mathscr{X}_N^e$ for this functor on $(N-1)$-filtered complexes was considered by Wall in [8], and the (relative) projectives were determined in [7]. The functor which assigns to an $(N-1)$-filtered complex its spectral sequence plus filtered limit factors through $\mathscr{X}_N^e$. Yet in many ways the objects of $\mathscr{X}_N^e$ are easier to deal with than spectral sequences.

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As an example, the reader will easily verify that, in the case $N = 3$, our results provide a homotopy classification theorem for short exact sequences of complexes of free abelian groups. Our classification theorem gives in this case a Künneth-like exact sequence

$$0 \to \mathrm{Ext}^1(HA, HB) \to H[A, B] \to \mathrm{Hom}(HA, HB) \to 0.$$

Here $H[A, B]$ is the group of homotopy classes of chain-map triples between the short exact sequences A, B of chain complexes; HA and HB are the long exact homology sequences; and Ext^1 is relative to a suitable projective class. In fact, the projectives are the long exact sequences all of whose terms are projective. Moreover, the above short exact sequence splits, as we show for a general N in §5; so that it does determine $H[A, B]$ in terms of HA and HB.

The results of §§4 and 5 admit an extension, corresponding in the above example to dropping the requirement that the components of B be free; this would give a 'universal coefficient theorem' alongside the above classification theorem; but we have omitted it to avoid complicating further the exposition. It is just a matter of extending the domain category, and accepting the fact that then only some objects (including the free B in the above case) admit projective resolutions; an extra 'five-lemma' argument is needed in §5.

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1. Triangulated categories

A *stable category* is an additive category $\mathscr{A}$ together with an additive automorphism $\Sigma: \mathscr{A} \to \mathscr{A}$ called the *suspension functor*. A *triangle* in $\mathscr{A}$ is a diagram

$$A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'.$$

An *arrow of triangles* is a commutative diagram (f', f, f''):

$$\begin{array}{ccccccc} A' & \xrightarrow{a''} & A & \xrightarrow{a'} & A'' & \xrightarrow{a} & \Sigma A' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & {\scriptstyle \Sigma f'}\downarrow \\ B' & \xrightarrow[b'']{} & B & \xrightarrow[b']{} & B'' & \xrightarrow[b]{} & \Sigma B' \end{array}$$

A class $\mathscr{T}$ of triangles of $\mathscr{A}$ will be called a *triangulation* of $\mathscr{A}$ when the following conditions are satisfied:

T0. *any triangle isomorphic to a triangle in* $\mathscr{T}$ *is in* $\mathscr{T}$;

T1. *for each* A *in* $\mathscr{A}$, $0 \to A \xrightarrow{1} A \to 0$ *is in* $\mathscr{T}$;

T2. *a triangle* $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'$ *is in* $\mathscr{T}$ *if and only if* $A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A' \xrightarrow{\Sigma a''} \Sigma A$ *is in* $\mathscr{T}$;

T3. *for each arrow* $a'\colon A \to A''$ *in* $\mathscr{A}$, *there exists a triangle* $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'$ *in* $\mathscr{T}$;

T4. *for triangles* $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'$, $B' \xrightarrow{b''} B \xrightarrow{b'} B'' \xrightarrow{b} \Sigma B'$ *in* $\mathscr{T}$ *and arrows* $f\colon A \to B, f''\colon A'' \to B''$ *in* $\mathscr{A}$ *such that* $f''a' = b'f$, *there exists an arrow* $f'\colon A' \to B'$ *in* $\mathscr{A}$ *such that* (f', f, f'') *is an arrow of triangles.*

Let ***Abg*** denote the category of abelian groups.

If $\mathscr{A}, \mathscr{X}$ are stable categories then a functor $F\colon \mathscr{A} \to \mathscr{X}$ is *stable* when it is additive and $F\Sigma = \Sigma F$.

If $\mathscr{A}$ has a triangulation $\mathscr{T}$ and $\mathscr{X}$ is an abelian category then a functor $F\colon \mathscr{A} \to \mathscr{X}$ is *homological* when, for each triangle

$$A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'$$

in $\mathscr{T}$, the sequence

$$FA' \xrightarrow{Fa''} FA \xrightarrow{Fa'} FA'' \xrightarrow{Fa} F\Sigma A'$$

is exact in $\mathscr{X}$.

The following properties of a triangulation $\mathscr{T}$ of $\mathscr{A}$ are mostly due to Puppe [6].

T5. *If* $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'$ *is in* $\mathscr{T}$, *then* $a'.a'' = 0$, $a.a' = 0$, $\Sigma a''.a = 0$.

T6. *For each object* B *of* $\mathscr{A}$, *the functors*

$$\mathscr{A}(B, -)\colon \mathscr{A} \to \boldsymbol{Abg}, \quad \mathscr{A}(-, B)\colon \mathscr{A} \to \boldsymbol{Abg}^{op}$$

are homological.

T7. *If* (f', f, f'') *is an arrow of triangles in* $\mathscr{T}$ *and any two of* f', f, f'' *are isomorphisms in* $\mathscr{A}$, *then so is the third.*

T8. *Suppose* $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A', B' \xrightarrow{b''} B \xrightarrow{b'} A'' \xrightarrow{b} \Sigma B'$ *are in* $\mathscr{T}$. *Then*:

(a) $A = 0$ *if and only if* a *is an isomorphism*;

(b) $a = 0$ *if and only if there exists a direct sum situation*

$$A' \underset{\longleftarrow}{\overset{a''}{\longrightarrow}} A \underset{\longleftarrow}{\overset{a'}{\longrightarrow}} A'';$$

(c) *if* $b.a' = 0$, $a.b' = 0$ *then* $A' \oplus B \cong A \oplus B'$.

T9. $\mathscr{T}$ *is closed under finite direct sums.*

T10. *If* $\mathscr{T}'$ *is a triangulation of* A *with* $\mathscr{T}' \subset \mathscr{T}$, *then* $\mathscr{T}' = \mathscr{T}$.

2. A general classification theorem

Suppose $\mathscr{A}$ is a stable category with a triangulation $\mathscr{T}$, suppose $\mathscr{X}$ is a stable abelian category, and suppose $F\colon \mathscr{A} \to \mathscr{X}$ is a stable homological functor. With this data we shall develop a relative homological algebra in $\mathscr{A}$.

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An object P of $\mathscr{A}$ is *F-projective* when FP is projective in $\mathscr{X}$ and the function

$$F: \mathscr{A}(P,A) \to \mathscr{X}(FP,FA)$$

is a bijection for all A in $\mathscr{A}$.

COMPARISON THEOREM. *If $A' \xrightarrow{a''} P \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'$ is in $\mathscr{T}$ with P F-projective, and if $B' \xrightarrow{b''} B \xrightarrow{b'} B'' \xrightarrow{b} \Sigma B'$ is in $\mathscr{T}$ with $Fb = 0$, then, for each arrow $f'': A'' \to B''$ in $\mathscr{A}$, there is an arrow (f', f, f''):*

$$\begin{array}{ccccccc} A' & \xrightarrow{a''} & P & \xrightarrow{a'} & A'' & \xrightarrow{a} & \Sigma A' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & {\scriptstyle \Sigma f'}\downarrow \\ B' & \xrightarrow[b'']{} & B & \xrightarrow[b']{} & B'' & \xrightarrow[b]{} & \Sigma B' \end{array}$$

of triangles. If (g', g, f'') is another such arrow of triangles, then there exist arrows $s', s: P \to B'$ such that $g' - f' = s'a''$, $g - f = b''s$.

