Compositio Math. 142 (2006) 1594–1614
doi:10.1112/S0010437X06002399

Bases for some reciprocity algebras III

Roger Howe and Soo Teck Lee

Abstract

We construct bases for the stable branching algebras for the symmetric pairs $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$, $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$ and $(\mathrm{O}_{2n}, \mathrm{GL}_n)$. Each basis element is expressed as a sum of products of pfaffians.

1. Introduction

Let G be a complex classical group and H a symmetric subgroup of G, that is, H is the subgroup of fixed points of an involution on G. Consider the problem of decomposing an irreducible finite-dimensional representation of G under restriction to H. Using classical invariant theory, one can construct an algebra $\mathfrak{A}$ with the following properties: G and a torus A_k act on $\mathfrak{A}$ by algebra automorphisms and, under the action of $G \times A_k$, we have

$$\mathfrak{A} = \bigoplus_i V_i \otimes L_i,$$

where each V_i is an irreducible G module and each L_i is a one-dimensional space on which A_k acts by a character ψ_i. In addition, both the V_i and the ψ_i are distinct (see, for example, [HTW04]). The subspace $V_i \otimes L_i$ is the ψ_i-eigenspace for A_k in $\mathfrak{A}$, and since L_i is one-dimensional, this subspace can also be regarded as a copy of the irreducible G-module V_i. Thus, if we ignore the action of A_k, then $\mathfrak{A}$ is a multiplicity free sum of irreducible representations of G. We now let U_H be the maximal unipotent subgroup of H. Then the subalgebra $\mathfrak{A}^{U_H}$ of U_H-invariants in $\mathfrak{A}$ is given by

$$\mathfrak{A}^{U_H} = \bigoplus_i (V_i)^{U_H} \otimes L_i.$$

This subalgebra carries an action by the maximal torus A_H of H. If ϕ_j is a dominant weight for H, then the $\phi_j \times \psi_i$-eigenspace of $A_H \times A_k$ in $\mathfrak{A}^{U_H}$ can be identified with the space of H highest weight vectors of weight ϕ_j in the G-module V_i. Thus, its dimension is equal to the multiplicity of the irreducible representation of H with highest weight ϕ_j in V_i. The representations V_i which occur in $\mathfrak{A}$ are of a special type, and they are said to be in the *stable range*. Thus, the algebra $\mathfrak{A}^{U_H}$ encodes all branching information for restricting representations of G in the stable range to H. In view of this property, we call $\mathfrak{A}^{U_H}$ a *stable branching algebra* for (G, H). As explained in [HTW04], $\mathfrak{A}^{U_H}$ also describes the branching rule for another symmetric pair, so it is also called a *reciprocity algebra*.

This paper is the third in a series of three which construct explicit bases for stable branching algebras. The most basic case $(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n)$ (that is, the case of the GL_n tensor product algebras) was treated in [HTW05b]. The paper [HL06a] deals with two variants of the GL_n tensor product algebras and [HL06b] treats the stable branching algebras for the pairs $(\mathrm{GL}_n, \mathrm{O}_n)$, $(\mathrm{O}_{n+m}, \mathrm{O}_n \times \mathrm{O}_m)$ and $(\mathrm{Sp}_{2n}, \mathrm{GL}_n)$. In this paper, we construct bases for the stable branching algebras for the symmetric pairs $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$, $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$ and $(\mathrm{O}_{2n}, \mathrm{GL}_n)$. Our results

Received 7 February 2006, accepted in final form 10 May 2006.
2000 Mathematics Subject Classification 22E46 (primary).
Keywords: branching rules, reciprocity algebras, SAGBI basis.
The second named author is partially supported by NUS grant R-146-000-067-112.

will serve two purposes. Firstly, our description of the H highest weight vectors in an irreducible G module V_i provides information on how H modules sit inside V_i. Secondly, these bases provide information about the algebra structure of $\mathfrak{A}^{U_H}$. In particular, we will be able to show that the stable branching algebras are flat deformations of the semigroup rings attached to certain lattice cones. In particular, they are flat deformations of toric varieties [HJLTW06].

We follow essentially the same approach as [HL06b]. Each of the stable branching algebras can be realized as an explicit algebra of polynomials on a space of matrices. The bases which we construct are indexed by certain sets of Littlewood–Richardson (LR) tableaux (or ordered pairs of LR tableaux). Each basis element is a specific polynomial, and with respect to a certain monomial order, it has a highest term which allows one to reconstruct the tableau (or the ordered pair of tableaux) from which it came. This shows, in particular, that these bases are *SAGBI bases* (Subalgebra Analog to Gröbner Bases for Ideals) [RS90, Stu96] for these stable branching algebras (see [HJLTW06]). The main difference from [HL06b] is that each basis element which we construct in this paper is a sum of products of pfaffians (instead of determinants). The main issue is to verify that statements on pfaffians parallel to those on determinants in [HL06b] hold in the new cases. We provide full details on how this can be done in the case of $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$, but for the other two cases we only state the results.

This paper is arranged as follows. In § 2, we introduce notation for the representations of GL_n, O_n and Sp_{2n}. We review the definition of pfaffians in § 3. We construct the bases for the stable branching algebras for $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$, $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$ and $(\mathrm{O}_{2n}, \mathrm{GL}_n)$ in §§ 4, 5 and 6, respectively.

2. Notation

2.1 Representations of GL_n, O_n and Sp_{2n}

We use the following standard notation: $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ for the general linear group of invertible $n \times n$ complex matrices; $\mathrm{O}_n = \mathrm{O}_n(\mathbb{C})$ for the orthogonal group, the subgroup of GL_n which preserves a non-degenerate symmetric bilinear form; and $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\mathbb{C})$, the symplectic group, for the subgroup of GL_{2n} which preserves a non-degenerate skew-symmetric bilinear form. We shall also identify a Young diagram D with its sequence of row lengths $D = (\lambda_1, \ldots, \lambda_k)$. The number of rows in D is denoted by $r(D)$. Young diagrams will be used to parametrize irreducible representations of the groups GL_n, O_n and Sp_{2n} (see, for example, [How95]).

GL_n: Let A_{GL_n} be the diagonal torus in GL_n. For Young diagrams $D = (\lambda_1, \ldots, \lambda_r)$ and $E = (\mu_1, \ldots, \mu_s)$ such that $r + s \leqslant n$, we let $\psi_n^{D,E} : A_{\mathrm{GL}_n} \to \mathbb{C}^\times$ be the character given by

$$\psi_n^{D,E}[\mathrm{diag}(a_1, \ldots, a_n)] = [a_1^{\lambda_1} \cdots a_r^{\lambda_r}][a_{n-s+1}^{-\mu_s} \cdots a_n^{-\mu_1}],$$

and let $\rho_n^{D,E}$ be the irreducible representation of GL_n with highest weight $\psi_n^{D,E}$. When $E = (0)$, we shall write $\psi_n^{D,(0)}$ and $\rho_n^{D,(0)}$ simply as ψ_n^D and ρ_n^D, respectively.

O_n: If D is a Young diagram such that the sum of the lengths of the first two columns of D does not exceed n, we let σ_n^D be the irreducible representation of O_n generated by the GL_n highest weight vector in ρ_n^D. If $r(D) \neq n/2$, the restriction of σ_n^D to $\mathrm{SO}_n = \mathrm{SO}_n(\mathbb{C})$ is irreducible. The maximal torus A_{SO_n} of SO_n is isomorphic to $(\mathbb{C}^\times)^m$, where $m = [n/2]$. If D is a Young diagram with $r(D) < n/2$, let $\phi_n^D : A_{\mathrm{SO}_n} \to \mathbb{C}^\times$ be the restriction of the character ψ_n^D to A_{SO_n}. Then as a SO_n module, σ_n^D has highest weight ϕ_n^D.

Sp_{2n}: The diagonal torus $A_{\mathrm{Sp}_{2n}}$ of Sp_{2n} is isomorphic to $(\mathbb{C}^\times)^n$. The highest weights and the irreducible finite-dimensional representations of Sp_{2n} are parametrized by Young diagrams with at most n rows. If D is such a Young diagram, we denote the corresponding highest weight and representation by χ_{2n}^D and τ_{2n}^D, respectively.

2.2 The matrix associated with a LR tableau

If E is a Young diagram, then by a *banal tableau* of shape E, we mean the tableau obtained by filling each column of E from top to bottom with consecutive positive integers starting from 1.

In [HTW05b], for a given LR tableau (see [Fu97]) T of shape F/D where D and F are Young diagrams and D is contained in F, a banal tableau BT and a 'content preserving' map from T to BT are defined, i.e. each cell of T is mapped to a cell in BT with the same value. The map can be visualized as the process of successively removing the 'vertical skew strips' from T and reassembling them into columns of BT. This process is called 'standard peeling'. Thus T is constructed by the reverse process of standard peeling. The contents of BT are moved to the skew diagram one column at a time, starting from the last column of BT. If BT has shape E, then we say 'T is a LR tableau of shape F/D and content E'. Readers may refer to [HTW05b] for a detailed description of this process. Using standard peeling, we can associate T with the matrix $M(T) = (m_{ij})$ of nonnegative integers where m_{ij} is the number of entries from the jth column of E that get put into the ith column of F/D.

3. Review on pfaffians

Let $A = (a_{ij})$ be a $2n \times 2n$ skew symmetric complex matrix. The *pfaffian* of A is defined by [GW98]

$$\mathrm{Pfaff}(A) = \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

Here S_{2n} denote the symmetric group on $\{1, 2, \ldots, 2n\}$. To remove the factor of $1/(n!2^n)$, we let $C(2n)$ be the subset of S_{2n} which contains all permutations σ such that:

(i) $\sigma(2i-1) < \sigma(2i)$ for all $1 \leqslant i \leqslant n$; and

(ii) $\sigma(2j) < \sigma(2j+2)$ for all $1 \leqslant j \leqslant n-1$.

