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CONDITIONS FOR THE UNIQUENESS OF THE FIXED POINT IN KAKUTANI'S THEOREM

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ABSTRACT. Kakutani's Theorem states that every point convex and usc multifunction ϕ defined on a compact and convex set in a Euclidean space has at least one fixed point. Some necessary conditions are given here which ϕ must satisfy if c is the unique fixed point of ϕ. It is e.g. shown that if the width of $\phi(c)$ is greater than zero, then ϕ cannot be lsc at c, and if in addition c lies on the boundary of $\phi(c)$, then there exists a sequence $\{x_k\}$ which converges to c and for which the width of the sets $\phi(x_k)$ converges to zero. If the width of $\phi(c)$ is zero, then the width of $\phi(x_k)$ converges to zero whenever the sequence $\{x_k\}$ converges to c, but in this case ϕ can be lsc at c.

1. **Introduction.** Let C be a compact and convex set contained in a Euclidean space, and $\phi : C \to C$ be a point convex and upper semi-continuous multifunction. Then Kakutani's fixed point theorem [7] asserts that ϕ has a fixed point, i.e. a point c with $c \in \phi(c)$. This theorem has important applications, in particular in game theory and mathematical economics, and hence the question is of interest whether the existing fixed point is also unique. Usually this is not the case.

Sufficient conditions which ϕ must satisfy in order to have a unique fixed point are hardly feasable without severe restrictions on ϕ, as they do not even exist if ϕ is single-valued. But we give here several necessary conditions which ϕ must satisfy if c is the unique (or even an isolated) fixed point of ϕ. They are local in character, and are concerned with the possible lower semicontinuity of ϕ, and the width of the images of ϕ, near the fixed point c. Theorems 1 and 2 deal with the case where the width of $\phi(c)$ is greater than zero, and Theorem 3 deals with the case where it is zero. The first two theorems are related to results by O. H. Hamilton [6], but they neither imply them, nor are they implied by them.

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Examples are given which show that these theorems cannot be sharpened into a result which covers all cases in a uniform way, and some open problems are stated at the end.

2. **Results.** A multifunction $\phi : X \to Y$ from a space X to a space Y is a correspondence which assigns to each point $x \in X$ a non-empty set $\phi(x)$ in Y. As all multifunctions in this paper satisfy continuity conditions, and as the definitions of the various types of continuity for multifunctions differ in the literature, we give the definitions used here.

DEFINITION 1. Let X and Y be topological spaces and $\phi : X \to Y$ be a multifunction.

(i) ϕ is called *upper semicontinuous* (usc) at the point $x \in X$ if $\phi(x)$ is closed, and if for every open V in Y with $\phi(x) \subset V$ there exists an open neighbourhood U of x with $\phi(U) \subset V$. ϕ is called usc if it is usc at every point.

(ii) ϕ is called *lower semicontinuous* (lsc) at the point $x \in X$ if for every open V in Y with $\phi(x) \cap V \neq \emptyset$ there exists an open neighbourhood U of x with $\phi(x') \cap V \neq \emptyset$ for every $x' \in U$. ϕ is called lsc if it is lsc at every point.

Note that our definition of usc is not the same as the one given in Kakutani [7], but it is equivalent. See e.g. [1], pp. 109–112.

The letter C will always denote a compact and convex set contained in a Euclidean space E, and we assume that C and E have the same (linear) dimension. If $A \subset C$, then ClA and $CL_E A$ are used for the closure of A in C resp. in E, and Int and Bd are used analogously for the interior and the boundary. The fixed point set $\{x \in C \mid x \in \phi(x)\}$ of the multifunction $\phi : C \to C$ is denoted by Fix ϕ.

We now give some necessary conditions which arise if the point convex and usc multifunction $\phi : C \to C$ has a fixed point which is unique. In the first result we only require that c is an *isolated* fixed point, i.e. that there exists a neighbourhood $U(c)$ of c with $ClU(c) \cap \text{Fix}\,\phi = \{c\}$.