For any object A of $\mathscr{A}$ and non-negative integer r, the statement $\dim_F A \leqq r$ is defined inductively as follows. Firstly, $\dim_F A = 0$ means A is F-projective. For $r > 0$, $\dim_F A \leqq r$ means there exists a triangle $A' \xrightarrow{a} P \xrightarrow{a'} A \xrightarrow{x} A'$ in $\mathscr{T}$ where $Fx = 0$, P is F-projective and $\dim_F A' \leqq r - 1$. Write $\dim_F A = r$ when $\dim_F A \leqq r$ but $\dim_F A \nleqq r - 1$. Write $\dim_F A = \infty$ if there exists no integer r such that $\dim_F A \leqq r$.

CLASSIFICATION THEOREM. *For objects A, B in $\mathscr{A}$, if $\dim_F A \leqq 1$, then there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathscr{X}}(F\Sigma A, FB) \to \mathscr{A}(A,B) \xrightarrow{F} \mathscr{X}(FA,FB) \to 0$$

of abelian groups.

PROOF. Since $\dim_F A \leqq 1$ there is a triangle $Q \xrightarrow{a} P \xrightarrow{q} A \xrightarrow{p} \Sigma Q$ in $\mathscr{T}$ where $Fp = 0$ and P, Q are F-projective. So we have exact sequences

$$\mathscr{A}(\Sigma P, B) \xrightarrow{\mathscr{A}(\Sigma a, 1)} \mathscr{A}(\Sigma Q, B) \to \mathscr{A}(A,B) \xrightarrow{\mathscr{A}(q,1)} \mathscr{A}(P,B) \xrightarrow{\mathscr{A}(a,1)} \mathscr{A}(Q,B)$$

$$0 \to FQ \xrightarrow{Fa} FP \to FA \to 0$$

$$0 \to F\Sigma Q \xrightarrow{F\Sigma a} F\Sigma P \to F\Sigma A \to 0$$

$$0 \to \mathscr{X}(FA,FB) \to \mathscr{X}(FP,FB) \xrightarrow{\mathscr{X}(Fa,1)} \mathscr{X}(FQ,FB)$$

$$\mathscr{X}(F\Sigma P, FB) \xrightarrow{\mathscr{X}(F\Sigma a,\, 1)} \mathscr{X}(F\Sigma Q, FB) \to \mathrm{Ext}^1_{\mathscr{X}}(F\Sigma A, FB) \to 0,$$

and commutative diagrams

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$$\begin{array}{ccc} \mathscr{A}(\Sigma P,B) & \xrightarrow{\mathscr{A}(\Sigma a,1)} & \mathscr{A}(\Sigma Q,B) \\ F\downarrow & & \downarrow F \\ \mathscr{X}(F\Sigma P,FB) & \xrightarrow[\mathscr{X}(F\Sigma a,1)]{} & \mathscr{X}(F\Sigma Q,FB) \end{array} \qquad \begin{array}{ccc} \mathscr{A}(P,B) & \xrightarrow{\mathscr{A}(a,1)} & \mathscr{A}(Q,B) \\ F\downarrow & & \downarrow F \\ \mathscr{X}(FP,FB) & \xrightarrow[\mathscr{X}(Fa,1)]{} & X(FQ,FB) \end{array}$$

in which the columns are bijections. So up to isomorphism, $\mathscr{X}(FA,FB)$ is the kernel of $\mathscr{A}(a,1)$, and $\mathrm{Ext}^1_{\mathscr{X}}(F\Sigma A,FB)$ is the kernel of $\mathscr{A}(\Sigma a,1)$. From the first exact sequence and commutative square

$$\begin{array}{ccc} \mathscr{A}(A,B) & \xrightarrow{\mathscr{A}(q,1)} & \mathscr{A}(P,B) \\ F\downarrow & & \downarrow F \\ \mathscr{X}(FA,FB) & \xrightarrow[\mathscr{X}(Fq,1)]{} & \mathscr{X}(FP,FB) \end{array}$$

the short exact sequence of the theorem follows. The proof of naturality uses the comparison theorem and is left to the reader.

For the remainder of this section we assume that there are *enough F-projectives*; that is, for each object A of $\mathscr{A}$, there exists an F-projective P and an arrow $a'\colon P \to A$ in $\mathscr{A}$ such that Fa' is an epimorphism in $\mathscr{X}$. This assumption is equivalent to the assumption that, for each object A of $\mathscr{A}$, there exists a triangle $A' \xrightarrow{a} P \xrightarrow{a'} A \xrightarrow{p} \Sigma A'$ in $\mathscr{T}$ with $Fp = 0$ and P F-projective.

LEMMA 1. *If the triangle $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} \Sigma A'$ in $\mathscr{T}$ is such that $Fa = 0$ and the function $F\colon \mathscr{A}(A'',A') \to \mathscr{X}(FA'',FA')$ is injective, then the triangle is isomorphic to the triangle*

$$A' \xrightarrow{\binom{1}{0}} A' \oplus A'' \xrightarrow{(0,1)} A'' \xrightarrow{0} \Sigma A'.$$

PROOF. From the commutative square

$$\begin{array}{ccc} \mathscr{A}(A'',A') & \xrightarrow{\mathscr{A}(a',1)} & \mathscr{A}(A,A') \\ F\downarrow & & \downarrow F \\ \mathscr{X}(FA'',FA') & \xrightarrow[\mathscr{X}(Fa',1)]{} & \mathscr{X}(FA,FA') \end{array}$$

we see that $\mathscr{A}(a',1)$ is injective since the left column is injective and Fa' is an epimorphism implies $\mathscr{X}(Fa',1)$ is injective. So $\mathscr{A}(a',1)a = aa' = 0$ implies $a = 0$. The result now follows from T8(b).

LEMMA 2. *The direct sum $P \oplus Q$ in $\mathscr{A}$ is F-projective if and only if P, Q are both F-projective.*

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Combining these two lemmas we have the following result (under, of course, the blanket assumption we have made for the rest of this section, that there are enough F-projectives).

THEOREM 3. *An object P of $\mathscr{A}$ is F-projective if and only if the function $F: \mathscr{A}(P, A) \to \mathscr{X}(FP, FA)$ is injective for all A in $\mathscr{A}$.*

A proof analogous to that given by Kelly in [5] yields:

THEOREM ON MAPS INDUCING ZERO MAPS. *In the diagram*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_r} A_{r+1}$$

n $\mathscr{A}$, let $f_i: A_i \to A_{i+1}$ be such that $Ff_i = 0$ for $0 \leqq i \leqq r$. If $\dim_F A_0 \leqq r$ then $f_r f_{r-1} \cdots f_0 = 0$.

SCHANUEL'S LEMMA. *If the triangles*

$$A' \xrightarrow{a} P \xrightarrow{a'} A \xrightarrow{p} \Sigma A'$$

$$C' \xrightarrow{c} Q \xrightarrow{c'} A \xrightarrow{q} \Sigma C'$$

are in $\mathscr{T}$, if $Fp = 0$ and $Fq = 0$, and if P, Q are F-projective, then $P \oplus C' \cong Q \oplus A'$.

PROOF. Since P is F-projective, $F(qa') = Fq.Fa' = 0.F.a' = 0$ implies $qa' = 0$. Similarly $pc' = 0$. So by T8(c) we have the result.

LEMMA 4. *Suppose P, A are objects of $\mathscr{A}$ and P is F-projective. Then* $\dim_F A = \dim_F(P \oplus A)$.