Then an elementary argument shows that

$$\mathrm{Pfaff}(A) = \sum_{\sigma \in C(2n)} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

In this paper, we only consider the pfaffian of those matrices A of the following form:

$$A = \begin{pmatrix} B & C \\ -C^{\mathrm{t}} & 0 \end{pmatrix}$$

where B is an $m \times m$ skew-symmetric matrix, C is as $m \times (2n-m)$ matrix and $n \leqslant m < 2n$. In this case,

$$\mathrm{Pfaff}(A) = \sum_{\sigma \in C(2n,m)} (\mathrm{sgn}\,\sigma) \prod_{i=1}^{m-n} a_{\sigma(2i-1),\sigma(2i)} \prod_{j=1}^{2n-m} a_{\sigma(2m-2n+2j-1),m+j} \tag{3.1}$$

where

$$C(2n,m) = \{\sigma \in C(2n) : \sigma(2i-1) \leqslant m \ \forall 1 \leqslant i \leqslant n\}. \tag{3.2}$$

Note that when $m = n$, then

$$\mathrm{Pfaff}(A) = (-1)^{n(n-1)/2} \det \begin{pmatrix} a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,2n} \\ a_{2,n+1} & a_{2,n+2} & \cdots & a_{2,2n} \\ \vdots & \vdots & & \vdots \\ a_{n,n+1} & a_{n,n+2} & \cdots & a_{n,2n} \end{pmatrix}.$$

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

4. A basis for the stable branching algebra for $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$

4.1 The algebra $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$

Let n and k be positive integers such that $k \leqslant n$, and let $\mathcal{P}(\mathrm{M}_{2n,k})$ be the algebra of polynomial functions on the space $\mathrm{M}_{2n,k} = \mathrm{M}_{2n,k}(\mathbb{C})$ of $2n \times k$ complex matrices. Let GL_{2n} and GL_k act on $\mathcal{P}(\mathrm{M}_{2n,k})$ by the formula:

$$[(g,h)f](X) = f(g^{\mathrm{t}} X h), \quad g \in \mathrm{GL}_{2n},\ h \in \mathrm{GL}_k,\ f \in \mathcal{P}(\mathrm{M}_{2n,k}),\ X \in \mathrm{M}_{2n,k}$$

and let Sp_{2n} act by the restriction of the action of GL_{2n}. By the $(\mathrm{GL}_{2n}, \mathrm{GL}_k)$-duality,

$$\mathcal{P}(\mathrm{M}_{2n,k}) = \sum_{r(F) \leqslant k} \rho_{2n}^F \otimes \rho_k^F.$$

Let U_{GL_k} and $U_{\mathrm{Sp}_{2n}}$ denote the standard maximal unipotent subgroups of GL_k and Sp_{2n}, respectively. Taking U_{GL_k}-invariants in $\mathcal{P}(\mathrm{M}_{2n,k})$ gives

$$\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{GL}_k}} = \sum_{r(F) \leqslant k} \rho_{2n}^F \otimes (\rho_k^F)^{U_{\mathrm{GL}_k}}.$$

This algebra contains one copy of each irreducible representation ρ_{2n}^F of GL_{2n} with $r(F) \leqslant k$. We consider the subalgebra of $U_{\mathrm{Sp}_{2n}}$-invariants in $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{GL}_k}}$:

$$\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}} \cong \sum_{r(F) \leqslant k} (\rho_{2n}^F)^{U_{\mathrm{Sp}_{2n}}} \otimes (\rho_k^F)^{U_{\mathrm{GL}_k}}.$$

This algebra is a module for $A_{\mathrm{Sp}_{2n}} \times A_{\mathrm{GL}_k}$, where $A_{\mathrm{Sp}_{2n}}$ and A_{GL_k}, are the diagonal torus of Sp_{2n} and GL_k, respectively. So, we can also write

$$\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}} = \bigoplus_{r(D), r(F) \leqslant k} W_{D,F},$$

where for each pair (D, F) of Young diagrams with at most k rows, $W_{D,F}$ is the $\chi_{2n}^D \times \psi_k^F$-eigenspace of $A_{\mathrm{Sp}_{2n}} \times A_{\mathrm{GL}_k}$. The elements in $W_{D,F}$ can be identified with the Sp_{2n} highest weight vectors in ρ_{2n}^F with weight χ_{2n}^D. Thus, the algebra structure of $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$ carries information on the branching rule from GL_{2n} to Sp_{2n} for the representations ρ_{2n}^F with $r(F) \leqslant k$. In view of this property, we call $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$ a *stable branching algebra* for $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$ (see [HTW04]). The goal of this section is to construct a basis for this algebra.

4.2 The $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ module structure of $\mathcal{P}(\mathrm{M}_{2n,k})$

We write a typical element of $\mathrm{M}_{2n,k}$ as $X = (x_{ij})$. For $1 \leqslant i, j \leqslant k$, let

$$\omega_{ij}(X) = \sum_{a=1}^{n} (x_{ai} x_{n+a,j} - x_{n+a,i} x_{aj}). \tag{4.1}$$

By the first fundamental theorem of invariant theory for Sp_{2n}, the algebra $\mathcal{P}(\mathrm{M}_{nk})^{\mathrm{Sp}_{2n}}$ of Sp_{2n} invariants in $\mathcal{P}(\mathrm{M}_{2n,k})$ is generated by the ω_{ij}. In fact, since $k \leqslant n$,

$$\mathcal{P}(\mathrm{M}_{2n,k})^{\mathrm{Sp}_{2n}} = \mathbb{C}[\omega_{ij} :\ 1 \leqslant i < j \leqslant k] \cong \mathcal{P}(\textstyle\bigwedge^2(\mathbb{C}^k)) \cong \displaystyle\sum_{E \in \mathcal{E}_k} \rho_k^E$$

as a GL_k module [How95], where

$$\mathcal{E}_k = \{(\mu_1, \mu_1, \mu_2, \mu_2, \dots, \mu_m, \mu_m) : \mu_1 \geqslant \dots \geqslant \mu_m \geqslant 0,\ 2m \leqslant k\} \tag{4.2}$$

is the set of all Young diagrams E such that E has at most k rows and all of its column lengths are even.

Next, for $1 \leqslant i, j \leqslant k$, we let

$$\Delta_{ij} = \sum_{a=1}^{n} \left(\frac{\partial^2}{\partial x_{ai} \partial x_{n+a,j}} - \frac{\partial^2}{\partial x_{n+a,i} \partial x_{aj}} \right),$$

and

$$\mathcal{H}(\mathrm{M}_{2n,k}, \mathrm{Sp}_{2n}) = \{ f \in \mathcal{P}(\mathrm{M}_{2n,k}) : \ \Delta_{ij}(f) = 0, \ \forall 1 \leqslant i, j \leqslant k \}.$$

Under the action of $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ (see [How95]),

$$\mathcal{H}(\mathrm{M}_{2n,k}, \mathrm{Sp}_{2n}) \cong \sum_{r(D) \leqslant k} \tau_{2n}^D \otimes \rho_k^D.$$

Since $k \leqslant n$, we have

$$\mathcal{P}(\mathrm{M}_{2n,k}) \cong \mathcal{H}(\mathrm{M}_{2n,k}, \mathrm{Sp}_{2n}) \otimes \mathcal{P}(\mathrm{M}_{2n,k})^{\mathrm{Sp}_{2n}} \tag{4.3}$$

as a $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ module. It follows from this and the LR rule that

$$\mathcal{P}(\mathrm{M}_{2n,k}) \cong \sum_{r(D), r(F) \leqslant k} \left(\sum_{E \in \mathcal{E}_k} c_{E,D}^F \right) \tau_{2n}^D \otimes \rho_k^F.$$

Here $c_{E,D}^F$ is the LR coefficient which counts the number of LR tableaux of shape F/E and content D (see [Fu97]). In particular, this implies that

$$\dim W_{D,F} = \sum_{E \in \mathcal{E}_k} c_{E,D}^F. \tag{4.4}$$

4.3 The algebra $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$

The algebra $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$ contains $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ highest weight vectors. In this subsection, we shall prove that $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$ is isomorphic to an algebra of $\mathrm{GL}_{2n} \times \mathrm{GL}_k$ highest weight vectors. This allows us to apply the results of [HL06b, § 5] in our construction of the basis.

Let $\mathrm{AM}_k = \mathrm{AM}_k(\mathbb{C})$ denote the space of all $k \times k$ skew-symmetric complex matrices. Let $\mathrm{GL}_{2n} \times \mathrm{GL}_k$ act on $\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k$ by

$$(g, h)(X, \xi) = ((g^{-1})^{\mathrm{t}} X h^{-1}, (h^{-1})^{\mathrm{t}} \xi h^{-1}), \quad g \in \mathrm{GL}_{2n}, \ h \in \mathrm{GL}_k, \ X \in \mathrm{M}_{2n,k}, \ \xi \in \mathrm{AM}_k.$$

This extends to an action of $\mathrm{GL}_{2n} \times \mathrm{GL}_k$ on the polynomial algebra $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$ on $\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k$ in the usual way. Let $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}}$ be the subalgebra of $U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}$-invariants in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$. We claim that the algebras $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$ and $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}}$ are isomorphic. In fact, the space $\mathcal{H}(\mathrm{M}_{2n,k}, \mathrm{Sp}_{2n})^{U_{\mathrm{Sp}_{2n}}}$ of $U_{\mathrm{Sp}_{2n}}$-invariants in $\mathcal{H}(\mathrm{M}_{2n,k}, \mathrm{Sp}_{2n})$ coincides with the space $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{GL}_{2n}}}$ of $U_{\mathrm{GL}_{2n}}$-invariants in $\mathcal{P}(\mathrm{M}_{2n,k})$. Using this and (4.3), we have

$$\begin{aligned} \mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}}} &\cong \mathcal{H}(\mathrm{M}_{2n,k}, \mathrm{Sp}_{2n})^{U_{\mathrm{Sp}_{2n}}} \otimes \mathcal{P}(\mathrm{M}_{2n,k})^{\mathrm{Sp}_{2n}} \\ &\cong \mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{GL}_{2n}}} \otimes \mathcal{P}(\mathrm{AM}_k) \\ &\cong \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}}}. \end{aligned}$$

So taking U_{GL_k}-invariants gives $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}} \cong \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}}$. An isomorphism $\alpha : \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}} \to \mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$ is defined as follows: if $f \in \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}}$ and $X \in \mathrm{M}_{2n,k}$, then

$$[\alpha(f)](X) = f(X, \omega(X)).$$

Here $\omega(X) = (\omega_{ij}(X))$ and $\omega_{ij}(X)$ is defined in (4.1). Observe that α sends the $\psi_{2n}^D \times \psi_k^F$-eigenspace $\widetilde{W}_{D,F}$ of $A_{\mathrm{GL}_{2n}} \times A_{\mathrm{GL}_k}$ in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{2n} \times U_k}$ to $W_{D,F}$. It follows from this and (4.4) that $\dim \widetilde{W}_{D,F} = \sum_{E \in \mathcal{E}_k} c_{E,D}^F$.

4.4 GL_k highest weight vectors in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$

We denote a typical element in $\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k$ by (X, ξ) where $X = (x_{ij})$ and $\xi = (\xi_{ij})$. So $\xi_{ij} = -\xi_{ji}$ and $\xi_{ii} = 0$ for all $1 \leqslant i, j \leqslant k$. Let m and p be positive integers such that $p \leqslant m \leqslant k$ and $m + p$ is even. Let $[\xi]_m$ be the following $m \times m$ submatrix of ξ:

$$[\xi]_m = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1m} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2m} \\ \vdots & \vdots & & \vdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mm} \end{pmatrix}.$$

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{C}^k$ where, for each $1 \leqslant i \leqslant p$,

$$\mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{ik}).$$

We set

$$[\mathbf{v}_1, \ldots, \mathbf{v}_p]_m = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & & \vdots \\ v_{p1} & v_{p2} & \cdots & v_{pm} \end{pmatrix}$$

and

$$P_m(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \mathrm{Pfaff}\left(\begin{array}{c|c} [\xi]_m & ([\mathbf{v}_1, \ldots, \mathbf{v}_p]_m)^{\mathrm{t}} \\ \hline -[\mathbf{v}_1, \ldots, \mathbf{v}_p]_m & 0 \end{array}\right).$$

Next, for $1 \leqslant i \leqslant 2n$, let

$$X_i = (x_{i1}, x_{i2}, \ldots, x_{ik}).$$

For $1 \leqslant j_1, \ldots, j_p \leqslant 2n$, consider the polynomial

$$P_m(X_{j_1}, \ldots, X_{j_p}) \tag{4.5}$$

on $\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k$.