THEOREM 1. *Let $\phi : C \to C$ be a point convex and usc multifunction. If c is an isolated fixed point and if $c \in \text{Int}_E\, \phi(c)$, then ϕ is not lsc at c.*

Proof. Let d be the Euclidean metric and $N(c, \varepsilon) = \{x \in C \mid d(x, c) < \varepsilon\}$. Determine $\varepsilon > 0$ so that $N(c, \varepsilon) \subset \phi(c)$. If n denotes the dimension of C and conv the convex hull, then we can select points $x_1, x_2, \ldots, x_{n+1}$ in $N(c, \varepsilon) \setminus \{c\}$ for which

$$N(c, \delta) \subset \text{conv}\{x_1, x_2, \ldots, x_{n+1}\} \subset N(c, \varepsilon)$$

for some $0 < \delta < \varepsilon$. For each $i = 1, 2, \ldots, n+1$ there exists a neighbourhood $V(x_i)$ of x_i, open in C, so that

$$N(c, \delta/2) \subset \text{conv}\{y_1, y_2, \ldots, y_{n+1}\}$$

whenever $y_i \in V(x_i)$ for $i = 1, 2, \ldots, n+1$.

Now assume, by way of contradiction, that ϕ is lsc at c. Then there exist neighbourhoods $U_i(c)$, open in C, with $\phi(x)\cap V(x_i)\neq\emptyset$ for all $x\in U_i(c)$. Let

$$U(c)=N(c,\delta/2)\cap[\cap(U_i(c)\mid i=1,2,\ldots,n+1)].$$

Then $x\in U(c)$ implies the existence of points $y_i=y_i(x)\in\phi(x)\cap V(x_i)$, and as $\phi(x)$ is convex, we have

$$N(c,\delta/2)\subset\operatorname{conv}\{y_1,y_2,\ldots,y_{n+1}\}\subset\phi(x).$$

But $U(c)\subset N(c,\delta/2)$, hence $x\in\phi(x)$ for all $x\in U(c)$, and c cannot be an isolated fixed point.

Next we consider the case where $c\notin\operatorname{Int}_E\phi(c)$, but where still $\operatorname{Int}_E\phi(c)\neq\emptyset$. This is equivalent to the assumption that $w\phi(c)>0$, where w denotes the *width* of the compact and convex set $\phi(c)$. (See [8], p. 157.) In this case we shall require that c is not only isolated, but essential according to Definition 3, which is modelled on the definition of an essential fixed point class. (See e.g. [2], p. 87.) Note that this definition does unfortunately not coincide, in the single-valued case, with the definition proposed by M. K. Fort, Jr. [4].

DEFINITION 2. (i) Let $\phi,\psi:C\to C$ be two point convex and usc multifunctions and $I=[0,1]$ the unit interval. We say that ϕ and ψ are *homotopic*, and write $\phi\sim\psi$, if there exists a point convex and usc multifunction $\Phi:C\times I\to C$ with $\Phi(x,0)=\phi(x)$ and $\Phi(x,1)=\psi(x)$ for all $x\in C$. Then Φ is called a *homotopy* from ϕ to ψ.

(ii) If $A\subset C$ and $\phi(a)=\psi(a)$ for all $a\in A$, then we say that ϕ is *homotopic to ψ relative A*, and write $\phi\sim\psi$ rel A, if there exists a homotopy Φ from ϕ to ψ such that $\Phi(a,t)=\phi(a)=\psi(a)$ for all $a\in A$ and $t\in I$.

DEFINITION 3. Let $\phi:C\to C$ be a point convex and usc multifunction and c be an isolated fixed point of ϕ. We say that c is an *inessential fixed point* if for every open neighbourhood $U(c)$ with $\operatorname{Fix}\phi\cap ClU(c)=\{c\}$ there exists a point convex and usc multifunction $\psi:C\to C$ so that

(i) $\phi(x)=\psi(x)$ for all $x\in C\backslash U(c)$,
(ii) $\phi\sim\psi$ rel $C\backslash U(c)$,
(iii) $\operatorname{Fix}\psi\cap ClU(c)=\emptyset$.