PROOF. Let $\dim_F A = r$. For $r = 0$ this is just Lemma 2. Suppose $r > 0$. There exists a triangle

$$A' \xrightarrow{a} Q \xrightarrow{a'} A \xrightarrow{q} \Sigma A'$$

in $\mathscr{T}$ with $Fq = 0$, Q F-projective and $\dim_F A' \leqq r - 1$. But then

$$A' \xrightarrow{\binom{0}{a}} P \oplus Q \xrightarrow{1 \oplus a'} P \oplus A \xrightarrow{(0, q)} \Sigma A'$$

is in $\mathscr{T}$; so $\dim_F(P \oplus A) \leqq r$. Suppose $\dim_F(P \oplus A) < r$. Then there exists a triangle

$$C \xrightarrow{c} R \xrightarrow{c'} P \oplus A \xrightarrow{r} C$$

in $\mathscr{T}$ with $Fr = 0$, R F-projective and $\dim_F C \leqq r - 2$. By Schanuel's lemma it follows that

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$$P \oplus Q \oplus C \cong R \oplus A'.$$

By the first part of the argument, $\dim_F C \leqq r - 2$ implies

$$\dim_F(P \oplus Q \oplus C) \leqq r - 2.$$

So $\dim_F(R \oplus A') \leqq r - 2$. Then from the triangle

$$R \oplus A' \xrightarrow{1 \oplus a} R \oplus Q \xrightarrow{(0,a')} A \xrightarrow{\binom{0}{y}} \Sigma(E \oplus A')$$

in $\mathscr{T}$ we deduce that $\dim_F A \leqq r - 1$, a contradiction. So $\dim_F(P \oplus A) = r$.

DIMENSION THEOREM. *Suppose the triangle* $A' \to P \to A \xrightarrow{p} \Sigma A'$ *is in* $\mathscr{T}$ *with* $Fp = 0$ *and* P *F-projective. If* A *is F-projective then so is* A'; *otherwise,*

$$\dim_F A' = \dim_F A - 1.$$

PROOF. If A is F-projective then $P \cong A' \oplus A$ by Lemma 1; but P is F-projective so A' is F-projective by Lemma 2. Suppose $\dim_F A = r > 0$. Then there exists a triangle

$$C' \to Q \to A \xrightarrow{q} \Sigma C'$$

in $\mathscr{T}$ with $Fq = 0$, Q F-projective and $\dim_F C' = r - 1$. By Schanuel's Lemma,

$$P \oplus C' \cong Q \oplus A'.$$

Then

$$r - 1 = \dim_F C' = \dim_F(P \oplus C') = \dim_F(Q \oplus A') = \dim_F A'$$

by repeated use of Lemma 4.

REMARK. The restriction $\dim_F A \leqq 1$ in the classification theorem may be weakened, but then we must settle for a spectral sequence instead of a short exact sequence. Given an object A_0 of $\mathscr{A}$ we can choose triangles

$$A_{n+1} \xrightarrow{a_n} P_n \xrightarrow{a_n'} A_n \xrightarrow{p_n} \Sigma A_{n+1}$$

for each integer $n \geqq 0$, where $Fp_n = 0$ and P_n is F-projective. Let $\mathscr{A}(A,B)$, $\mathscr{A}(P,B)$ denote the graded abelian groups with n-th components

$$\mathscr{A}(\Sigma^n A_n, B),\ \mathscr{A}(\Sigma^n P_n, B)$$

respectively for B in $\mathscr{A}$. Then we have a Massey exact couple

$$\begin{array}{ccc} \mathscr{A}(A,B) & \xrightarrow{\mathscr{A}(p,1)} & \mathscr{A}(A,B) \\ {\scriptstyle \mathscr{A}(a,1)} \nwarrow & & \swarrow {\scriptstyle \mathscr{A}(a',1)} \\ & \mathscr{A}(P,B) & \end{array}$$

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of graded abelian groups. The second term of the spectral sequence of this exact couple is of the form $\mathrm{Ext}_{\mathscr{X}}^{q-p}(F\Sigma A_0, FB)$. The spectral sequence converges to $\mathscr{A}(\Sigma^n A_0, B)$ when $\dim_F A_0 \leqq r$ for some integer r.

3. The homology functor

In this section we identify our concepts in a familiar case; and thereby provide a starting-point for our inductive arguments in later sections.

Let $\mathscr{B}$ denote an additive category with direct sums. *A complex A* over $\mathscr{B}$ is a diagram

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}^A} A_n \xrightarrow{d_n^A} A_{n-1} \longrightarrow \cdots$$

in which $d_n^A \cdot d_{n+1}^A = 0$ for all integers n. For complexes A, C over $\mathscr{B}$, a complex $[A,C]$ over $\boldsymbol{Abg}$ is defined as follows:

$$[A,C]_n = \prod_{r \in \mathbb{Z}} \mathscr{B}(A_r, C_{r+n}),$$

$$(d_n^{[A,C]} f)_r = d_{r+n}^C \cdot f_r - (-1)^n f_{r-1} \cdot d_r^A.$$

If $\mathscr{G}$ is an abelian category then each complex A over $\mathscr{G}$ gives rise to objects B_nA, Z_nA, H_nA for $n \in \mathbb{Z}$ as shown in the following short exact sequences

$$0 \longrightarrow Z_nA \xrightarrow{i} A_n \xrightarrow{\eta} B_{n-1}A \to 0$$

$$0 \longrightarrow B_nA \xrightarrow{j} Z_nA \xrightarrow{\zeta} H_nA \to 0$$

where $d = ij\eta$. In particular this applies when $\mathscr{G} = \boldsymbol{Abg}$.

For complexes A, C over $\mathscr{B}$, elements of $Z_0[A,C]$ are called *chain arrows* from A to C. Two chain arrows $f, g\colon A \to C$ are *homotopic* when there exists an element s of $B_1[A,C]$ such that $f - g = d(s)$; we write $s\colon f \simeq g$.

The category whose objects are complexes over $\mathscr{B}$ and whose arrows are chain arrows will be denoted by $C\mathscr{B}$; the category whose objects are complexes over $\mathscr{B}$ and whose arrows are homotopy classes of chain arrows will be denoted by $K\mathscr{B}$. Define $\Sigma\colon C\mathscr{B} \to C\mathscr{B}$ as follows:

$$(\Sigma A)_n = A_{n-1},$$

$$d^{\Sigma A} = -d^A,$$

$$(\Sigma f)_n = f_{n-1}.$$

This makes $C\mathscr{B}$ stable, and induces stability on $K\mathscr{B}$.

Given a chain arrow $f\colon A \to B$ over $\mathscr{B}$, the *cone* of f is the complex Cf over $\mathscr{B}$ defined as follows:

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$$(Cf)_n = B_n \oplus A_{n-1},$$

$$d_n^{Cf} = \begin{pmatrix} d_n^A & f_{n-1} \\ 0 & -d_{n-1}^B \end{pmatrix}$$

Then $\binom{1}{0}: B \to Cf$ and $(0,1): Cf \to \Sigma A$ are chain arrows. It is well known (see [3] for example) that there is a triangulation $\mathscr{T}$ of $K\mathscr{B}$ which contains all the triangles of the form

$$A \xrightarrow{[f]} B \xrightarrow{[{}^0_1]} Cf \xrightarrow{[0,1]} \Sigma A;$$

we call this the *canonical triangulation* of $K\mathscr{B}$. Moreover, for each sequence $A \xrightarrow{i} B \xrightarrow{p} C$ of chain arrows such that we have a direct sum situation

$$A_n \underset{p'_n}{\overset{i_n}{\rightleftarrows}} B_n \underset{i'_n}{\overset{p_n}{\rightleftarrows}} C_n$$

for each n, there exists a chain arrow $\delta: C \to \Sigma A$ such that the triangle

$$A \xrightarrow{[i]} B \xrightarrow{[p]} C \xrightarrow{[\delta]} \Sigma A$$

is in $\mathscr{T}$; $[\delta]$ is called the *deviation class* of the sequence $A \xrightarrow{i} B \xrightarrow{p} C$.