Lemma 4.4.1. *The polynomial $P_m(X_{j_1}, \ldots, X_{j_p})$ is a GL_k highest weight vector of weight $\psi_k^{\mathbf{1}_m}$ in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$ where*

$$\mathbf{1}_m = (\overbrace{1, \ldots, 1}^{m}, 0, \ldots, 0).$$

Proof. By (3.1),

$$P_m(X_{j_1}, \ldots, X_{j_p}) = \sum_{\sigma \in C(p+m,m)} (\mathrm{sgn}\ \sigma) \prod_{i=1}^{(m-p)/2} \xi_{\sigma(2i-1),\sigma(2i)} \prod_{b=1}^{p} x_{j_b,\sigma(m-p+2b-1)} \tag{4.6}$$

where $C(m+p, m)$ is the subset of the symmetric group S_{m+p} defined in (3.2). From this expression, it is easy to see that $P_m(X_{j_1}, \ldots, X_{j_p})$ is a GL_k weight vector of weight $\psi_k^{\mathbf{1}_m}$.

We consider the derived action of the Lie algebra $\mathfrak{gl}_k = \mathfrak{gl}_k(\mathbb{C})$ of GL_k on $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$. Let E_{ab} be the element in $\mathfrak{gl}_k$ such that its (a, b)th entry is 1 and 0 elsewhere. Then E_{ab} acts by the differential operator

$$E_{ab} = \sum_{j=1}^{2n} x_{ja} \frac{\partial}{\partial x_{jb}} + \sum_{i<b} \xi_{ia} \frac{\partial}{\partial \xi_{ib}} + \sum_{j>b} \xi_{aj} \frac{\partial}{\partial \xi_{bj}}. \tag{4.7}$$

We need to show that $E_{ab}[P_m(X_{j_1}, \ldots, X_{j_p})] = 0$ for $1 \leqslant a < b \leqslant k$. Note that

$$E_{ab}[P_m(X_{j_1}, \ldots, X_{j_p})^2] = 2P_m(X_{j_1}, \ldots, X_{j_p})\{E_{ab}[P_m(X_{j_1}, \ldots, X_{j_p})]\}.$$

Thus, it suffices to show that $E_{ab}[P_m(X_{j_1}, \ldots, X_{j_p})^2] = 0$.

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

Let

$$A = \left(\begin{array}{c|c} [\xi]_m & ([X_{j_1}, \ldots, X_{j_p}]_m)^{\mathrm{t}} \\ \hline -[X_{j_1}, \ldots, X_{j_p}]_m & 0 \end{array}\right).$$

Then

$$[P_m(X_{j_1}, \ldots, X_{j_p})]^2 = \det A.$$

First we note that if $b > m$, then from the expression for the operator E_{ab} given in (4.7), it is clear that $E_{ab}(\det A) = 0$. So we assume that $1 \leqslant a < b \leqslant m$. For each $1 \leqslant i, j \leqslant m+p$, let $A(i, j)$ be the matrix obtained from A by deleting its ith row and jth column, and for each $1 \leqslant l \leqslant m+p$, let $A[l]$ be the matrix obtained from A by replacing its lth row by its image R_l under E_{ab}. Specifically,

$$R_l = \begin{cases} (\overbrace{0, \ldots, 0, \xi_{la}}^{b}, 0, \ldots, 0) & 1 \leqslant l \leqslant m,\ l \neq a, b \\ (\overbrace{0, \ldots, 0, 0}^{b}, 0, \ldots, 0) & l = a \\ (\overbrace{0, \ldots, 0, x_{j_{l-m},a}}^{b}, 0, \ldots, 0) & m+1 \leqslant l \leqslant m+p, \end{cases}$$

and

$$R_b = (\underbrace{\overbrace{\xi_{a1}, \ldots, \xi_{a,a-1}, 0}^{a}, \xi_{a,a+1}, \ldots, \xi_{a,b-1}, 0}_{b}, \xi_{a,b+1}, \ldots, \xi_{am}, x_{j_1,a}, \ldots, x_{j_p,a}).$$

By expanding the determinant of $A[l]$ along its lth row, we obtain

$$\det A[l] = \begin{cases} (-1)^{b+l}\xi_{la} \det A(l, b) & 1 \leqslant l \leqslant m,\ l \neq a, b \\ 0 & l = a \\ -(-1)^{b+l} x_{j_{l-m},a} \det A(l, b) & m+1 \leqslant l \leqslant m+p. \end{cases}$$

By adding the (-1) multiple of the ath row of $A[b]$ to its bth row, the bth row becomes

$$(\overbrace{0, \ldots, 0, -\xi_{ab}}^{b}, 0, \ldots, 0) = (\overbrace{0, \ldots, 0, \xi_{ba}}^{b}, 0, \ldots, 0).$$

So

$$\det A[b] = (-1)^{b+l}\xi_{ba} \det A(b, b).$$

Consequently,

$$\begin{aligned} E_{ab}(\det A) &= \sum_{l=1}^{m+p} \det A[l] \\ &= \sum_{l=1}^{m} (-1)^{b+l}\xi_{la} \det A(l, b) + \sum_{l=m+1}^{m+p} (-1)^{b+l}(-x_{j_{l-m},a}) \det A(l, b). \end{aligned}$$

Observe that the right-hand side of the above equation is the expansion of the determinant of the matrix obtained from A by replacing its bth column by its ath column. Since this matrix has two equal columns,

$$E_{ab}(\det A) = 0. \qquad \square$$

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

4.5 The action by $\mathfrak{gl}_{2n}$

We now investigate how the Lie algebra $\mathfrak{gl}_{2n}$ of GL_{2n} acts on the polynomial $P_m(X_{j_1}, X_{j_2}, \ldots, X_{j_p})$ which is defined in (4.5).

LEMMA 4.5.1. *If $T \in \mathfrak{gl}_{2n}$ and $1 \leqslant j_1, \ldots, j_p \leqslant 2n$, then*

$$T[P_m(X_{j_1}, X_{j_2}, \ldots, X_{j_p})] = \sum_{l=1}^{p} P_m(X_{j_1}, \ldots, X_{j_{l-1}}, T(X_{j_l}), X_{j_{l+1}}, \ldots, X_{j_p}),$$

where, for each $1 \leqslant l \leqslant p$,

$$T(X_{j_l}) = (T(x_{j_l,1}), T(x_{j_l,2}), \ldots, T(x_{j_l,k})).$$

Proof. Using the formula for $P_m(X_{j_1}, X_{j_2}, \ldots, X_{j_p})$ given in (4.6), we obtain

$$\begin{aligned}
&T[P_m(X_{j_1}, X_{j_2}, \ldots, X_{j_p})] \\
&\quad = \sum_{\sigma \in C(p+m,m)} (\mathrm{sgn}\ \sigma) \prod_{a=1}^{(m-p)/2} \xi_{\sigma(2a-1),\sigma(2a)} T\left(\prod_{b=1}^{p} x_{j_b,\sigma(m-p+2b-1)}\right) \\
&\quad = \sum_{\sigma \in C(p+m,m)} (\mathrm{sgn}\ \sigma) \prod_{a=1}^{(m-p)/2} \xi_{\sigma(2a-1),\sigma(2a)} \left[\sum_{l=1}^{p} T(x_{j_l,\sigma(m-p+2l-1)}) \prod_{\substack{1\leqslant b\leqslant p\\ b\neq l}} x_{j_b,\sigma(m-p+2b-1)}\right] \\
&\quad = \sum_{l=1}^{p} \left\{\sum_{\sigma \in C(p+m,m)} (\mathrm{sgn}\ \sigma) \prod_{a=1}^{(m-p)/2} \xi_{\sigma(2a-1),\sigma(2a)} [T(x_{j_l,\sigma(m-p+2l-1)})] \prod_{\substack{1\leqslant b\leqslant p\\ b\neq l}} x_{j_b,\sigma(m-p+2b-1)}\right\} \\
&\quad = \sum_{l=1}^{p} P_m(X_{j_1}, \ldots, X_{j_{l-1}}, T(X_{j_l}), X_{j_{l+1}}, \ldots, X_{j_p}).
\end{aligned}$$

□

COROLLARY 4.5.2. *Let $1 \leqslant p \leqslant m \leqslant k$ and let $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ be the standard basis for $\mathbb{C}^{2n}$. Then the linear map*

$$\Phi : \bigwedge^p \mathbb{C}^{2n} \to \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$$

specified by

$$\Phi(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p}) = P_m(X_{j_1}, \ldots, X_{j_p})$$

is a GL_{2n} isomorphism from $\bigwedge^p \mathbb{C}^{2n}$ onto its image.

Proof. First we note that the linear map $(\mathbb{C}^{2n})^p \to \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$ specified by

$$(\varepsilon_{j_1}, \ldots, \varepsilon_{j_p}) \to P_m(X_{j_1}, \ldots, X_{j_p})$$

is alternating. In fact, if $s < t$ and we switch the positions of X_{j_s} and X_{j_t} in $P_m(X_{j_1}, \ldots, X_{j_p})$, then we obtain

$$\begin{aligned}
P_m(X_{j_1}, \ldots, X_{j_t}, \ldots, X_{j_s}, \ldots, X_{j_p}) &= \sum_{\sigma \in C(p+m,m)} (\mathrm{sgn}\ \sigma) \left[\prod_{i=1}^{(m-p)/2} \xi_{\sigma(2i-1),\sigma(2i)}\right] \\
&\quad \times x_{j_t,\sigma(m-p+2s-1)} x_{j_s,\sigma(m-p+2t-1)} \prod_{\substack{1\leqslant b\leqslant p\\ b\neq s,t}} x_{j_b,\sigma(m-p+2b-1)}.
\end{aligned}$$

Now for each $\sigma \in C(p+m,m)$, we let $\tilde{\sigma} = \sigma \circ \tau$ where τ is the transposition $(m-p+2s-1, m-p+2t-1)$. Then

$$\begin{aligned} &P_m(X_{j_1},\dots,X_{j_t},\dots,X_{j_s},\dots,X_{j_p}) \\ &\quad = \sum_{\tilde{\sigma}\in C(p+m,m)} (\operatorname{sgn}\tilde{\sigma})(\operatorname{sgn}\tau)\Bigg[\prod_{i=1}^{(m-p)/2} \xi_{\tilde{\sigma}(2i-1),\tilde{\sigma}(2i)}\Bigg] \\ &\qquad\quad \times x_{j_t,\tilde{\sigma}(m-p+2t-1)} x_{j_s,\tilde{\sigma}(m-p+2s-1)} \prod_{\substack{1\leqslant b\leqslant p\\ b\neq s,t}} x_{j_b,\tilde{\sigma}(m-p+2b-1)} \\ &\quad = -P_m(X_{j_1},\dots,X_{j_s},\dots,X_{j_t},\dots,X_{j_p}). \end{aligned}$$