Otherwise we say that c is an *essential fixed point.*

THEOREM 2. *Let $\phi:C\to C$ be a point convex and usc multifunction. If c is an essential fixed point of ϕ, if $w\phi(c)>0$ and $c\in\operatorname{Bd}_E\phi(c)$, then ϕ is not lsc at c, and there exists a sequence $\{x_k\}$ converging to c so that $\{w\phi(x_k)\}$ converges to zero.*

Proof. (i) We shall first show that there exists a sequence $\{x_k\}$ with the desired properties. This is clearly the case if we can show that for every positive integer k the open ball $N(c,1/k)$ contains a point x_k with $w\phi(x_k)<1/k$.

Assume by way of contradiction that for some $k_0 \in N^+$ we have $w\phi(x) \geq 1/k_0$ for all $x \in N(c, 1/k_0)$. We shall show that then c is inessential. For this purpose, let $U_1(c)$ be any open neighbourhood of c with $\text{Fix}\ \phi \cap ClU_1(c) = \{c\}$. As ϕ is usc, there exists an open neighbourhood $U_2(c)$ with $\phi(U_2(c)) \subset N(\phi(c), \frac{1}{2}k_0)$, where

$$N(\phi(c), \tfrac{1}{2}k_0) = \{x \in C \mid d(x, \phi(c)) < \tfrac{1}{2}k_0\}.$$

Now let H be a support hyperplane of $\phi(c)$ at c, and H' the hyperplane which is parallel to H, intersects $\phi(c)$, and is at the distance $m = \min\{w\phi(c)/2, \frac{1}{3}k_0\}$ from H. The Euclidean space E containing C is divided by H' into two closed half spaces E^c and E^{c-}; let E^c be the one which contains c. With $U(c) = U_1(c) \cap U_2(c) \cap N(c, 1/k_0)$ define a multifunction $\psi : C \to C$ by

$$\psi(x) = \begin{cases} \phi(x) \cap E^{c-} & \text{if} \quad x \in U(c), \\ \phi(x) & \text{if} \quad x \notin U(c). \end{cases}$$

We first show that ψ is well defined, i.e. that $\psi(x) \neq \emptyset$ for all $x \in C$. If $x \in C \backslash U(c)$, this is obvious; if $x = c$, then $w\phi(c) > m$ implies $\psi(c) = \phi(c) \cap E^{c-} \neq \emptyset$. If $x \in U(c) \backslash \{c\}$, then $\phi(x)$ is contained in the parallel set

$$\bar{N}_E(\phi(c), \tfrac{1}{2}k_0) = \{x \in E \mid d(x, \phi(c)) \leq \tfrac{1}{2}k_0\}.$$

But the width of the compact and convex set $\bar{N}_E(\phi(c), \frac{1}{2}k_0) \cap E^c$ is $\leq m + \frac{1}{2}k_0 < 1/k_0$, and thus $w\phi(x) \geq 1/k_0$ implies $\psi(x) = \phi(x) \cap E^{c-} \neq \emptyset$.

ψ is point convex. As the multifunction $\phi' : C \to C$ given by

$$\phi'(x) = \begin{cases} C \cap E^{c-} & \text{if} \quad x \in U(c), \\ C & \text{if} \quad x \notin U(c) \end{cases}$$

is clearly usc, and as $\psi = \phi \cap \phi'$, we see that ψ is also usc ([1], pp. 111–112).

In order to see that $\phi \sim \psi$ rel $C \backslash U(c)$, let H'' be the support hyperplane of $\bar{N}_E(\phi(c), \frac{1}{2}k_0)$ which is parallel to H and for which H lies between H' and H''. For $0 \leq t \leq 1$, let H_t be the hyperplane between H' and H'' which is parallel to H and whose distance from H'' is $d(H_t, H'') = t \cdot d(H', H'')$. Define E_t^c and E_t^{c-} correspondingly to E^c and E^{c-} (i.e. so that $E^{c-} \subset E_t^{c-}$ for all t), and let $\phi_t : C \to C$ be given by

$$\phi_t(x) = \begin{cases} \phi(x) \cap E_t^{c-} & \text{if} \quad x \in U(c), \\ \phi(x) & \text{if} \quad x \notin U(c). \end{cases}$$

Then the multifunction $\Phi : C \times I \to C$ defined by $\Phi(x, t) = \phi_t(x)$ for all $x \in C$ and $t \in I$ provides a homotopy rel $C \backslash U(c)$ between $\phi_0 = \phi$ and $\phi_1 = \psi$. As $\text{Fix}\ \psi \subset \text{Fix}\ \phi \diagdown \{c\}$, we have $\text{Fix}\ \psi \cap \text{Cl}\ U(c) = \emptyset$, and c is inessential in contradiction to the assumptions of Theorem 2.