Let $\mathscr{G}$ denote an abelian category with enough projectives. Let $G\mathscr{G}$ denote the category of graded objects over $\mathscr{G}$; that is, the full subcategory of $C\mathscr{G}$ consisting of those complexes A with $d_A = 0$. For each complex A over $\mathscr{G}$, B_nA, Z_nA, H_nA determine objects BA, ZA, HA of $G\mathscr{G}$. Functors $B, Z, H: C\mathscr{G} \to G\mathscr{G}$ are induced. The functor H equalizes homotopic chain arrows and so we induce a functor $H: K\mathscr{G} \to G\mathscr{G}$. Let $P\mathscr{G}$ denote the full subcategory of $\mathscr{G}$ consisting of the projective objects of $\mathscr{G}$; it is additive with finite direct sums. Let

$$A_2 = KP\mathscr{G}, \mathscr{X}_2 = G\mathscr{G} \text{ and } F_2 = (KP\mathscr{G} \subset K\mathscr{G} \xrightarrow{H} G\mathscr{G}).$$

Then $F_2: \mathscr{A}_2 \to \mathscr{X}_2$ is a stable homological functor where $\mathscr{A}_2$ has the canonical triangulation.

An object A of $C\mathscr{G}$ will be called *CE-projective* when ZA and HA are projective in $G\mathscr{G}$. A sequence $A' \to A \to A''$ in $C\mathscr{G}$ will be called *CE-exact* when the sequences

$$0 \to ZA' \to ZA \to ZA'' \to 0$$

and

$$0 \to HA' \to HA \to HA'' \to 0$$

are exact in $G\mathscr{G}$. It is proven in [2] that CE-projectives and CE-exactness give a projective class in $C\mathscr{G}$; moreover, it is shown there that A is CE-projective if and only if it is isomorphic to a complex $C \oplus P$ (the direct sum *as complexes*) where C

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is a contractible complex of projective objects over $\mathscr{G}$ and P is a complex of projective objects over $\mathscr{G}$ with zero differential.

THEOREM 5. *An object A of $\mathscr{A}_2$ (that is a complex of projective objects over $\mathscr{G}$) is F_2-projective if and only if it is CE-projective.*

PROOF. Suppose A is F_2-projective. Let $P = HA$. Then $ZA \cong BA \oplus P$. We can suppose $ZA = BA \oplus P$, that the sequence

$$0 \longrightarrow ZA \xrightarrow{\ i\ } A \xrightarrow{\ \eta\ } BA \longrightarrow 0 \text{ is } 0 \longrightarrow BA \oplus P \xrightarrow{(i',i'')} A \xrightarrow{\ \eta\ } BA \longrightarrow 0,$$

and that A has differential $(i',i'')\binom{1}{0}\eta = i'\eta$. But $F_2\colon \mathscr{A}_2(A,P) \to \mathscr{X}_2(HA,HP)$ is an isomorphism and is induced by composition with i''; so there exists a chain arrow $p\colon A \to P$ with $pi'' = 1$. So i'' is a retract, and we may put $A = C \oplus P$, $i'' = \binom{0}{1}$, $p = (0,1)$. But p is a chain arrow, so $pi'\eta = 0$; so $pi' = 0$; so i' has the form

$$\begin{pmatrix} i'_0 \\ 0 \end{pmatrix} : BA \to C \oplus P.$$

Moreover, $\eta(i',i'') = 0$ implies η of the form $(\eta_0, 0)\colon C \oplus P \to BA$. So $A = C \oplus P$ then has differential

$$\begin{pmatrix} i'_0 \\ 0 \end{pmatrix}(\eta_0, 0) = \begin{pmatrix} i'_0\eta_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So A is the direct sum of the complex C with differential $i'_0\eta_0$, and P with zero differential. Then

$$0 \longrightarrow BA \oplus P \xrightarrow{i'_0 \oplus 1} C \oplus P \xrightarrow{(\eta_0, 0)} BA \longrightarrow 0$$

exact implies

$$0 \longrightarrow BA \xrightarrow{\ i'_0\ } C \xrightarrow{\ \eta_0\ } BA \longrightarrow 0$$

exact. So $HC = 0$. But C is F_2-projective; so

$$F_2\colon \mathscr{A}_2(C,C) \to \mathscr{X}_2(HC,HC)$$

is an isomorphism. So $\mathscr{A}_2(C,C) = 0$; that is, C is contractible. So A is CE-projective.

For the converse, it suffices to prove that contractible complexes C and complexes P of projective objects with zero differential are F_2-projective. Since $HC = 0$ and $\mathscr{A}_2(C,A) = 0$ for all complexes A, C is F_2-projective. Also $P = HP$ is projective, so it suffices to prove that $F_2\colon \mathscr{A}_2(P,A) \to \mathscr{X}_2(P,HA)$ is an isomorphism for all A. Take $f\colon P \to HA$; since P is projective and ζ is epimorphic, $f = \zeta \cdot f'$ for some f'. Let $g = i \cdot f'$; then $dg = ij\eta g = ij\eta if' = 0$, and $Hg = f$. This

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proves surjectivity. Suppose $Hg = 0$ for some chain arrow $g: P \to A$. Then $\zeta \cdot Zg = 0$, so $Zg = j \cdot k$ for some $k: P \to BA$. But P is projective and η is epimorphic, so $k = \eta \cdot h$ for some $h: P \to A$. Then $g = i \cdot Zg = ijk = ij\eta h = dh$, so $g \simeq 0$. This proves injectivity.

For any complex A over $\mathscr{G}$, the *CE-dimension* of A is the maximum of the projective dimensions of ZA and HA in $G\mathscr{G}$. Given any A in $\mathscr{A}_2$, since CE-projectives form a projective class, there exists a CE-exact sequence

$$0 \longrightarrow A' \xrightarrow{a} P \xrightarrow{a'} A \longrightarrow 0$$

where P is CE-projective. Since each A_n is projective, in each dimension this sequence is isomorphic to a direct sum situation. Thus there exists a triangle

$$A' \xrightarrow{[a]} P \xrightarrow{[a']} A \xrightarrow{[p]} \Sigma A'$$

in the canonical triangulation of $\mathscr{A}_2$. From this we have the following result.

THEOREM 6. *There are enough F_2-projectives.*

THEOREM 7. *For an object A of $\mathscr{A}_2$, $\dim_{F_2} A$ is equal to the CE-dimension of A.*

PROOF. We use induction on r to show that $\dim_{F_2} A \leqq r$ if and only if CE-dim $A \leqq r$. For $r = 0$ this is Theorem 5. Suppose $r > 0$ and the result true for $r - 1$. Let

$$0 \to A' \to P \to A \to 0$$

be CE-exact with P CE-projective. This gives a canonical triangle

$$A' \to P \to A \to \Sigma A'.$$

If $\dim_{F_2} A \leqq r$ then, by the Dimension Theorem, $\dim_{F_2} A' \leqq r - 1$. By induction, CE-dim $A' \leqq r - 1$; so CE-dim $A \leqq r$. Conversely, if CE-dim $A \leqq r$ then CE-dim $A' \leqq r - 1$. By induction, $\dim_{F_2} A' \leqq r - 1$; then by the Dimension Theorem, $\dim_{F_2} A \leqq r$.