Hence, the linear map Φ exists. By Lemma 4.5.1, Φ is a GL_{2n} map. Since $\bigwedge^p\mathbb{C}^{2n}$ is an irreducible GL_{2n} module, Φ is a GL_{2n} isomorphism from $\bigwedge^p\mathbb{C}^{2n}$ onto its image. ☐

Corollary 4.5.3. *The function $P_m(X_1,\dots,X_p)$ is a joint $\mathrm{GL}_{2n}\times\mathrm{GL}_k$ highest weight vector of weight $\psi_{2n}^{\mathbf{1}_p}\times\psi_k^{\mathbf{1}_m}$.*

Proof. This is because $\varepsilon_1\wedge\cdots\wedge\varepsilon_p$ is the GL_{2n} highest weight vector in $\bigwedge^p\mathbb{C}^{2n}$ of weight $\psi_{2n}^{\mathbf{1}_p}$ and $P_m(X_1,\dots,X_p)=\Phi(\varepsilon_1\wedge\cdots\wedge\varepsilon_p)$. ☐

4.6 $\mathbf{GL_{2n}\times GL_k}$ highest weight vectors in $\boldsymbol{\mathcal{P}(\mathrm{M}_{2n,k}\oplus \mathrm{AM}_k)}$

Recall that for fixed D and F, the $\psi_{2n}^D\times\psi_k^F$-eigenspace $\widetilde{W}_{D,F}$ in $\mathcal{P}(\mathrm{M}_{2n,k}\oplus\mathrm{AM}_k)^{U_{2n}\times U_k}$ has dimension $\sum_{E\in\mathcal{E}_k} c_{E,D}^F$. We now let

$$\Omega(D,F) \tag{4.8}$$

be the set of all ordered pairs (E,T) where $E\in\mathcal{E}_k$ and T is a LR tableau of shape F/E and content D. Then the number of elements in $\Omega(D,F)$ coincides with $\dim\widetilde{W}_{D,F}$. We shall use the set $\Omega(D,F)$ to index a basis for $\widetilde{W}_{D,F}$.

We now fix $(E,T)\in\Omega(D,F)$ and will construct a $\mathrm{Sp}_{2n}\times\mathrm{GL}_k$ highest weight vector $\zeta_{(E,T)}$ in $\mathcal{P}(\mathrm{M}_{2n,k}\oplus\mathrm{AM}_k)$ with weight $\psi_{2n}^D\times\psi_k^F$. Let

$$\begin{aligned} D^{\mathrm{t}} &= (d_1,\dots,d_r)\\ E^{\mathrm{t}} &= (e_1,\dots,e_s)\\ F^{\mathrm{t}} &= (f_1,\dots,f_t), \end{aligned}\tag{4.9}$$

the conjugate diagrams of D, E and F, respectively. For each $1\leqslant i\leqslant t$, let

$$p_i = f_i - e_i.$$

Recall that T is constructed by the reverse process of standard peeling (see § 2.2). Let $M(T)=(m_{ij})$ be the $t\times r$ matrix where m_{ij} is the number of elements from the jth column of D get put into the ith column of F. For $1\leqslant i\leqslant t$ and $1\leqslant j\leqslant r$, let

$$a_{ij} = m_{ij}+m_{i+1,j}+\cdots+m_{tj},$$

and

$$a_{t+1,j}=0.$$

Then the entries from the jth column of D get put into the ith column of F are

$$a_{i+1,j}+1, a_{i+1,j}+2,\dots,a_{i,j}.$$

We now let $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_r)\in S_{d_1}\times\cdots\times S_{d_r}$. Here S_{d_i} denotes the symmetric group on $\{1,\dots,d_i\}$.

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

For each $1 \leqslant i \leqslant t$, let $\mathcal{X}_i(\boldsymbol{\sigma})$ be the $p_i \times f_i$ matrix

$$\mathcal{X}_i(\boldsymbol{\sigma}) = \begin{pmatrix} \mathcal{X}_{i1}(\sigma_1) \\ \mathcal{X}_{i2}(\sigma_2) \\ \vdots \\ \mathcal{X}_{ir}(\sigma_r) \end{pmatrix}$$

where for each j, $\mathcal{X}_{ij}(\sigma_l)$ is the $m_{il} \times f_i$ matrix defined by

$$\mathcal{X}_{ij}(\sigma_j) = \begin{pmatrix} x_{\sigma_j(a_{i+1,j}+1),1} & x_{\sigma_j(a_{i+1,j}+1),2} & \cdots & x_{\sigma_j(a_{i+1,j}+1),f_i} \\ x_{\sigma_j(a_{i+1,j}+2),1} & x_{\sigma_j(a_{i+1,j}+2),2} & \cdots & x_{\sigma_j(a_{i+1,j}+2),f_i} \\ \vdots & \vdots & & \vdots \\ x_{\sigma_j(a_{i,j}),1} & x_{\sigma_j(a_{i,j}),2} & \cdots & x_{\sigma_j(a_{i,j}),f_i} \end{pmatrix}.$$

Set

$$\zeta_{(E,T)} = \sum_{\boldsymbol{\sigma} \in S_{d_1} \times \cdots \times S_{d_r}} \left(\prod_{i=1}^{r} \operatorname{sgn}(\sigma_i) \right) \cdot \left[\prod_{i=1}^{t} \operatorname{Pfaff} \left(\begin{array}{c|c} [\xi]_{f_i} & [\mathcal{X}_i(\boldsymbol{\sigma})]^{\mathrm{t}} \\ \hline -\mathcal{X}_i(\boldsymbol{\sigma}) & 0 \end{array} \right) \right]. \tag{4.10}$$

LEMMA 4.6.1. *The polynomial $\zeta_{(E,T)}$ defined in (4.10) is a $\mathrm{GL}_{2n} \times \mathrm{GL}_k$ highest weight vector in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$ of weight $\psi_{2n}^D \times \psi_k^F$.*

Proof. It is easy to verify that the weight of $\zeta_{(E,T)}$ is $\psi_{2n}^D \times \psi_k^F$. That $\zeta_{(E,T)}$ is a GL_k highest weight vector of weight ψ_k^F follows from Lemma 4.4.1. To see that $\zeta_{(E,T)}$ is a GL_{2n} highest weight vector, let

$$\Phi : \textstyle\bigwedge^{p_1}(\mathbb{C}^{2n}) \otimes \bigwedge^{p_2}(\mathbb{C}^{2n}) \otimes \cdots \otimes \bigwedge^{p_t}(\mathbb{C}^{2n}) \longrightarrow \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$$

be the GL_{2n} map defined by

$$\Phi(u_1 \otimes \cdots \otimes u_t) = \Phi_1(u_1) \cdots \Phi_t(u_t)$$

where for each $1 \leqslant i \leqslant t$, $u_i \in \bigwedge^{p_i}(\mathbb{C}^{2n})$ and $\Phi_i : \bigwedge^{p_i}(\mathbb{C}^{2n}) \to \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$ is the GL_{2n}-map defined in Corollary 4.5.2. We now observe that $\zeta_{(E,T)}$ is the image of the highest weight vector Γ_N given in [HL06b, Lemma 6.2.1], where the matrix N is the transpose of $M(T)$. So it is a GL_{2n} highest weight vector of weight ψ_{2n}^D. $\square$

4.7 The leading monomial of $\zeta_{(E,T)}$

Recall that typical elements of $\mathrm{M}_{2n,k}$ and AM_k are written as $X = (x_{ij})$ and $\xi = (\xi_{ij})$, respectively. We define a monomial ordering τ_1 on $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)$ as follows: it is the graded lexicographic order [CLO97] such that

$$\begin{aligned} &\xi_{12} > \xi_{13} > \cdots > \xi_{1k} > \xi_{23} > \cdots > \xi_{2n} > \xi_{34} > \cdots > \xi_{k-1,k} \\ &\quad > x_{11} > x_{12} > \cdots > x_{1k} > x_{21} > \cdots > x_{2n,k}. \end{aligned} \tag{4.11}$$

We shall compute the leading monomial of $\zeta_{(E,T)}$ with respect to τ_1.

We first consider the function

$$P_m(X_{j_1}, \ldots, X_{j_p}) = \operatorname{Pfaff} \left(\begin{array}{c|c} [\xi]_m & ([X_{j_1}, \ldots, X_{j_p}]_m)^{\mathrm{t}} \\ \hline -[X_{j_1}, \ldots, X_{j_p}]_m & 0 \end{array} \right),$$

which is a building block of the highest weight vector $\zeta_{(E,T)}$. We note that in the expansion of $P_m(X_{j_1}, \ldots, X_{j_p})$ given in (4.6), exactly $(m-p)/2$ entries in each term need to be chosen from the submatrix $[\xi]_m$. It is clear from the definition of τ_1 that $\xi_{12}\xi_{23} \cdots \xi_{m-p-1,m-p}$ has the highest possible order among all possible choices.

