MEROMORPHIC FUNCTIONS WITH ONE DEFICIENT VALUE

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1. Introduction

Let $f(z)$ be a meromorphic function and write

$$\delta(a, f) = 1-\limsup_{r\to\infty} \frac{N(r, a)}{T(r, f)}, \quad \Delta(a, f) = 1-\liminf_{r\to\infty} \frac{N(r, a)}{T(r, f)}.$$

Here $N(r, a)$ and $T(r, f)$ have their usual meanings (see [4], [5]) and $0 \leqq |a| \leqq \infty$. If $\delta(a, f) > 0$ then a is said to be an exceptional (or deficient) value in the sense of Nevanlinna (N.e.v.), and if $\Delta(a, f) > 0$ then a is said to be an exceptional value in the sense of Valiron (V.e.v.). The Weierstrass $p(z)$ function has no exceptional value N or V. Functions of zero order can have atmost one N.e.v. [4, p. 114], but may have more than one V.e.v. (see [6], [8]). In this note we consider functions satisfying some regularity conditions and having one and only one exceptional value V.

2. Functions of zero order

THEOREM 1. *Let $f(z)$ be a meromorphic function such that, as $r \to \infty$,*

$$T(r, f) = O((\log r)^2) \tag{1.1}$$

and

$$N(r, a) = o(T(r, f)) \tag{1.2}$$

for some a, finite or infinite.
Then

$$N(r, b) \sim T(r, f) \tag{1.3}$$

for every $b \neq a$.

PROOF. If (1.1) holds then [9, p. 30] for $a \neq b$

$$\max_r \{N(r, a), N(r, b)\} = (1+o(1))T(r, f). \tag{1.4}$$

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But the left hand expression

$$\leqq N(r, a)+N(r, b) \leqq N(r, b)+o(T(r, f))$$

and (1.3) follows.

For functions f not satisfying (1.1) but the condition (1.5) below, Theorem 2 below gives the same conclusion. The proof of Theorem 2 is similar to that of Theorem 1 of [7] and will be omitted.

THEOREM 2. *Let $f(z)$ be a non-constant meromorphic function. Let $\Psi(r)$ and $\theta(r)$ be two functions tending to ∞ with r, and let $\phi(r)$ be any function tending to ∞, however slowly, with r. Let $\Psi(r\theta(r)) = O(\Psi(r))$ $(r \to \infty)$, and suppose that for all large r, $\Psi(r)$ and $\theta(r)$ are non-decreasing functions of r.*
If ultimately

$$\frac{\Psi(r)\phi(r)}{\log \theta(r)} \leqq T(r, f) \leqq \Psi(r), \tag{1.5}$$

and if

$$N(r, a) = o(T(r, f)) \tag{1.6}$$

for some a finite or infinite, then

$$N(r, b) \sim T(r, f) \tag{1.7}$$

for every $b \neq a$, and

$$n(r, b) = o(T(r, f)) \tag{1.8}$$

for every b.

3. Functions with no finite deficient value

The results given in theorem 1 of [7] and in the above theorems cannot in general be extended to functions of positive order (cf.: [7]). However it can be proved that there exists an entire function with asymptotically prescribed growth and having no finite deficient value. If for any entire function f, $\log M(r, f) \sim T(r, f)$ and $T(r, f)$ satisfies a growth regularity condition, then also f has no finite deficient value. More precisely we have

THEOREM 3. *Let $\Lambda(r)$ be an increasing function of r and a convex function of $\log r$ with $\Lambda(r) \neq O$ $(\log r)$. Assume further that*

$$\Lambda(r) = O(r^k) \qquad (r \to \infty) \tag{3.1}$$

for some $k > 0$. Then there exists an entire function $f(z)$ of finite order such that

$$\log M(r, f) \sim \Lambda(r) \sim T(r, f) \sim N(r, a) \qquad (r \to \infty) \tag{3.2}$$

for every finite a.

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THEOREM 4. *Let $f(z) \not\equiv 0$ be entire and let there exist constants $\sigma > 1$, $B > 1$ such that*

$$T(\sigma r, f) < BT(r, f) \qquad (r \geqq r_0). \tag{3.3}$$

Suppose also that

$$\log M(r, f) \sim T(r, f). \tag{3.4}$$

Then $\log M(r, f) \sim N(r, a)$ *for every finite a.*

These two theorems 3 and 4 are due to Professor Albert Edrei. If we do not assume (3.1), then f, in theorem 3, may not be of finite order. (See [1], [2].)

PROOF OF THEOREM 3. Let $F(z)$ be an entire function such that $T(r, F) \sim \Lambda(r) \sim \log M(r, F)$. (See theorem 1 of [3].) Let $f(z) = F(z) - \alpha z$ and select the constant α so that

$$\lim_{r\to\infty} N(r, 1/(f-\tau))/T(r, f) = 1$$

for every finite fixed τ. This is possible by the proposition on p. 386 of [3]. Since f and F are not polynomials,

$$\log M(r, f) \sim \log M(r, F),\ T(r, f) \sim T(r, F),$$

and the theorem is proved.

PROOF OF THEOREM 4. It is known [3; pp. 393—4] that if $f(z)$ be entire and c any complex number, then for $1 < r < R$,

$$m\left(r, \frac{1}{f-c}\right) \leqq \frac{11R}{R-r}\, T\left(R, \frac{1}{f-c}\right) \mu(r) \left\{1 + \log^+ \frac{1}{\mu(r)}\right\}.$$

Here $\mu(r)$ is the measure of θ for which $|f(re^{i\theta}) - c| < 1$. Hence

$$m\left(r, \frac{1}{f-c}\right) \leqq \frac{12R}{R-r}\, T(R, f)\mu(r) \left(1 + \log^+ \frac{1}{\mu(r)}\right) \qquad (r_0 < r < R). \tag{3.5}$$

Now

$$2\pi T(r, f) \leqq \mu(r) \log (1+|c|) + (2\pi - \mu(r)) \log M(r, f)$$

and so

$$\mu(r) \left\{1 - \frac{\log (1+|c|)}{\log M(r)}\right\} \leqq 2\pi \left\{1 - \frac{T(r, f)}{\log M(r)}\right\}.$$

Hence by (3.4), $\mu(r) = o(1)$. Choose in (3.5), $R = \sigma r$. Then we have

$$\begin{aligned} m\left(r, \frac{1}{f-c}\right) &\leqq \frac{12\sigma}{|1-\sigma|}\, T(\sigma r, f)\mu(r) \left(1 + \log^+ \frac{1}{\mu(r)}\right) \\ &< \frac{12\sigma B}{|1-\sigma|}\, T(r, f)\, o(1). \end{aligned}$$

Hence

$$m\left(r, \frac{1}{f-c}\right) \leqq o(T(r, f))$$

and the result follows.

In conclusion I must thank Professor A. Edrei for allowing me to include theorems 3 and 4.

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