All of §2 now applies to the homology functor $F_2: \mathscr{A}_2 \to \mathscr{X}_2$; the definitions, assumptions and conditions of theorems stated there have interpretations for this functor in terms of properties which can be calculated by more familiar techniques. These results all appear in [1] for which reason we adopt the prefix CE.

4. The functors $F_N: \mathscr{A}_N \to \mathscr{X}_N$

Let $\mathscr{G}$ denote an abelian category with enough projectives, and let N denote an integer greater than or equal to 2.

The categories $\mathscr{B}_N$, $\mathscr{A}_N$ are defined as follows. The objects

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$$A = (A^1, A^2, \cdots, A^{N-1})$$

of $\mathscr{B}_N$ are $(N-1)$-tuples of objects A^i of $\mathscr{G}$ such that each A^i is a subobject of A^{i+1} and each of the objects A^1, A^{i+1}/A^i is *projective* in $\mathscr{G}$. For such an object we put $A^0 = 0$ and $A^{pq} = A^p/A^q$ for $0 \leqq q < p < N$; note that each of these objects is projective in $\mathscr{G}$. It will be convenient to have notations for the inclusion $\omega_r^{qp}: A^{qr} \to A^{pr}$ and the canonical epimorphism $\omega_{rq}^p: A^{pr} \to A^{pq}$. The arrows $: A \to B$ of $\mathscr{B}_N$ are $(N-1)$-tuples of arrows $f^i: A^i \to B^i$ of $\mathscr{G}$ which commute with the inclusions. Such an arrow induces arrows $f^{pq}: A^{pq} \to B^{pq}$ of $\mathscr{G}$ which commute with the ω's. The category $\mathscr{B}_N$ is additive with finite direct sums. Let $\mathscr{A}_N$ denote the category $K\mathscr{B}_N$; it is stable with canonical triangulation $\mathscr{T}_N$.

Next we define a category $\mathscr{X}_N$. The objects of $\mathscr{X}_N$ are diagrams D in $\mathscr{G}$ given as follows. The objects in the diagram D are objects D_{uv} of $\mathscr{G}$, one for each ordered pair of integers u, v such that $u - N < v < u$. The arrows in the diagram D are arrows

$$d_{uv}^{st}: D_{st} \to D_{uv}$$

of $\mathscr{G}$, one such arrow for each ordered quadruplet of integers s, t, u, v such that $s - N < t < s$, $u - N < v < u$, and the arrow d_{uv}^{st} is zero unless $u - N < t \leqq v < s \leqq u$. The arrows in the diagram D satisfy the commuting condition

$$d_{wx}^{uv} \cdot d_{uv}^{st} = d_{wx}^{st}. \tag{1}$$

The arrows of $\mathscr{X}_N$ are just arrows of diagrams. The category $\mathscr{X}_N$ is abelian and stable; the suspension functor $\Sigma: \mathscr{X}_N \to \mathscr{X}_N$ is

$$(\Sigma D)_{uv} = D_{N+v,u}$$

$$(\Sigma d)_{uv}^{st} = d_{N+v,u}^{N+t,s}.$$

An object D of $\mathscr{X}_N$ is called *exact* when each of the sequences

$$D_{uv} \xrightarrow{d_{tv}^{uv}} D_{tv} \xrightarrow{d_{tu}^{tv}} D_{tu} \xrightarrow{d_{N+v,u}^{tu}} D_{N+v,u} \tag{2}$$

$t - N < v < u < t$ is exact in $\mathscr{G}$. Note that $\mathscr{X}_2$ may be identified with the $\mathscr{X}_2$ of the last section.

For $0 \leqq q < p < N$, define $E_{pq}: X_N \to X_2$ on an object D of $\mathscr{X}_N$ by the equations

$$(E_{pq}D)_{2n} = D_{p-nN,q-nN}$$

$$(E_{pq}D)_{2n-1} = D_{q-(n-1)N,p-nN},$$

and similarly on arrows. Note that, for any u, v such that $u - N < v < u$, there exist unique p, q, n such that $0 \leqq q < p < N$ and either $u = p - nN$, $v = q - nN$,

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or $u = q - (n-1)N$, $v = p - nN$. So every object D_{uv} of D is of the form $(E_{pq}D)_n$ for unique p, q, n. The left adjoint

$$J_{pq}\colon \mathscr{X}_2 \to \mathscr{X}_N$$

of E_{pq} is given as follows. If there exists n such that $u - N < q - nN \leqq v < p - nN \leqq u$, then

$$(J_{pq}X)_{uv} = X_{2n};$$

if there exists n such that $u - N < p - nN \leqq v < q - (n-1)N \leqq u$, then

$$(J_{pq}X)_{uv} = X_{2n-1};$$

otherwise $(J_{pq}X)_{uv} = 0$. The arrows of the diagram $J_{pq}X$ are the identity arrows of the X_n wherever it is possible to put them in, and the other arrows are zero.

Now we come to the definition of the functor $F_N\colon \mathscr{A}_N \to \mathscr{X}_N$. For each object $A = (A^1, A^2, \cdots, A^{N-1})$ of $\mathscr{A}_N$ (note that each A^i is a complex of projective objects over $\mathscr{G}$), the object $D = F_N A$ of $\mathscr{X}_N$ is defined as follows. For any integer n and for integers p, q such that $0 \leqq q < p < N$,

$$(E_{pq}D)_n = H_n A^{pq}.$$

This determines D_{uv} uniquely for all $u - N < v < u$. The sequences (2) arise by taking the long exact homology sequence of short exact sequences of the form

$$0 \longrightarrow A^{qr} \xrightarrow{\omega_r^{qp}} A^{pr} \xrightarrow{\omega_{rq}^{p}} A^{pq} \longrightarrow 0 \qquad 0 \leqq r < q < p < N.$$

The arrows d_{uv}^{st} which do not occur in such sequences are determined by the commuting condition (1). Then $F_N A$ is an exact object of $\mathscr{X}_N$. Next F_N becomes a functor from $C\mathscr{B}_N$ to $\mathscr{X}_N$ on defining $F_N f\colon F_N A \to F_N B$, for a chain arrow $f\colon A \to B$ over $\mathscr{B}_N$, by the component arrows

$$H_n f^{pq}\colon H_n A^{pq} \to H_n B^{pq}$$

in $\mathscr{G}$. If $f \simeq 0$ then $F_N f = 0$. So F_N is indeed a functor from $\mathscr{A}_N$ to $\mathscr{X}_N$. From the corresponding properties of the homology functor, it is clear that $F_N\colon \mathscr{A}_N \to \mathscr{X}_N$ is a *stable homological functor*. For $N = 2$ this definition agrees with the definitions of the last section.

We shall study the functors F_N using induction on N. As a tool for this induction, we define, for each complex A over $\mathscr{B}_N$ complexes ΩA, ΓA over $\mathscr{B}_{N-1}$ by

$$(\Omega A)^i = A^i,\ (\Gamma A)^i = A^{i+1,1} \text{ for } 0 < i < N-1.$$

For A, C in $C\mathscr{B}_N$, define chain arrows

$$\phi\colon [\Gamma A, \Omega C] \to [A, C]$$

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$$\psi: [A,C] \to \prod_{0<i<N} [A^{i,i-1}, C^{i,i-1}]$$

over ***Abg*** as follows. For $g \in [\Gamma A, \Omega C]$, $\phi(g)^{i+1}$ is the composite

$$A^{i} \xrightarrow{\omega_{01}^{i+1}} A^{i+1,1} \xrightarrow{g^{i}} C^{i} \xrightarrow{\omega_{0}^{i,i+1}} C^{i+1}.$$

For $f \in [A,C]$, $\psi(f)^{i} = f^{i,i-1}$.