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

Lemma 4.7.1. *Let $\tilde{P}_m(X_{j_1},\ldots,X_{j_p})$ be the sum of all of the terms in $P_m(X_{j_1},\ldots,X_{j_p})$ containing the factor $\xi_{12}\xi_{23}\cdots\xi_{m-p-1,m-p}$. Then*

$$\tilde{P}_m(X_{j_1},\ldots,X_{j_p}) = (-1)^{p(p-1)/2}(\xi_{12}\xi_{23}\cdots\xi_{m-p-1,m-p})\begin{vmatrix} x_{j_1,m-p+1} & x_{j_1,m-p+2} & \cdots & x_{j_1,m} \\ x_{j_2,m-p+1} & x_{j_2,m-p+2} & \cdots & x_{j_2,m} \\ \vdots & \vdots & & \vdots \\ x_{j_p,m-p+1} & x_{j_p,m-p+2} & \cdots & x_{j_p,m} \end{vmatrix}.$$

Proof. By (4.6),

$$P_m(X_{j_1},\ldots,X_{j_p}) = \sum_{\sigma\in C(p+m,m)} (\mathrm{sgn}\ \sigma) \prod_{i=1}^{(m-p)/2} \xi_{\sigma(2i-1),\sigma(2i)} \prod_{b=1}^{p} x_{j_b,\sigma(m-p+2b-1)}$$

From this, it is clear that if $\sigma \in C(2n,m)$, then

$$\xi_{12}\xi_{23}\cdots\xi_{m-p-1,m-p} = \prod_{i=1}^{(m-p)/2} \xi_{\sigma(2i-1),\sigma(2i)}$$

if and only if

$$\sigma(j) = j, \quad j = 1,\ldots,m-p.$$

For such a permutation σ, there exists a unique permutation $\tau \in S_p$ such that

$$\sigma(m-p+2j-1) = m-p+\tau(j), \quad j = 1,\ldots,p.$$

Conversely, any permutation $\tau \in S_p$ uniquely determines an element σ of $C(m+p,m)$ with the property $\sigma(j) = j$ for $j = 1,\ldots,m-p$. In addition,

$$\mathrm{sgn}\ \sigma = (-1)^{p(p-1)/2}\,\mathrm{sgn}\ \tau.$$

It follows that

$$\begin{aligned}\tilde{P}_m(X_{j_1},\ldots,X_{j_p}) &= (-1)^{p(p-1)/2}(\xi_{12}\xi_{23}\cdots\xi_{m-p-1,m-p}) \sum_{\tau\in S_p} (\mathrm{sgn}\ \tau) \prod_{b=1}^{p} x_{j_b,m-p+\tau(b)} \\ &= (-1)^{p(p-1)/2}(\xi_{12}\xi_{23}\cdots\xi_{m-p-1,m-p})\begin{vmatrix} x_{j_1,m-p+1} & x_{j_1,m-p+2} & \cdots & x_{j_1,m} \\ x_{j_2,m-p+1} & x_{j_2,m-p+2} & \cdots & x_{j_2,m} \\ \vdots & \vdots & & \vdots \\ x_{j_p,m-p+1} & x_{j_p,m-p+2} & \cdots & x_{j_p,m} \end{vmatrix}.\end{aligned}$$

□

Recall that E is the Young diagram given in (4.9). Let

$$\xi_E = \prod_{j=1}^{s} (\xi_{12}\xi_{23}\cdots\xi_{e_j-1,e_j}), \tag{4.12}$$

and consider the sum of all of the terms in $\zeta_{(E,T)}$ which contain ξ_E as a factor. It is of the form

$$\xi_E \cdot (\zeta_{(E,T)}[\xi_E]) \tag{4.13}$$

where $\zeta_{(E,T)}[\xi_E]$ is a polynomial in the variables (x_{ij}). Note that the leading monomial of $\zeta_{(E,T)}$ is the product of ξ_E and the leading monomial of $\zeta_{(E,T)}[\xi_E]$.

Lemma 4.7.2. *The leading monomial of $\zeta_{(E,T)}[\mu_E]$ is given by*

$$x_T = \prod_{\mathbf{b}\in T} x_{c(\mathbf{b})a(\mathbf{b})}$$

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

where for each box $\mathbf{b}$ *in* T, $a(\mathbf{b})$ *is the row of* F *in which the box* $\mathbf{b}$ *lies and* $c(\mathbf{b})$ *is the entry in* $\mathbf{b}$. *Consequently, the leading monomial of* $\zeta_{(E,T)}$ *is* $x_T\xi_E$.

Proof. We first set up some notation. For each $1 \leqslant i \leqslant t$, let $\hat{\mathcal{X}}_i(\boldsymbol{\sigma})$ be the $p_i \times p_i$ matrix

$$\hat{\mathcal{X}}_i(\boldsymbol{\sigma}) = \begin{pmatrix} \hat{\mathcal{X}}_{i1}(\sigma_1) \\ \hat{\mathcal{X}}_{i2}(\sigma_2) \\ \vdots \\ \hat{\mathcal{X}}_{ir}(\sigma_r) \end{pmatrix} \tag{4.14}$$

where, for each j, $\hat{\mathcal{X}}_{ij}(\sigma_j)$ is the $m_{ij} \times p_i$ matrix

$$\hat{\mathcal{X}}_{ij}(\sigma_j) = \begin{pmatrix} x_{\sigma_j(a_{i+1,j}+1),e_i+1} & x_{\sigma_j(a_{i+1,j}+1),e_i+2} & \cdots & x_{\sigma_j(a_{i+1,j}+1),f_i} \\ x_{\sigma_j(a_{i+1,j}+2),e_i+1} & x_{\sigma_j(a_{i+1,j}+2),e_i+2} & \cdots & x_{\sigma_j(a_{i+1,j}+2),f_i} \\ \vdots & \vdots & & \vdots \\ x_{\sigma_j(a_{i,j}),e_{i+1}} & x_{\sigma_j(a_{i,j}),e_{i+2}} & \cdots & x_{\sigma_j(a_{i,j}),f_i} \end{pmatrix}.$$

Note that $\hat{\mathcal{X}}_i(\boldsymbol{\sigma})$ is obtained from the matrix $\mathcal{X}_i(\boldsymbol{\sigma})$ by removing its first e_i columns. It is now clear from the expression of $\zeta_{(E,T)}$ given in (4.10) and Lemma 4.7.1 that

$$\zeta_{(E,T)}[\xi_E] = \pm \sum_{\boldsymbol{\sigma} \in S_{d_1} \times \cdots \times S_{d_r}} \left(\prod_{i=1}^{r} \operatorname{sgn}(\sigma_i) \right) \prod_{j=1}^{t} \det \hat{\mathcal{X}}_j(\boldsymbol{\sigma}). \tag{4.15}$$

We now need some results of [HTW05b] and [HL06b]. In the paper [HTW05b], a polynomial $\delta_T(Y)$ in the variables $Y = (y_{ij})$ is defined in (3.4). An ordering on the set of monomials in the variables y_{ij} is defined in § 3.4.1, and the leading monomial of $\delta_T(Y)$ with respect to this monomial ordering is given in Lemma 3.2. On the other hand, an expansion formula for $\delta_T(Y)$ is given in [HL06b, Lemma 5.7.2]. We observe that if we identify x_{ij} with the variable y_{ji}, then a comparison of (4.15) and the formula given in [HL06b, Lemma 5.7.2] reveals that $\zeta_{(E,T)}[\xi_E]$ coincides with a multiple of $\delta_T(Y)$. In addition, under this identification, the monomial orderings defined in [HTW05b] and τ_1 are identical. Consequently, the leading monomial of $\zeta_{(E,T)}[\xi_E]$ is the image of the leading monomial of $\delta_T(Y)$. Thus, the lemma follows. □

4.8 A basis for $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$

We are now ready to state and prove the main theorem of this section.

Theorem 4.8.1. *The set*

$$\bigcup_{r(D),r(F) \leqslant k} \{\zeta_{(E,T)} : (E,T) \in \Omega(D,F)\}$$

is a basis for $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}}$. *Thus, its image in* $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$ *is a basis for* $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{GL}_k}}$.

Proof. By Lemma 4.7.2, the leading monomial of $\zeta_{(E,T)}$ is $x_T\xi_E$. Now ξ_E determines the diagram E, x_T determines the diagram D, and ξ_E and x_T together determine the diagram F and the tableau T. This shows that $(D,F,E,T) \to x_T\xi_E$ is one-to-one. Since the highest weight vectors $\zeta_{(E,T)}$ have distinct leading monomials, they form a linearly independent set. Moreover, for fixed D and F, the number of the highest weight vectors we obtain in this way is equal to the dimension of the $\psi_{2n}^D \times \psi_k^F$-eigenspace $\widetilde{W}_{D,F}$ of $A_{\mathrm{GL}_{2n}} \times A_{\mathrm{GL}_k}$ in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}}$. Hence, they form a basis for this eigenspace. It follows that by varying the diagrams D and F we obtain a basis for $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_k}}$. □

5. A basis for the stable branching algebras for $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$

In this section, we construct a basis for the stable branching algebra for the symmetric pair $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$. Since the construction is similar to the case of $(\mathrm{O}_{n+m}, \mathrm{O}_n \times \mathrm{O}_m)$ which has been done in [HL06b], we omit most of the details.

5.1 The subgroup $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ of $\mathrm{Sp}_{2(n+m)}$

Recall that $\mathrm{Sp}_{2(n+m)}$ is the subgroup of GL_{2n+2m} which leaves the symplectic form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{2n+2m}$ invariant, where for $\mathbf{a} = (a_1, \ldots, a_{2n+2m}), \mathbf{b} = (b_1, \ldots, b_{2n+2m}) \in \mathbb{C}^{2n+2m}$,

$$\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{i=1}^{n+m} (a_i b_{n+m+i} - a_{n+m+i} b_i).$$

Let $\{\varepsilon_1, \ldots, \varepsilon_{2n+2m}\}$ be the standard basis for $\mathbb{C}^{2n+2m}$. Let W_1 and W_2 be the subspaces of $\mathbb{C}^{2n+2m}$ spanned by $\{\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+m+1}, \ldots, \varepsilon_{2n+m}\}$ and by $\{\varepsilon_{n+1}, \ldots, \varepsilon_{n+m}, \varepsilon_{2n+m+1}, \ldots, \varepsilon_{2n+2m}\}$, respectively. Then the subgroup of all elements in $\mathrm{Sp}_{2(n+m)}$ which leave both W_1 and W_2 stable is isomorphic to $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$. We abuse notation and denote this subgroup also by $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$.

5.2 The algebra $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}}$

Let $k \leqslant \min(n, m)$ and let $\mathrm{M}_{2(n+m),k} = \mathrm{M}_{2(n+m),k}(\mathbb{C})$ be the space of all $2(n+m) \times k$ complex matrices. For $1 \leqslant i, j \leqslant k$ and $T \in \mathrm{M}_{2(n+m),k}$, we let

$$\xi_{ij}(T) = \langle T_i, T_j\rangle,$$

where T_i and T_j are the ith and the jth column of T, respectively. Let

$$\mathcal{V} = \{T \in \mathrm{M}_{2(n+m),k} : \xi_{ij}(T) = 0 \ \forall 1 \leqslant i, j \leqslant k\},$$

and let $\mathcal{R}(\mathcal{V})$ be the algebra of regular functions on $\mathcal{V}$. Let $\mathrm{Sp}_{2(n+m)} \times \mathrm{GL}_k$ act on $\mathcal{R}(\mathcal{V})$ by

$$[(g,h).f](T) = f(g^{\mathrm{t}}\, Th), \quad g \in \mathrm{Sp}_{2(n+m)},\ h \in \mathrm{GL}_k,\ f \in \mathcal{R}(\mathcal{V}),\ T \in \mathcal{V}.$$

Then as a $\mathrm{Sp}_{2(n+m)} \times \mathrm{GL}_k$ module [How95],

$$\mathcal{R}(\mathcal{V}) \cong \sum_{r(H) \leqslant k} \tau^H_{2(n+m)} \otimes \rho^H_k.$$

Let U_{GL_k}, $U_{\mathrm{Sp}_{2n}}$ and $U_{\mathrm{Sp}_{2m}}$ be the maximal unipotent subgroups of GL_k, Sp_{2n} and Sp_{2m}, respectively. Taking U_{GL_k} invariants in $\mathcal{R}(\mathcal{V})$ gives

$$\mathcal{R}(\mathcal{V})^{U_{\mathrm{GL}_k}} \cong \sum_{r(H) \leqslant k} \tau^H_{2(n+m)} \otimes (\rho^H_k)^{U_{\mathrm{GL}_k}}.$$

This algebra contains one copy of each irreducible representation τ^H_{2n+2m} of $\mathrm{Sp}_{2(n+m)}$ with $r(H) \leqslant k$. We consider the subalgebra of $U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}}$ invariants in $\mathcal{R}(\mathcal{V})^{U_{\mathrm{GL}_k}}$, which is

$$\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}} \cong \sum_{r(H) \leqslant k} (\tau^H_{2(n+m)})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}}} \otimes (\rho^H_k)^{U_{\mathrm{GL}_k}}.$$

This algebra encodes all information on the branching rule from $\mathrm{Sp}_{2(n+m)}$ to $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ for all representations $\tau^H_{2(n+m)}$ with $r(H) \leqslant k$. In view of this property, we call $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}}$ a *stable branching algebra* for $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$ (see [HTW04]). The goal of this section is to construct a basis for this algebra.