(ii) We now show that ϕ cannot be lsc at c. As $w\phi(c) > 0$, there exist points $x_1, x_2, \ldots, x_{n+1}$ in $\phi(c)$ (where again n is the dimension of C) for which conv$\{x_1, x_2, \ldots, x_{n+1}\}$ has a width which is greater than zero. Let $V(x_i)$ be open neighbourhoods of x_i so that $y_i \in V(x_i)$ implies that $w\,\text{conv}\{y_1, y_2, \ldots, y_{n+1}\} \geq w_0 > 0$, for some constant w_0. We now proceed again indirectly, and assume that ϕ is lsc at c. As $V(x_i) \cap \phi(c) \neq \emptyset$, there exist open neighbourhoods $U_i(c)$ with $\phi(x) \cap V(x_i) \neq \emptyset$ for all $x \in U_i(c)$. Hence if $x \in 0(c) = \cap\{U_i(c) \mid i = 1, 2, \ldots, n+1\}$, then there exist points $y_i \in V(x_i) \cap \phi(x)$ for all $i = 1, 2, \ldots, n+1$, and as $\phi(x)$ is convex, conv$\{y_1, y_2, \ldots, y_{n+1}\} \subset \phi(x)$. So $w\phi(x) \geq w_o > 0$ for all $x \in 0(c)$, contradicting the existence of a sequence $\{x_k\}$ converging to c for which $\{w\phi(x_k)\}$ converges to zero.

If the fixed point of $\phi : C \to C$ is unique, then it is clearly isolated and essential. Hence we can summarise Theorems 1 and 2 as follows.

COROLLARY. *Let $\phi : C \to C$ be a point convex and usc multifunction. If c is the unique fixed point of ϕ and if $w\phi(c) > 0$, then ϕ is not lsc at c, and if in addition $c \in \text{Bd}_E\, \phi(c)$, then there exists a sequence $\{x_k\}$ converging to c so that $\{w\phi(x_k)\}$ converges to zero.*

It remains to consider the case where $w\phi(c) = 0$.

THEOREM 3. *Let $\phi : C \to C$ be a point convex and usc multifunction. If c is an isolated fixed point of ϕ and $w\phi(c) = 0$, then $\{w\phi(x_k)\}$ converges to zero for every sequence $\{x_k\}$ converging to c.*

Proof. Let $\{x_k\}$ converge to c, and $\varepsilon > 0$. As ϕ is usc, there exists an open neighbourhood $U(c)$ with $\phi(U(c)) \subset N(\phi(c), \varepsilon/2) \subset \bar{N}_E(\phi(c), \varepsilon/2)$, and as $\{x_k\}$ converges to c, there exists a $k_0 \in N^+$ with $x_k \in U(c)$ for all $k \geq k_0$. Now $w\phi(c) = 0$ implies $w\bar{N}_E(\phi(c), \varepsilon/2)) = \varepsilon$, hence $w\phi(x_k) \leq \varepsilon$ for all $k \geq k_0$, and Theorem 3 holds.

3. **Remarks.** We give here some examples to show that the conclusions in Theorems 1, 2 and 3 cannot be strengthened, and the assumptions in Theorem 2 cannot be weakened, to give the same result in all cases.

EXAMPLE 1. There exists a multifunction which satisfies all assumptions of Theorem 1 but not all conclusions of Theorem 2. For let the multifunction ϕ_1 be defined on the interval $[-1, 1]$ by

$$\phi_1(x) = \begin{cases} [\frac{1}{2}, 1] & \text{if } -1 \leq x < 0, \\ [-1, 1] & \text{if } x = 0, \\ [-1, -\frac{1}{2}] & \text{if } 0 < x \leq 1. \end{cases}$$

ϕ_1 is usc, point convex, and has 0 as an essential fixed point situated in the interior of $\phi_1(0)$. It is not lsc at 0, but $\{w\phi_1(x_k)\}$ converges to $\frac{1}{2}$ for every sequence $\{x_k\}$ which converges to 0.