LEMMA 8. *For complexes A, C over $\mathscr{B}_N$, the sequence*

$$0 \to [\Gamma A, \Omega C] \xrightarrow{\phi} [A,C] \xrightarrow{\psi} \prod_{0<i<N} [A^{i,i-1}, C^{i,i-1}] \to 0$$

of complexes over ***Abg*** *is exact. The connecting arrow*

$$\Delta: \prod_{0<i<N} H_n[A^{i,i-1}, C^{i,i-1}] \to H_{n-1}[\Gamma A, \Omega C]$$

of the long exact homology sequence is given by

$$\Delta[h] = [g] \text{ where } g^{i} = \delta h^{i+1} \omega_{1i}^{i+1} - (-1)^{n} \omega_{0}^{1i} h^{1} \partial$$

and $[\delta]$, $[\partial]$ are the deviation classes of the sequences

$$C^{i} \xrightarrow{\omega_{0}^{i,i+1}} C^{i+1} \xrightarrow{\omega_{0i}^{i+1}} C^{i+1,i}, A^{1} \xrightarrow{\omega_{0}^{1,i+1}} A^{i+1} \xrightarrow{\omega_{01}^{i+1}} A^{i+1,1}.$$

For a complex A over $\mathscr{B}_2$, let $\gamma(A)$ denote the cone of $1: \Sigma^{-1}A \to \Sigma^{-1}A$. For $0 \leqq q < p < N$ we define $J_{pq}A$ in $C\mathscr{B}_N$ by the equations

$$\begin{aligned} (J_{p0}A)^{i} &= 0 \text{ for } 1 \leqq i < p, \\ &= A \text{ for } p \leqq i < N; \\ (J_{pq}A)^{i} &= 0 \quad \text{ for } 1 \leqq i < q, \\ &= \Sigma^{-1}A \text{ for } q \leqq i < p, \\ &= \gamma(A) \quad \text{for } p \leqq i < N, \text{ where } q > 0. \end{aligned}$$

Note that $(J_{pq}A)^{pq} = A$ and $(J_{pq}A)^{p-1,q-1} = \Sigma^{-1}A$ (the convention $C^{p-1,-1} = \Sigma^{-1}C^{N-1,p-1}$ covers the case $q = 0$). In the obvious way, $J_{pq}: C\mathscr{B}_2 \to C\mathscr{B}_N$ becomes a functor which preserves homotopies. Let $E_{pq}: C\mathscr{B}_N \to C\mathscr{B}_2$ be given by $E_{pq}A = A^{pq}$, $E_{pq}f = f^{pq}$.

LEMMA 9. *For A in $C\mathscr{B}_2$ and C in $C\mathscr{B}_N$, the chain arrows*

$$E_{pq}: [J_{pq}A, C] \to [A, C^{pq}],\ E_{p-1,q-1}: [C, J_{pq}A] \to [C^{p-1,q-1}, \Sigma^{-1}A]$$

for $0 \leqq q < p < N$ have right chain inverses which are left homotopy inverses.

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For $D \in \mathscr{X}_N$, define ΩD, ΓD in $\mathscr{X}_{N-1}$ as follows. For $0 \leqq q < p < N-1$,

$$E_{pq}\Omega D = E_{pq}D,\ E_{pq}\Gamma D = E_{p+1,q+1}D.$$

The arrows of ΩD, ΓD are the arrows of D which are between objects of D which are in positions belonging to objects of ΩD, ΓD respectively. For X, D in $\mathscr{X}_N$, define a sequence

$$\prod_{0<i<N} \mathscr{X}_2(E_{i,i-1}X, E_{i,i-1}\Sigma^{-1}D) \xrightarrow{\theta_1} \mathscr{X}_{N-1}(\Gamma X, \Omega D) \xrightarrow{\phi} \mathscr{X}_N(X, D)$$

(3)

$$\xrightarrow{\psi} \prod_{0<i<N} \mathscr{X}_2(E_{i,i-1}X, E_{i,i-1}D) \xrightarrow{\theta_1} \mathscr{X}_{N-1}(\Gamma X, \Omega\Sigma D)$$

in $\boldsymbol{Abg}$ as follows. For $\alpha \in \mathscr{X}_{N-1}(\Gamma X, \Omega D)$, $E_{pq}\phi(\alpha)$ is the composite

$$E_{pq}X \xrightarrow{d} E_{p+1,q+1}X \xrightarrow{E_{pq}\alpha} E_{pq}D.$$

For $\beta \in \mathscr{X}_N(X, D)$, $\psi(\beta)^i = E_{i,i-1}\beta$. For

$$\gamma \in \prod_{0<i<N} \mathscr{X}_2(E_{i,i-1}X, E_{i,i-1}\Sigma^n D),$$

$E_{pq}\theta_n(\gamma)$ is the difference between the composite

$$E_{p+1,q+1}X \xrightarrow{d} E_{p+1,p}X \xrightarrow{\gamma^{p+1}} E_{p+1,p}\Sigma^n D \xrightarrow{d} E_{pq}\Sigma^{n+1}D$$

and $(-1)^n$ of the composite

$$E_{p+1,q+1}X \xrightarrow{d} E_{q+1,q}X \xrightarrow{\gamma^{q+1}} E_{q+1,q}\Sigma^{n+1}D \xrightarrow{d} E_{pq}\Sigma^{n+1}D.$$

LEMMA 10. *If X is a projective object of $\mathscr{X}_N$ and D is an exact object of $\mathscr{X}_N$ then the sequence* (3) *is exact.*

PROOF. It suffices to prove this lemma for $X = J_{pq}Y$ where $0 \leqq q < p < N$ and Y is projective in $\mathscr{X}_2$, since every projective object of $\mathscr{X}_N$ is a retract of a coproduct of such X.

(a) Suppose $q > 0$. Then (3) becomes:

$$(Y, E_{p,p-1}\Sigma^{-1}D) \oplus (Y, E_{q,q-1}D) \to (Y, E_{p-1,q-1}D) \to (Y, E_{pq}D)$$

$$\to (Y, E_{p,p-1}D) \oplus (Y, E_{q,,-1}\Sigma D) \to (Y, E_{p-1,q-1}\Sigma D).$$

The diagram

$$\begin{array}{ccccccccccc}
\cdots \to & E_{p,p-1}\Sigma^{-1}D & \to & E_{p-1,q}D & \to & E_{pq}D & \to & E_{p,p-1}D & \to & E_{p-1,q}\Sigma D & \to \cdots \\
& \downarrow & & \| & & \downarrow & & \downarrow & & \| & \\
\cdots \to & E_{p-1,q-1}D & \to & E_{p-1,q}D & \to & E_{q,q-1}\Sigma D & \to & E_{p-1,q-1}\Sigma D & \to & E_{p-1,q}\Sigma D & \to \cdots
\end{array}$$

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commutes with exact rows, so the Mayer-Vietoris sequence

$$\cdots \to E_{p-1,q-1}D \to E_{pq}D \to E_{p,p-1}D \oplus E_{q,q-1}\Sigma D \to E_{p-1,q-1}\Sigma D \to \cdots$$

is exact in $\mathscr{X}_2$. Since Y is projective in $\mathscr{X}_2$, the result follows by taking $(Y, -)$ of the last sequence.

(b) Suppose $q = 0$. Then (3) becomes the sequence obtained by applying $(Y, -)$ to the Mayer-Vietoris sequence

$$E_{p,p-1}\Sigma^{-1} \oplus E_{N-1,0}\Sigma^{-1} \to E_{N-1,p-1}\Sigma^{-1} \to E_{p0} \to E_{p,p-1} \oplus E_{N-1,0} \to E_{N-1,p-1}$$

where the D's have been omitted.