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

5.3 The algebra $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_{2m}} \times U_{\mathrm{GL}_k}}$

As in the case of $(\mathrm{O}_{n+m}, \mathrm{O}_n \times \mathrm{O}_m)$, we can replace the stable branching algebra $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}}$ by a subalgebra of a polynomial algebra which we now describe. Let $\mathrm{GL}_{2n} \times \mathrm{GL}_{2m} \times \mathrm{GL}_k$ act on $\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k$ by

$$(g_1, g_2, h)(X, Y, N) = ((g_1^{-1})^{\mathrm{t}} X h^{-1}, (g_2^{-1})^{\mathrm{t}} Y h^{-1}, (h^{-1})^{\mathrm{t}} N h^{-1})$$

where $(g_1, g_2, h) \in \mathrm{GL}_{2n} \times \mathrm{GL}_{2m} \times \mathrm{GL}_k$, $X \in \mathrm{M}_{2n,k}$, $Y \in \mathrm{M}_{2m,k}$ and $N \in \mathrm{AM}_k$. This induces an action of $\mathrm{GL}_{2n} \times \mathrm{GL}_{2m} \times \mathrm{GL}_k$ on the polynomial algebra $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)$. We consider the subalgebra

$$\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_{2m}} \times U_{\mathrm{GL}_k}}$$

of $U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_{2m}} \times U_{\mathrm{GL}_k}$-invariants in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)$. This algebra is isomorphic to $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}}$. We now describe an isomorphism

$$\alpha : \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_{2m}} \times U_{\mathrm{GL}_k}} \to \mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}}.$$

Recall that, in § 5.1, we defined the symplectic form $\langle \cdot, \cdot \rangle$ and the subspaces W_1 and W_2 of $\mathbb{C}^{2n+2m}$. Note that $\mathbb{C}^{2n+2m} = W_1 \oplus W_2$, and for $i = 1, 2$, we let $\pi_i : \mathbb{C}^{2n+2m} \to W_i$ be the corresponding linear projection. Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be the restrictions of $\langle \cdot, \cdot \rangle$ to W_1 and W_2, respectively. For $1 \leqslant i, j \leqslant k$ and $T \in \mathrm{M}_{2(n+m),k}$, let

$$\nu_{ij}(T) = \langle \pi_1(T_i), \pi_1(T_j) \rangle_1 - \langle \pi_2(T_i), \pi_2(T_j) \rangle_2$$

where T_i and T_j are the ith and the jth column of T. Now if

$$f \in \mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_{2m}} \times U_{\mathrm{GL}_k}}$$

and $T \in \mathcal{V}$, then αf is the function on $\mathcal{V}$ defined by

$$(\alpha f)(T) = f(X, Y, N) \tag{5.1}$$

where

$$T = \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathrm{M}_{2n,k}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathrm{M}_{2m,k}, \quad \text{and} \quad N = (\nu_{ij}(T)).$$

5.4 An index set for the basis

We shall construct a basis for the algebra

$$\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_{2m}} \times U_{\mathrm{GL}_k}}.$$

Then its image under the isomorphism α given in (5.1) will be a basis for $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}}$. First we note that under the action by $A_{\mathrm{GL}_{2n}} \times A_{\mathrm{GL}_{2m}} \times A_{\mathrm{GL}_k}$,

$$\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}} \times U_{\mathrm{GL}_{2m}} \times U_{\mathrm{GL}_k}} = \bigoplus_{r(D), r(E), r(H) \leqslant k} \widetilde{W}_{D,E,H},$$

where $\widetilde{W}_{D,E,H}$ is the $\psi_{2n}^D \times \psi_{2m}^E \times \psi_k^H$-eigenspace of $A_{\mathrm{GL}_{2n}} \times A_{\mathrm{GL}_{2m}} \times A_{\mathrm{GL}_k}$. It is clear that the image of $\widetilde{W}_{D,E,H}$ under α is precisely the $\chi_{2n}^D \times \chi_{2m}^E \times \psi_k^H$-eigenspace $W_{D,E,H}$ of $A_{\mathrm{Sp}_{2n}} \times A_{\mathrm{Sp}_{2m}} \times A_{\mathrm{GL}_k}$ in $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_{\mathrm{GL}_k}}$. Since the elements in $W_{D,E,H}$ can be identified with the $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ highest weight vectors in τ_{2n+2m}^H with weight $\chi_{2n}^D \times \chi_{2m}^E$, the dimension of $W_{D,E,H}$ is equal to the multiplicity of the $\tau_{2n}^D \otimes \tau_{2m}^E$ in τ_{2n+2m}^H. This is given by (see [HTW05a])

$$\dim \widetilde{W}_{D,E,H} = \dim W_{D,E,H} = \sum_{F \in \mathcal{E}_k, r(G) \leqslant k} c_{F,D}^G c_{G,E}^H.$$

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

We now let

$$\Omega(D,E,H) \tag{5.2}$$

be the set of all ordered 4-tuples (F,G,T_1,T_2) such that $F\in\mathcal{E}_k$, G is a Young diagram with at most k rows, and T_1 (respectively T_2) is a LR tableaux of shape G/F (respectively H/G) with content D (respectively E). Note that the number of elements in $\Omega(D,E,H)$ coincides with $\dim\widetilde{W}_{D,E,H}$, so we will use it to index a basis for $\widetilde{W}_{D,E,H}$.

5.5 Highest weight vectors in $\widetilde{W}_{D,E,H}$

We now fix $(F,G,T_1,T_2)\in\Omega(D,E,H)$ and construct a $\mathrm{GL}_{2n}\times\mathrm{GL}_{2m}\times\mathrm{GL}_k$ highest weight vector $\omega_{(F,G,T_1,T_2)}$ of weight $\psi_{2n}^{D}\times\psi_{2m}^{E}\times\psi_k^{H}$.

First, suppose that the conjugate diagrams of D, E, F, G and H are given by

$$\begin{aligned}D^{\mathrm{t}}&=(d_1,\dots,d_r)\\E^{\mathrm{t}}&=(e_1,\dots,e_s)\\F^{\mathrm{t}}&=(f_1,\dots,f_t)\\G^{\mathrm{t}}&=(g_1,\dots,g_u)\\H^{\mathrm{t}}&=(h_1,\dots,h_v).\end{aligned}$$

Consider the reverse process of standard peeling in the construction of T_1 and T_2 (see § 2.2). Let $M(T_1)=(m_{ij}^{(1)})$ and $M(T_2)=(m_{ij}^{(2)})$ where:

- $m_{ij}^{(1)}$ is the number of elements from the jth column of D get put into the ith column of G;
- $m_{ij}^{(2)}$ is the number of elements from the jth column of E get put into the ith column of H.

For $1\leqslant i\leqslant u$ and $1\leqslant j\leqslant r$, let

$$a_{ij}^{(1)}=m_{ij}^{(1)}+m_{i+1,j}^{(1)}+\dots+m_{uj}^{(1)},$$

and

$$a_{u+1,j}^{(1)}=0,$$

and, for $1\leqslant i\leqslant v$ and $1\leqslant j\leqslant s$, let

$$a_{ij}^{(2)}=m_{ij}^{(2)}+m_{i+1,j}^{(2)}+\dots+m_{vj}^{(2)},$$

and

$$a_{v+1,j}^{(2)}=0,$$

Then:

- the entries from the jth column of D that get put into the ith column of G/F are
$$a_{i+1,j}^{(1)}+1,a_{i+1,j}^{(1)}+2,\dots,a_{i,j}^{(1)};$$
- the entries from the jth column of E that get put into the ith column of H/G are
$$a_{i+1,j}^{(2)}+1,a_{i+1,j}^{(2)}+2,\dots,a_{i,j}^{(2)}.$$

We are now ready to define the highest weight vector $\omega_{(F,G,T_1,T_2)}$. Write a typical element $\mathrm{M}_{2n,k}\oplus\mathrm{M}_{2m,k}\oplus\mathrm{AM}_k$ as (X,Y,N) where $X=(x_{ij})$, $Y=(y_{ij})$ and $N=(\nu_{ij})$. Let $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_r)\in S_{d_1}\times\dots\times S_{d_r}$ and $\boldsymbol{\tau}=(\tau_1,\dots,\tau_s)\in S_{e_1}\times\dots\times S_{e_s}$. For each $1\leqslant i\leqslant u$, let

$$p_i=g_i-f_i$$

and $\mathcal{X}_i(\boldsymbol{\sigma})$ be the $p_i \times h_i$ matrix

$$\mathcal{X}_i(\boldsymbol{\sigma}) = \begin{pmatrix} \mathcal{X}_{i1}(\sigma_1) \\ \mathcal{X}_{i2}(\sigma_2) \\ \vdots \\ \mathcal{X}_{ir}(\sigma_r) \end{pmatrix},$$

where, for each $1 \leqslant j \leqslant r$,

$$\mathcal{X}_{ij}(\sigma_j) = \begin{pmatrix} x_{\sigma_j(a^{(1)}_{i+1,j}+1),1} & x_{\sigma_j(a^{(1)}_{i+1,j}+1),2} & \cdots & x_{\sigma_j(a^{(1)}_{i+1,j}+1),h_i} \\ x_{\sigma_j(a^{(1)}_{i+1,j}+2),1} & x_{\sigma_j(a^{(1)}_{i+1,j}+2),2} & \cdots & x_{\sigma_j(a^{(1)}_{i+1,j}+2),h_i} \\ \vdots & \vdots & & \vdots \\ x_{\sigma_j(a^{(1)}_{i,j}),1} & x_{\sigma_j(a^{(1)}_{i,j}),2} & \cdots & x_{\sigma_j(a^{(1)}_{i,j}),h_i} \end{pmatrix},$$

and, for $1 \leqslant i \leqslant v$, let

$$q_i = h_i - g_i$$

and $\mathcal{Y}_i(\boldsymbol{\tau})$ be the $q_i \times h_i$ matrix

$$\mathcal{Y}_i(\boldsymbol{\tau}) = \begin{pmatrix} \mathcal{Y}_{i1}(\tau_1) \\ \mathcal{Y}_{i2}(\tau_2) \\ \vdots \\ \mathcal{Y}_{is}(\tau_s) \end{pmatrix}$$

where, for each $1 \leqslant l \leqslant s$,

$$\mathcal{Y}_{il}(\tau_l) = \begin{pmatrix} y_{\tau_l(a^{(2)}_{i+1,l}+1),1} & y_{\tau_l(a^{(2)}_{i+1,l}+1),2} & \cdots & y_{\tau_l(a^{(2)}_{i+1,l}+1),h_i} \\ y_{\tau_l(a^{(2)}_{i+1,l}+2),1} & y_{\tau_l(a^{(2)}_{i+1,l}+2),2} & \cdots & y_{\tau_l(a^{(2)}_{i+1,l}+2),h_i} \\ \vdots & \vdots & & \vdots \\ y_{\tau_l(a^{(2)}_{i,l}),1} & y_{\tau_l(a^{(2)}_{i,l}),2} & \cdots & y_{\tau_l(a^{(2)}_{i,l}),h_i} \end{pmatrix}.$$