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EXAMPLE 2. There exists a multifunction ϕ_2 which satisfies all assumptions of Theorem 3 but which is lsc at an isolated fixed point. For let the multifunction ϕ_2 be defined on the interval $[-1, 1]$ by

$$\phi_2(x)=\begin{cases}[0,-x] & \text{if } -1\le x<0,\\ \{0\} & \text{if } x=0,\\ [-x,0] & \text{if } 0<x\le 1.\end{cases}$$

ϕ_2 is again usc and point convex, 0 is its unique fixed point, and $w\phi_2(0)=0$. But ϕ_2 is lsc at 0.

EXAMPLE 3. There exists a multifunction which satisfies all assumptions of Theorem 2 apart from the fact that the fixed point c is only isolated but not essential, and for which both conclusions of Theorem 2 are false. For let the multifunction ϕ_3 be defined on the interval $[-1, 1]$ by

$$\phi_3(x)=\begin{cases}[(3x+1)/2, 1] & \text{if } -1\le x\le 0,\\ [(x+1)/2, 1] & \text{if } 0\le x\le 1.\end{cases}$$

Then ϕ_3 is usc and point convex. It has two isolated fixed points at $x=-1$ and $x=1$, of which the first is inessential and the second essential. It is lsc at -1, and $\{w\phi_3(x_k)\}$ converges to 2 for every sequence $\{x_k\}$ which converges to -1.

4. **Some open questions.** Let diam A denote the diameter of A. If A is compact and convex, then diam $A\ge wA$, hence we have

PROBLEM 1. Can, in Theorems 2 and 3, and under the same assumptions, $w\phi(x_k)$ be replaced by diam $\phi(x_k)$?

We give a partial, negative, result in this direction, using the dimension $\dim\phi(c)$ of $\phi(c)$. Note that diam $\phi(c)=0$ is equivalent to $\dim\phi(c)=0$, and $w\phi(c)=0$ is equivalent to $\dim\phi(c)<n$, if again C is n-dimensional and contained in an n-dimensional Euclidean space.

PROPOSITION. *Let $\phi: C\to C$ be a point convex and usc multifunction. If* diam $\phi(c)=r$, *where $0<r\le n$, and if ϕ is lsc at c, then there exists no sequence $\{x_k\}$ which converges to c and for which* {diam $\phi(x_k)$} *converges to* 0.

Proof. If $\dim\phi(c)=r$, then there exist points $x_1, x_2, \ldots, x_{r+1}$ in $\phi(c)$ which span an r-simplex. Analogous to part (ii) of the proof of Theorem 2 we can find an open neighbourhood $0(c)=\cap\{U_i(c)\mid i=1,2,\ldots,r+1\}$ so that $x\in 0(c)$ implies the existence of points $y_1, y_2, \ldots, y_{r+1}$ with $\text{conv}\{y_1, y_2, \ldots, y_{r+1}\}\subset \phi(x)$ and diam $\text{conv}\{y_1, y_2, \ldots, y_{r+1}\}\ge d_0>0$ for some constant d_0, and thus prove the Proposition.

Therefore $w\phi(x_k)$ cannot be replaced by diam $\phi(x_k)$ in Theorem 3 if the answer to the next question is positive.

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PROBLEM 2. Does there exist a point convex and usc multifunction $\phi : C \to C$ which has an isolated fixed point c with $0 < \dim \phi(c) < \dim C$ and which is lsc at c?

Finally we note that Kakutani's theorem has been extended to compact and convex sets C contained in locally convex topological linear spaces by K. Fan [3] and I. L. Glicksberg [5]. (See also [1], p. 251.) This gives rise to the next question.

PROBLEM 3. How can the results of Theorems 1, 2 and 3 be extended to the case where C is a compact and convex subset of a locally convex topological linear space?

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