THEOREM 11. *An object A of $\mathscr{A}_N$ is F_N-projective if and only if each of the complexes A^p/A^q over $\mathscr{G}$ is CE-projective for $0 \leqq q < p < N$.*

PROOF. Suppose A is F_N-projective. Then F_NA is projective in $\mathscr{X}_N$, and so, by Theorem 1 of [7], each $(F_NA)_{uv}$ is projective. So HA^{pq} is projective in $\mathscr{X}_2$. For any C in $\mathscr{A}_2$, put

$$C^{N-1,q} = J_{q+1,0}C,\ D^{N-1,q} = J_{q+1,0}HC \text{ for } 0 \leqq q \leqq N-1,$$

$$C^{pq} = J_{p+1,q+1}\Sigma C,\ D^{pq} = J_{p+1,q+1}\Sigma HC \text{ for } 0 \leqq q < p < N-1.$$

Then we have a commutative diagram

$$\begin{array}{ccc} A_N(A, C^{pq}) & \xrightarrow[\cong]{F_N} & \mathscr{X}_N(F_NA, D^{pq}) \\ \cong\downarrow & & \downarrow\cong \\ \mathscr{A}_2(A^{pq}, C) & \xrightarrow[F_2]{} & \mathscr{X}_2(HA^{pq}, HC). \end{array}$$

The vertical isomorphisms come from Lemma 9 and the adjunction

$$J_{pq} \dashv E_{pq}: (\mathscr{X}_N, \mathscr{X}_2).$$

So the lower row is an isomorphism. Thus A^{pq} is F_2-projective.

Theorem 5 covers the case $N = 2$. Assume the theorem is true for $N-1$ where $N > 2$. Take A in $\mathscr{A}_N$ such that each A^{pq} is CE-projective. Then each $(\Gamma A)^{pq}$, $(\Omega A)^{pq}$ is CE-projective. So by induction ΓA, ΩA are F_{N-1}-projective. So $F_{N-1}\Gamma A, F_{N-1}\Omega A$ are projective in $\mathscr{X}_{N-1}$. By Theorem 1 of [7], all the arrows in $F_{N-1}A$, $F_{N-1}A$ have kernels which are projective objects of $\mathscr{G}$. Every arrow in F_NA is such an arrow. So every arrow of the exact object F_NA has projective kernel. By Theorem 1 of [7], F_NA is a projective object of $\mathscr{X}_N$. Take C in $\mathscr{A}_N$. Consider the five terms of the homology sequence of the short exact sequence of Lemma 8 around $H_0[A, C] = \mathscr{A}_N(A, C)$. This is exact. Consider also the sequence (3) with $X = F_NA, D = F_NC$. This is exact by Lemma 10. Moreover, the five arrows

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$$\prod_{0<i<N} F_2, F_{N-1}, F_N, \quad \prod_{0<i<N} F_2, F_{N-1}$$

on hom-sets give an arrow from the first sequence into the second. By Theorem 5 each of the F_2's is an isomorphism and by induction the F_{N-1}'s are isomorphisms. So by the "five lemma", $F_N: \mathscr{A}_N(A, C) \to \mathscr{X}_N(X, D)$ is an isomorphism; whence A is F_N-projective.

THEOREM 12. *There are enough F_N-projectives.*

PROOF. Suppose A is an object of $\mathscr{A}_N$. For $0 \leqq q < p < N$ choose a *CE*-projective Y_{pq} and a *CE*-exact sequence

$$Y_{pq} \xrightarrow{\eta_{pq}} A^{pq} \longrightarrow 0.$$

Choose $\bar{\eta}_{pq}: J_{pq}Y_{pq} \to A$ in $\mathscr{A}_N$ corresponding to η_{pq} under the isomorphism

$$\mathscr{A}_N(J_{pq}Y_{pq}, A) \cong \mathscr{A}_2(Y_{pq}, A^{pq})$$

of Lemma 9. Put

$$P = \sum_{0 \leqq q < p < N} J_{pq}Y_{pq},$$

and let $\varepsilon: P \to A$ be the unique arrow of $\mathscr{A}_N$ determined by the $\bar{\eta}_{pq}$. Then each of the sequences

$$HP^{pq} \xrightarrow{H\varepsilon^{pq}} HA^{pq} \longrightarrow 0$$

is exact, and each P^{pq} is *CE*-projective. So $F_N\varepsilon$ is an epimorphism and, since $F_N J_{pq} = J_{pq}H$ (or, if you like, by Theorem 11), P is F_N-projective.

THEOREM 13. *For an object A of $\mathscr{A}_N$,* $\dim_{F_N} A$ *is equal to the maximum of the CE-dimensions of the complexes A^p/A^q, $0 \leqq q < p < N$, over $\mathscr{G}$.*

PROOF. By Theorem 12, there exists an arrow $\varepsilon: P \to A$ in $\mathscr{A}_N$ where P is F_N-projective and $F_N\varepsilon$ is an epimorphism. By the cone construction, this gives rise to a triangle

$$C \to P \xrightarrow{\varepsilon} A \to \Sigma C$$

in $\mathscr{T}_N$ such that each of the triangles

$$C^{pq} \longrightarrow P^{pq} \xrightarrow{\varepsilon^{pq}} A^{pq} \longrightarrow \Sigma C^{pq}$$

is in $\mathscr{T}_2$, p^{pq} is F_2-projective (Theorem 11) and $F_2\varepsilon^{pq}$ is an epimorphism. Using Theorem 11 to start the induction, we can employ the Dimension Theorems for F_N and F_2 to prove that $\dim_{F_N} A \leqq r$ if and only if $\dim_{F_2} A^{pq} \leqq r$ for all $0 \leqq q < p < N$. Then the result follows from Theorem 7.

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So §2 applies to the functors $F_N: \mathscr{A}_N \to \mathscr{X}_N$. Moreover, the conditions of the theorems of §2 have, in this section, been put into a more familiar form.

The Classification Theorem involves the functor Ext for the category $\mathscr{X}_N$. The following theorems help in the calculation of this functor when $\mathscr{G}$ had finite projective dimension.

THEOREM 14. *Suppose $\mathscr{G}$ has finite projective dimension. An object X of $\mathscr{X}_N$ is projective if and only if it is exact and each of the objects X_{uv}, $u - N < v < u$, is projective in $\mathscr{G}$.*

PROOF. By Theorem 1 of [7], X is projective if and only if it is exact and the kernel of each of its arrows is projective. Then each X_{uv} is projective. Suppose X is exact and each X_{uv} is projective. Consider an arrow x of X which appears in one of the exact sequences (2) for X. Let k be the dimension of $\mathscr{G}$. The sequence (2) for X then give an exact sequence

$$0 \to \ker x \to P_0 \to P_1 \to \cdots \to P_k \to P_{k+1}$$

where each P_i is an object in the diagram X and hence projective. So $\ker x$ is projective. Every arrow of X is a composite xx' of such arrows x, x' of X. From the short exact sequence

$$0 \to \ker x' \to \ker(xx') \to \ker x \to 0$$

we deduce that $\ker(xx')$ is projective. So X is a projective object of $\mathscr{X}_N$.

THEOREM 15. *Suppose $\mathscr{G}$ has finite projective dimension k. An object of $\mathscr{X}_N$ has projective dimension at most k if and only if it is exact.*

PROOF. Consider a short exact sequence

$$0 \to D' \to X \to D \to 0$$

in $\mathscr{X}_N$ where X is exact. Consider as complexes the sequences (2) for D, X, D'; then we have short exact sequences of complexes where the middle complex has zero homology; so the homologies of the outside complexes are isomorphic. It follows that D is exact if and only if D' is exact.