We also let, for $1 \leqslant m \leqslant k$,

$$[\nu]_m = \begin{pmatrix} \nu_{11} & \nu_{12} & \cdots & \nu_{1m} \\ \nu_{21} & \nu_{22} & \cdots & \nu_{2m} \\ \vdots & \vdots & & \vdots \\ \nu_{m1} & \nu_{m2} & \cdots & \nu_{m,m} \end{pmatrix}.$$

Finally, we let

$$\omega_{(F,G,T_1,T_2)} = \sum_{\substack{\boldsymbol{\sigma}\in S_{d_1}\times\cdots\times S_{d_r} \\ \boldsymbol{\tau}\in S_{e_1}\times\cdots\times S_{e_s}}} \left(\prod_{i=1}^{r} \operatorname{sgn}(\sigma_i)\right)\left(\prod_{j=1}^{s} \operatorname{sgn}(\tau_j)\right) \times \left[\prod_{i=1}^{v} \operatorname{Pfaff}\left(\begin{array}{c|c|c} [\nu]_{h_i} & [\mathcal{X}_i(\boldsymbol{\sigma})]^{\mathrm{t}} & [\mathcal{Y}_i(\boldsymbol{\tau})]^{\mathrm{t}} \\ \hline -\mathcal{X}_i(\boldsymbol{\sigma}) & 0 & 0 \\ \hline -\mathcal{Y}_i(\boldsymbol{\sigma}) & 0 & 0 \end{array}\right)\right]. \tag{5.3}$$

This is a $\mathrm{GL}_{2n} \times \mathrm{GL}_{2m} \times \mathrm{GL}_k$ highest weight vector in $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)$ of weight $\psi^D_{2n} \times \psi^E_{2m} \times \psi^H_k$.

5.6 The leading monomial of $\omega_{(F,G,T_1,T_2)}$

Recall that typical elements of $\mathrm{M}_{2n,k}$, $\mathrm{M}_{2m,k}$ and AM_k are written $X = (x_{ij})$, $Y = (y_{ij})$ and $N = (\nu_{ij})$, respectively. We define a monomial ordering τ_2 on $\mathcal{P}(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k)$ as follows:

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

it is the graded lexicographic order [CLO97] such that

$$\begin{aligned}\nu_{12} > \nu_{13} > \cdots > \nu_{1k} > \nu_{23} > \nu_{24} > \cdots > \nu_{k-1,k}\\ > x_{11} > x_{12} > \cdots > x_{1k} > x_{21} > \cdots > x_{2n,k}\\ > y_{11} > y_{12} > \cdots > y_{1k} > y_{21} > \cdots > y_{2m,k}.\end{aligned} \tag{5.4}$$

Lemma 5.6.1. *The leading monomial of $\omega_{(F,G,T_1,T_2)}$ with respect to τ_2 is given by*

$$\biggl(\prod_{\mathbf{b}_1\in T_1} x_{c_1(\mathbf{b}_1)a_1(\mathbf{b}_1)}\biggr)\biggl(\prod_{\mathbf{b}_2\in T_2} y_{c_2(\mathbf{b}_2)a_2(\mathbf{b}_2)}\biggr)\biggl(\prod_{j=1}^{t}\prod_{i=1}^{f_j/2}\nu_{2i-1,2i}\biggr)$$

where:

(i) *for each box $\mathbf{b}_1$ in T_1, $a_1(\mathbf{b}_1)$ is the row of G in which the box $\mathbf{b}_1$ lies and $c_1(\mathbf{b}_1)$ is the entry in $\mathbf{b}_1$; and*

(ii) *for each box $\mathbf{b}_2$ in T_2, $a_2(\mathbf{b}_2)$ is the row of H in which the box $\mathbf{b}_2$ lies and $c_2(\mathbf{b}_2)$ is the entry in $\mathbf{b}_2$.*

The proof, which we omit, is similar to the proof of Lemma 8.8.1 in [HL06b]. An immediate consequence of this lemma is that the leading monomials of the $\omega_{(F,G,T_1,T_2)}$ are all distinct. In particular, it implies the following.

Theorem 5.6.2. *The set*

$$\bigcup_{r(D),r(E),r(H)\leqslant k}\{\omega_{(F,G,T_1,T_2)} : (F,G,T_1,T_2)\in\Omega(D,E,H)\}$$

is a basis for $\mathcal{P}(\mathrm{M}_{2n,k}\oplus\mathrm{M}_{2m,k}\oplus\mathrm{AM}_k)^{U_{\mathrm{GL}_{2n}}\times U_{\mathrm{GL}_{2m}}\times U_{\mathrm{GL}_k}}$. Thus, its image under α forms a basis for $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}}\times U_{\mathrm{Sp}_{2m}}\times U_{\mathrm{GL}_k}}$.

6. A basis for the stable branching algebras for $(\mathbf{O}_{2n},\mathbf{GL}_n)$

In this section, we construct a basis for the stable branching algebra for the symmetric pair $(\mathrm{O}_{2n},\mathrm{GL}_n)$. Since the construction is very similar to the case of $(\mathrm{Sp}_{2n},\mathrm{GL}_n)$ which has been done in [HL06b], we omit most of the details.

6.1 The even orthogonal group

Let $(\cdot,\cdot)$ be the symmetric bilinear form on $\mathbb{C}^{2n}$ given by

$$(u,v)=\sum_{i=1}^{2n}u_iv_{2n-i+1},\quad u=(u_1,\ldots,u_{2n}),\ v=(v_1,\ldots,v_{2n})\in\mathbb{C}^{2n}.$$

Let

$$\mathrm{O}_{2n}=\mathrm{O}_{2n}(\mathbb{C})=\{g\in\mathrm{GL}_{2n} : (gu,gv)=(u,v),\ \forall u,v\in\mathbb{C}^{2n}\}.$$

Then

$$\left\{\begin{pmatrix}a & 0\\ 0 & J(a^{-1})^{\mathrm{t}}J\end{pmatrix} : a\in\mathrm{GL}_n\right\}$$

is a subgroup of O_{2n} and it is isomorphic to GL_n. Here J is the $n\times n$ matrix with 1 on its antidiagonal and 0 elsewhere. In this section, we abuse notation and denote this subgroup also by GL_n.

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

6.2 The algebra $\mathcal{R}(\mathcal{N})^{U_{\mathrm{GL}_n}\times U_{\mathrm{GL}_k}}$

Let $2k \leqslant n$. For $1 \leqslant i, j \leqslant k$ and $T \in \mathrm{M}_{2n,k}$, we let

$$r^2_{ij}(T) = (T_i, T_j),$$

where T_i and T_j are the ith and the jth column of T, respectively. Let

$$\mathcal{N} = \{T \in \mathrm{M}_{2n,k} :\ r^2_{ij}(T) = 0\ \forall i, j\}.$$

and $\mathcal{R}(\mathcal{N})$ be the algebra of regular functions on $\mathcal{N}$. Let $\mathrm{O}_{2n} \times \mathrm{GL}_k$ act on $\mathcal{R}(\mathcal{N})$ by

$$[(g, h)f](T) = f(g^{\mathrm{t}}Th), \quad g \in \mathrm{O}_{2n},\ h \in \mathrm{GL}_k,\ f \in \mathcal{R}(\mathcal{N}),\ T \in \mathcal{N}.$$

Then as an $\mathrm{O}_{2n} \times \mathrm{GL}_k$ module [How95],

$$\mathcal{R}(\mathcal{N}) \cong \sum_{r(H)\leqslant k} \sigma^H_{2n} \otimes \rho^H_k.$$

Taking U_{GL_k} invariants gives

$$\mathcal{R}(\mathcal{N})^{U_{\mathrm{GL}_k}} \cong \sum_{r(H)\leqslant k} \sigma^H_{2n} \otimes (\rho^H_k)^{U_{\mathrm{GL}_k}}.$$

So this algebra contains one copy of each irreducible representation σ^H_{2n} of O_{2n} with $r(H) \leqslant k$. We consider its subalgebra of U_{GL_n} invariants

$$\mathcal{R}(\mathcal{N})^{U_{\mathrm{GL}_n}\times U_{\mathrm{GL}_k}} \cong \sum_{r(H)\leqslant k} (\sigma^H_{2n})^{U_{\mathrm{GL}_n}} \otimes (\rho^H_k)^{U_{\mathrm{GL}_k}}.$$

This algebra encodes information on the branching rule from O_{2n} to GL_n for all representations σ^H_{2n} with $r(H) \leqslant k$. In view of this property, we call $\mathcal{R}(\mathcal{N})^{U_{\mathrm{GL}_n}\times U_{\mathrm{GL}_k}}$ a *stable branching algebra* for $(\mathrm{O}_{2n}, \mathrm{GL}_n)$ (see [HTW04]).

6.3 An index set for the basis

The algebra $\mathcal{R}(\mathcal{N})^{U_{\mathrm{GL}_n}\times U_{\mathrm{GL}_k}}$ is an $A_{\mathrm{GL}_n} \times A_{\mathrm{GL}_k}$ module, and

$$\mathcal{R}(\mathcal{N})^{U_{\mathrm{GL}_n}\times U_{\mathrm{GL}_k}} = \bigoplus_{r(D),r(E),r(H)\leqslant k} W_{(D,E),H}.$$

Here for Young diagrams D, E and H, all with at most k rows, $W_{(D,E),H}$ is the $\psi^{D,E}_n \times \psi^H_k$-eigenspace of $A_{\mathrm{GL}_n} \times A_{\mathrm{GL}_k}$. Its dimension is equal to the multiplicity of the representation $\rho^{D,E}_n$ of GL_n in the representation σ^H_{2n} of O_{2n}, which is given by [HTW05a]

$$\sum_{F\in\mathcal{E}_k, r(G)\leqslant k} c^G_{F,D} c^H_{G,E}.$$

Let $\Omega(D, E, H)$ be the set defined in (5.2). Since the number of elements in $\Omega(D, E, H)$ coincides with $\dim W_{(D,E),H}$, we use it to index a basis for $W_{(D,E),H}$.