Let D be an object of $\mathscr{X}_N$ and choose an exact sequence

$$0 \to Y \to X^{k-1} \to \cdots \to X^1 \to X^0 \to D \to 0$$

in $\mathscr{X}_N$ in which each X^i is projective. If D is exact then, from the last paragraph, Y is exact. Since $\mathscr{G}$ has dimension k, each Y_{uv} is projective in $\mathscr{G}$. By Theorem 14, Y is projective. So D has dimension at most k.

Conversely, suppose D has dimension at most k. Then there exists an exact sequence as above with Y also projective. So Y is exact. Then D is exact.

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5. The split classification sequence

For this section we suppose that $\mathscr{G}$ is the category of abelian groups (or any abelian category with projective dimension 1). Every object of $\mathscr{A}_2$ has *CE*-dimension at most 1. By Theorem 13 then, every object A of $\mathscr{A}_N$ has $\dim_{F_N} A \leqq 1$. So for any objects A, B of $\mathscr{A}_N$, the Classification Theorem yields a natural short exact sequence.

$$(4) \qquad 0 \to \operatorname{Ext}^1_{\mathscr{X}_N}(F_N A, F_N B) \to \mathscr{A}_N(A,B) \xrightarrow{F_N} \mathscr{X}_N(F_N A, F_N B) \to 0$$

of abelian groups. In the present section we shall outline why this short exact sequence splits.

Any projective object X of $\mathscr{X}_2$ may be regarded as an object R_2X of $\mathscr{A}_2$ with zero differential and $F_2R_2X = X$. The right chain inverse, left homotopy inverse of Lemma 9 for the chain arrow

$$E_{pq}\colon [J_{pq}R_2X, J_{rs}R_2Y] \to [R_2X, E_{pq}J_{rs}R_2Y]$$

may be chosen naturally in $X, Y \in \mathscr{X}_2$. So we have a natural arrow

$$\Lambda\colon C\mathscr{B}_2(R_2X, E_{pq}J_{rs}R_2Y) \to C\mathscr{B}_N(J_{pq}R_2X, J_{rs}R_2Y)$$

which induces an isomorphism when $C\mathscr{B}_N$ is replaced by $\mathscr{A}_N$ and which has "evaluation at p, q" as a left inverse. For projective X in $\mathscr{X}_2$, note that

$$R_2E_{pq}J_{rs}X = E_{pq}J_{rs}R_2X$$

unless $0 \leqq q < s < r \leqq p < N$ in which case the left side is 0 and the right side is $\gamma(R_2X)$.

The category $\mathscr{Z}_N$ is defined as follows. The objects X of $\mathscr{Z}_N$ are families of projective objects X^{pq}, $0 \leqq q < p < N$, of $\mathscr{X}_2$. An arrow $\alpha\colon X \to Y$ in $\mathscr{Z}_N$ is an arrow $\alpha\colon \Sigma J_{pq}X^{pq} \to \Sigma J_{pq}Y^{pq}$ of $\mathscr{X}_N$ where the sums are over $0 \leqq q < p < N$. So $\mathscr{Z}_N$ may be identified with a full subcategory of $\mathscr{X}_N$.

The functor $R_N\colon \mathscr{Z}_N \to C\mathscr{B}_N$ is defined on objects by the equation

$$R_NX = \Sigma J_{pq}R_2X^{pq}.$$

On arrows R_N is given by the composite

$$\begin{aligned}
\mathscr{Z}_N(X,Y) &= \Sigma\mathscr{X}_N(J_{pq}X^{pq}, J_{rs}Y^{rs})\\
&\cong \Sigma\mathscr{X}_2(X^{pq}, E_{pq}J_{rs}Y^{rs})\\
&= \Sigma C\mathscr{B}_2(R_2X^{pq}, R_2E_{pq}J_{rs}Y^{rs})\\
&\leqq \Sigma C\mathscr{B}_2(R_2X^{pq}, E_{pq}J^{rs}R_2Y^{rs})\\
&\xrightarrow{\Gamma} \Sigma C\mathscr{B}_N(J_{pq}R_2X^{pq}, J_{rs}R_2Y^{rs})
\end{aligned}$$

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$$= C\mathscr{B}_N(R_N X, R_N Y).$$

Then $F_N R_N\colon \mathscr{Z}_N \to \mathscr{X}_N$ is the inclusion.

Let $\mathscr{X}_N^e$ denote the full subcategory of $\mathscr{X}_N$ consisting of the exact objects of $\mathscr{X}_N$. By Theorem 15, the objects of $\mathscr{X}_N$ have projective dimension at most 1 with respect to $\mathscr{Z}_N$. For each object D of $\mathscr{X}_N^e$, choose a short exact sequence

$$0 \to Y \xrightarrow{\kappa} X \to D \to 0$$

in $\mathscr{X}_N$, where X, Y are in $\mathscr{Z}_N$. Then $R_N\kappa\colon R_N Y \to R_N X$ is in $C\mathscr{B}_N$. Let $V_N D$ denote the cone $CR_N\kappa$ of the chain arrow $R_N\kappa$. Do the same for D' in $\mathscr{X}_N$. Suppose $\alpha\colon D \to D'$ is an arrow of $\mathscr{X}_N$. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & Y & \to & X & \to & D & \to 0 \\ & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & \\ 0 \to & Y' & \to & X' & \to & D' & \to 0 \end{array}$$

in $\mathscr{X}_N$. Let $f\colon V_N D \to V_N D'$ denote the chain arrow over $\mathscr{B}_N$ which, as a graded arrow over $\mathscr{B}_N$, is

$$\begin{pmatrix} R_N\beta & 0 \\ 0 & \Sigma R_N\gamma \end{pmatrix}$$

If different β, γ are chosen as above, this leads to a chain arrow over $\mathscr{B}_N$ which is homotopic to f. So let

$$V_N\alpha = [f]\colon V_N D \to V_N D'$$

in $\mathscr{A}_N$. So, after choosing suitable resolutions of objects of $\mathscr{X}_N$, we have defined a functor $V_N\colon \mathscr{X}_N^e \to \mathscr{A}_N$ which extends $R_N\colon \mathscr{Z}_N \to \mathscr{A}_N$. Since F_N is homological we have an exact sequence

$$\cdots \to \Sigma^{-1} F_N V_N D \to F_N R_N Y \xrightarrow{F_N R_N \kappa} F_N R_N X \to F_N V_N D \to \cdots$$

in $\mathscr{X}_N$. But $F_N R_N\kappa = \kappa$ is a monomorphism. So the sequence

$$0 \to Y \to X \to F_N V_N D \to 0$$

is exact in $\mathscr{X}_N$. So $F_N V_N D \cong D$, and this isomorphism is natural in $D \in \mathscr{X}_N$.

Theorem 16. ($\mathscr{G} = \boldsymbol{Abg}$). *The functor $F_N\colon \mathscr{A}_N \to \mathscr{X}_N^e$ has a right inverse V_N up to natural isomorphism.*

Corollary 17. ($\mathscr{G} = \boldsymbol{Abg}$). *There is a natural short exact sequence* (4) *for $A, B \in \mathscr{A}_N$; moreover,*

$$\mathscr{A}_N(A, B) \cong \mathscr{X}_N(F_N A, F_N B) \oplus \mathrm{Ext}^1_{\mathscr{X}_N}(F_N \Sigma A, F_N B).$$

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