6.4 The highest weight vector $\eta_{(F,G,T_1,T_2)}$

In this subsection, we fix $(F, G, T_1, T_2) \in \Omega(D, E, H)$ and construct a $\mathrm{GL}_n \times \mathrm{GL}_k$ highest weight vector $\eta_{(F,G,T_1,T_2)}$ in the polynomial algebra $\mathcal{P}(\mathrm{M}_{2n,k})$. The restrictions of all such highest weight vectors to $\mathcal{N}$ form a basis for $W_{(D,E),H}$.

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

Let the standard coordinates on $\mathrm{M}_{2n,k}$ be given by

$$\begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1k} \\ \vdots & \vdots & & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nk} \\ w_{11} & w_{12} & \cdots & w_{1k} \\ \vdots & \vdots & & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nk} \end{pmatrix}.$$

For $1 \leqslant a, b \leqslant k$, we also let

$$\mu_{ab} = \sum_{i=1}^{n} (z_{ia} w_{n-i+1,b} - z_{ib} w_{n-i+1,a}).$$

We shall also use the notation related to D, E, F, G, H, T_1 and T_2 defined in § 5.5. Let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r) \in S_{d_1} \times \cdots \times S_{d_r}$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_s) \in S_{e_1} \times \cdots \times S_{e_s}$. For each $1 \leqslant i \leqslant u$, let

$$p_i = g_i - f_i,$$

and $Z_i(\boldsymbol{\sigma})$ be the $p_i \times h_i$ matrix

$$Z_i(\boldsymbol{\sigma}) = \begin{pmatrix} Z_{i1}(\sigma_1) \\ Z_{i2}(\sigma_2) \\ \vdots \\ Z_{ir}(\sigma_r) \end{pmatrix},$$

where, for each $1 \leqslant j \leqslant r$,

$$Z_{ij}(\sigma_j) = \begin{pmatrix} z_{\sigma_j(a^{(1)}_{i+1,j}+1),1} & z_{\sigma_j(a^{(1)}_{i+1,j}+1),2} & \cdots & z_{\sigma_j(a^{(1)}_{i+1,j}+1),h_i} \\ z_{\sigma_j(a^{(1)}_{i+1,j}+2),1} & z_{\sigma_j(a^{(1)}_{i+1,j}+2),2} & \cdots & z_{\sigma_j(a^{(1)}_{i+1,j}+2),h_i} \\ \vdots & \vdots & & \vdots \\ z_{\sigma_j(a^{(1)}_{i,j}),1} & z_{\sigma_j(a^{(1)}_{i,j}),2} & \cdots & z_{\sigma_j(a^{(1)}_{i,j}),h_i} \end{pmatrix},$$

and, for $1 \leqslant i \leqslant v$, let

$$q_i = h_i - g_i$$

and let $W_i(\boldsymbol{\tau})$ be the $q_i \times h_i$ matrix

$$W_i(\boldsymbol{\tau}) = \begin{pmatrix} W_{i1}(\tau_1) \\ W_{i2}(\tau_2) \\ \vdots \\ W_{is}(\tau_s) \end{pmatrix},$$

where, for each $1 \leqslant l \leqslant s$, $W_{il}(\tau_l)$ is the $m^{(2)}_{il} \times h_i$ matrix

$$W_{il}(\tau_l) = \begin{pmatrix} w_{\tau_l(a^{(2)}_{i+1,l}+1),1} & w_{\tau_l(a^{(2)}_{i+1,l}+1),2} & \cdots & w_{\tau_l(a^{(2)}_{i+1,l}+1),h_i} \\ w_{\tau_l(a^{(2)}_{i+1,l}+2),1} & w_{\tau_l(a^{(2)}_{i+1,l}+2),2} & \cdots & w_{\tau_l(a^{(2)}_{i+1,l}+2),h_i} \\ \vdots & \vdots & & \vdots \\ w_{\tau_l(a^{(2)}_{i,l}),1} & w_{\tau_l(a^{(2)}_{i,l}),2} & \cdots & w_{\tau_l(a^{(2)}_{i,l}),h_i} \end{pmatrix}.$$

https://doi.org/10.1112/S0010437X06002399 Published online by Cambridge University Press

We also let, for $1 \leqslant m \leqslant k$,

$$[\mu]_m = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1m} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2m} \\ \vdots & \vdots & & \vdots \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mm} \end{pmatrix}.$$

Finally, we let

$$\eta_{(F,G,T_1,T_2)} = \sum_{\substack{\boldsymbol{\sigma} \in S_{d_1} \times \cdots \times S_{d_r} \\ \boldsymbol{\tau} \in S_{e_1} \times \cdots \times S_{e_s}}} \left(\prod_{i=1}^{r} \operatorname{sgn}(\sigma_i) \right) \left(\prod_{j=1}^{s} \operatorname{sgn}(\tau_j) \right) \times \left[\prod_{i=1}^{v} \operatorname{Pfaff} \left(\begin{array}{c|c|c} [\mu]_{h_i} & [Z_i(\boldsymbol{\sigma})]^{\mathrm{t}} & [W_i(\boldsymbol{\tau})]^{\mathrm{t}} \\ \hline -Z_i(\boldsymbol{\sigma}) & 0 & 0 \\ \hline -W_i(\boldsymbol{\sigma}) & 0 & 0 \end{array} \right) \right]. \tag{6.1}$$

This is a $\mathrm{GL}_n \times \mathrm{GL}_k$ highest weight vector of weight $\psi_n^{D,E} \times \psi_k^H$.

6.5 The leading monomial of $\eta_{(F,G,T_1,T_2)}$

Let

$$S = \{z_{ij}, w_{ij} : \ 1 \leqslant i, j \leqslant k\} \cup \{\mu_{ij} : 1 \leqslant i < j \leqslant k\}$$

and $V = \operatorname{Span}(S)$. Then S is an algebraically independent subset of $\mathcal{P}(\mathrm{M}_{2n,k})$, so the subalgebra of $\mathcal{P}(\mathrm{M}_{2n,k})$ generated by S is a polynomial algebra. We denote this subalgebra by $\mathcal{P}(V)$. We observe that the highest weight vector $\eta_{(F,G,T_1,T_2)}$ belongs to this subalgebra $\mathcal{P}(V)$. In this subsection, we introduce a monomial ordering τ_3 in $\mathcal{P}(V)$ and compute the leading monomial of the highest weight vector $\eta_{(F,G,T_1,T_2)}$ with respect to this ordering.

The monomial ordering τ_3 is defined to be the graded lexicographic order [CLO97] such that

$$\begin{aligned} \mu_{12} > \mu_{13} > \cdots > \mu_{1k} > \mu_{23} > \mu_{24} > \cdots > \mu_{k-1,k} \\ > z_{11} > z_{12} > \cdots > z_{1k} > z_{21} > \cdots > z_{kk} \\ > w_{11} > w_{12} > \cdots > w_{1k} > w_{21} > \cdots > w_{kk}. \end{aligned} \tag{6.2}$$

LEMMA 6.5.1. *The leading monomial of $\eta_{(F,G,T_1,T_2)}$ with respect to τ_3 is given by*

$$\left(\prod_{\mathbf{b}_1 \in T_1} z_{c_1(\mathbf{b}_1) a_1(\mathbf{b}_1)} \right) \left(\prod_{\mathbf{b}_2 \in T_2} w_{c_2(\mathbf{b}_2) a_2(\mathbf{b}_2)} \right) \left(\prod_{j=1}^{t} \prod_{i=1}^{f_j/2} \mu_{2i-1,2i} \right)$$

where:

(i) *for each box $\mathbf{b}_1$ in T_1, $a_1(\mathbf{b}_1)$ is the row of G in which the box $\mathbf{b}_1$ lies and $c_1(\mathbf{b}_1)$ is the entry in $\mathbf{b}_1$; and*

(ii) *for each box $\mathbf{b}_2$ in T_2, $a_2(\mathbf{b}_2)$ is the row of H in which the box $\mathbf{b}_2$ lies and $c_2(\mathbf{b}_2)$ is the entry in $\mathbf{b}_2$.*

The proof is similar to the proof of Lemma 9.4.1 in [HL06b]. We note that the leading monomials of the $\eta_{(F,G,T_1,T_2)}$ are all distinct, so they form a linearly independent set.

THEOREM 6.5.2. *Let $\tilde{\eta}_{(F,G,T_1,T_2)}$ denote the restriction of $\eta_{(F,G,T_1,T_2)}$ to $\mathcal{N}$. Then the set*

$$\bigcup_{r(D), r(E), r(H) \leqslant k} \{\tilde{\eta}_{(F,G,T_1,T_2)} : \ (F, G, T_1, T_2) \in \Omega(D, E, H)\}$$

is a basis for $\mathcal{R}(\mathcal{N})^{U_{\mathrm{GL}_n} \times U_{\mathrm{GL}_k}}$.

References

CLO97 D. Cox, J. Little and D. O'Shea, *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra*, second edition, Undergraduate Texts in Mathematics, vol. 13 (Springer, New York, 1997).

Fu97 W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35 (Cambridge University Press, Cambridge, 1997).

GW98 R. Goodman and N. R. Wallach, *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, vol. 68 (Cambridge University Press, Cambridge, 1998).

How95 R. Howe, *Perspectives on invariant theory*, in *The Schur Lectures*, eds I. Piatetski-Shapiro and S. Gelbart, Israel Mathematical Conference Proceedings, vol. 8 (Bar-Ilan University, Ramat Gan, 1995), 1–182.

HJLTW06 R. Howe, S. Jackson, S. Lee, E.-C. Tan and J. Willenbring, *Toric degeneration of branching algebras*, Preprint (2006).

HL06a R. Howe and S. Lee, *Bases for some reciprocity algebras I*, Trans. Amer. Math. Soc., to appear.

HL06b R. Howe and S. Lee, *Bases for some reciprocity algebras II*, Adv. Math. **206** (2006), 145–210.

HTW04 R. Howe, E.-C. Tan and J. Willenbring, *Reciprocity algebras and branching for classical symmetric pairs*, Preprint (2004), math.RT/0407467.

HTW05a R. Howe, E.-C. Tan and J. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2005), 1601–1626.

HTW05b R. Howe, E.-C. Tan and J. Willenbring, *A basis for the* GL_n *tensor product algebra*, Adv. Math. **196** (2005), 531–564.

RS90 L. Rubbiano and M. Sweedler, *Subalgebra bases*, in *Commutative algebra*, Salvador, 1988, Lecture Notes in Mathematics, vol. 1430 (Springer, Berlin, 1990), 61–87.

Stu96 B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8 (American Mathematical Society, Providence, RI, 1996).

Roger Howe howe@math.yale.edu
Department of Mathematics, Yale University, New Haven, CT 06520-8283, USA

Soo Teck Lee matleest@nus.edu.sg
Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore