

## CONDITIONAL ORDERING OF $k$ -OUT-OF- $n$ SYSTEMS WITH INDEPENDENT BUT NONIDENTICAL COMPONENTS

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### Abstract

By considering  $k$ -out-of- $n$  systems with independent and nonidentically distributed components, we discuss stochastic monotone properties of the residual life and the inactivity time. We then present some stochastic comparisons of two systems based on the residual life and inactivity time.

*Keywords:* Hazard rate order; inactivity time; likelihood ratio order; residual life; reversed hazard rate order; stochastic order

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### 1. Introduction

As a very popular type of redundancy in fault-tolerant structures, the  $k$ -out-of- $n$  system, which works if at least  $k$  components work or, equivalently, at most  $n - k$  components fail, has been widely studied in electrical engineering, aviation industry, and weapon manufacturing, and has received considerable attention during the past few decades. Readers are referred to Kuo and Zuo (2002) for a comprehensive discussion on this structure and its properties. For  $1 \leq i \leq n$ , let  $X_i$  be the lifetime of the  $i$ th component, and let  $X_{k:n}$  be the lifetime of the  $k$ th failed component when failures are observed sequentially, and so it is the total lifetime of an  $(n - k + 1)$ -out-of- $n$  system. Hence, the study of order statistics naturally plays an important role in the study of  $k$ -out-of- $n$  systems.

In order to investigate the behavior of the lifetime of the system before it fails, many authors have discussed the ageing properties of

$$X_{k:n} - X_{k-1:n} \mid (X_{k-1:n} = t),$$

the conditional residual life of an  $(n - k + 1)$ -out-of- $n$  system, given that the  $(k - 1)$ th failure occurred at time  $t$ . We refer the reader to Langberg *et al.* (1980), Belzunce *et al.* (1999), Li and Zuo (2002), Li and Chen (2004), and Li and Zhao (2006), (2008) for some results in this regard. Since in some practical situations the accurate time to failure of the component of a system is often not observable and the only information that may be available is the total number of failed

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components, some authors also paid attention to

$$X_{k:n} - t \mid (X_{l:n} > t), \quad n \geq k > l \geq 1, \tag{1.1}$$

the *general residual life* of the system, given that the total number of failures till time  $t$  is not greater than  $l$ . Bairamov *et al.* (2002) studied the mean residual life of a parallel system, given no failure before time  $t \geq 0$ . For some further work in this direction, interested readers are referred to Asadi and Bairamov (2005), Li and Zhao (2006), and Khaledi and Shaked (2006).

For a system, which can be regarded as a black box in the sense that the exact failure times of its components cannot be observed, it is often of great importance for engineers and reliability analysts to make inference on the *inactivity time*  $t - X \mid (X \leq t)$ , the time elapsed since the failure of the system. This notion has a close connection with the so-called *autopsy* data which can be viewed as the information obtained by examining the status of components of a failed system. For more details about autopsy data, we refer the reader to Meilijson (1981) and Gåsemyr and Natvig (1998), (2001). Asadi (2006) studied the *mean inactivity time*  $E(t - X_{n:n} \mid X_{n:n} \leq t)$  of a parallel system given that the system failed at or before time  $t > 0$ . More generally, under the assumption that the remaining  $(n - k)$  components continue to work and are still subject to failure after the failure of the system, Khaledi and Shaked (2006) investigated

$$t - X_{l:n} \mid (X_{k:n} \leq t), \quad 1 \leq l \leq k \leq n, \tag{1.2}$$

the *general inactivity time* of the  $l$ th component of an  $(n - k + 1)$ -out-of- $n$  system given that the failure of the system occurred at or before time  $t > 0$ . Recently, Li and Zhao (2008) carried out a stochastic comparison on (1.1) and (1.2) of two  $(n - k + 1)$ -out-of- $n$  systems and generalized the main results of Khaledi and Shaked (2006). It is worth mentioning here that Hu *et al.* (2007) and Xie and Hu (2008) discussed conditional ordering of generalized order statistics, which includes (1.1) and (1.2) as special cases and, hence, extends some of the related results.

It is important to mention here that all the above results are derived under the assumption that components of systems are independent and identically distributed (i.i.d.). Due to the complicated expression of the distribution in the non-i.i.d. case, very few results in this case can be found in the literature. Under this more general setup, Sadegh (2008) first obtained some properties of the mean residual life function and the mean inactivity time function of parallel systems. Recently, Xu (2008) examined further the more general residual life of  $k$ -out-of- $n$  systems and the results derived there generalized some of the results of Li and Zhao (2006), Khaledi and Shaked (2006), and Sadegh (2008).

In this paper, under the general non-i.i.d. setup, we study

$$RL_{l,k,n}(t) = X_{k:n} - t \mid (X_{l:n} \leq t < X_{l+1:n}),$$

the *residual life* (RL) of an  $(n - k + 1)$ -out-of- $n$  system given that the  $l$ th ( $1 \leq l < k \leq n$ ) component has failed but the  $(l + 1)$ th component is working at time  $t \geq 0$ . If a system is still working at time  $t \geq 0$ , the reliability engineer may initiate some preventive maintenance policy or replacement policy to prevent the system from being damaged on a large scale or incurring a catastrophic loss. For this reason, it will be of interest for the engineer to have a knowledge of the properties of the conditional residual life of such systems so as to make a better decision about the system's design. Moreover, we also consider

$$IT_{l,k,n}(t) = t - X_{l:n} \mid (X_{k:n} \leq t < X_{k+1:n}),$$

the *inactivity time* (IT) of an  $(n - k + 1)$ -out-of- $n$  system given that the system had failed but the  $(k + 1)$ th ( $1 \leq l < k \leq n$ ) component is working at time  $t \geq 0$ . This kind of IT is also of importance in engineering reliability, since a knowledge of it may help the reliability engineer to initiate preventive maintenance or a replacement of the whole system at some reasonable epoch.

The rest of this paper is organized as follows. In Section 2 we present some stochastic monotone properties of  $RL_{l,k,n}(t)$  and  $IT_{l,k,n}(t)$  with respect to parameters  $l, k$ , and  $n$ . In Section 3 we discuss stochastic comparisons of the RLs and ITs from two similar  $(n - k + 1)$ -out-of- $n$  systems.

Throughout this paper, the term *increasing* stands for *monotone nondecreasing* and the term *decreasing* stands for *monotone nonincreasing*.

### 2. Monotone properties

Before proceeding to the main results, let us first recall some stochastic orders that are most pertinent to the main results developed here.

**Definition 2.1.** For two random variables  $X$  and  $Y$ , with their densities  $f$  and  $g$  and distribution functions  $F$  and  $G$ , respectively, let  $\bar{F} = 1 - F$  and  $\bar{G} = 1 - G$  be their survival functions. As the ratios in the statements below are well defined,  $X$  is said to be smaller than  $Y$  in

- (a) likelihood ratio order (denoted by  $X \leq_{lr} Y$ ) if  $g(x)/f(x)$  is increasing in  $x$ ;
- (b) hazard rate order (denoted by  $X \leq_{hr} Y$ ) if  $\bar{G}(x)/\bar{F}(x)$  is increasing in  $x$ ;
- (c) reversed hazard rate order (denoted by  $X \leq_{rh} Y$ ) if  $G(x)/F(x)$  is increasing in  $x$ ;
- (d) stochastic order (denoted by  $X \leq_{st} Y$ ) if  $\bar{G}(x) \geq \bar{F}(x)$ .

For a comprehensive discussion on these stochastic orders, we refer the reader to Shaked and Shanthikumar (2007) and Müller and Stoyan (2002).

Since the joint density (distribution) function of order statistics from independent and nonidentical observations can be represented as a permanent, we will now present a brief introduction to permanents.

The permanent function was first introduced by Binet and Cauchy (independently) as early as 1812, more or less simultaneously with the determinant function. We refer the reader to Bapat and Beg (1989), Bapat and Kochar (1994), Hu and Zhu (2003), and Hu *et al.* (2006) for some related discussion on this topic, and the recent review article by Balakrishnan (2007), which serves as a nice reference source for readers who are interested in the theory of permanent and its close connection to order statistics. If  $A = (a_{i,j})$  is a square matrix of order  $n$  then the *permanent* of  $A$  is calculated as  $\sum_{\sigma} \prod_{i=1}^n a_{i,\sigma(i)}$ , where the summation is taken over all permutations  $\sigma = (\sigma(1), \dots, \sigma(n))$  of  $(1, \dots, n)$ . If  $d_i \in \mathbb{R}^n$  for  $i = 1, 2, \dots, n$ , we denote by  $[d_1, \dots, d_n]$  the permanent of the  $n \times n$  matrix  $(d_1 \cdots d_n)$ . For convenience,

$$[\underbrace{d_1}_{r_1}, \underbrace{d_2}_{r_2}, \dots]$$

denotes the permanent having  $r_1$  copies of  $d_1$ ,  $r_2$  copies of  $d_2$ , and so on, and

$$[\underbrace{d_1}_{r_1}, \underbrace{d_2}_{r_2}]_A$$

denotes the permanent having those rows only in set  $A$ . If  $r_i = 1$ , it is omitted in the notation above. If  $r_i = 0$ ,  $d_i$  is understood not to appear in the permanent; if  $r_i < 0$  for some  $i$  then the permanent is understood to be 0.

In this section we will further assume that the life  $X_i$ ,  $i = 1, \dots, n$ , of the components of the  $(n - k + 1)$ -out-of- $n$  system are mutually independent, nonnegative random variables having their respective underlying distribution functions  $F_i$  and survival functions  $\bar{F}_i = 1 - F_i$ . For  $i = 1, \dots, n$ , let  $X_{i,t} = X_i - t \mid (X_i > t)$  denote the residual life of the component  $X_i$  with distribution function  $F_{i,t}(x)$  and survival function  $\bar{F}_{i,t}(x)$ , and let  $\phi_i(t) = F_i(t)/\bar{F}_i(t)$ . The column vector  $(F_{1,t}(x), \dots, F_{n,t}(x))^T$  will be denoted simply by  $F_t(x)$ , and  $\bar{F}_t(x)$  and  $\phi(t)$  are similarly defined.

Now, the main results of this section are as follows.

**Theorem 2.1.** For  $1 \leq l < k \leq n$  and  $t \geq 0$ ,  $RL_{l,k,n}(t)$  has the survival function

$$\bar{H}_{l,k,n,t}(x) = \frac{\sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{P_l^c}(x)}{\sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t)} \tag{2.1}$$

with  $\phi_{j_m}(t) = F_{j_m}(t)/\bar{F}_{j_m}(t)$  and

$$\bar{H}_{k-l,n-l,t}^{P_l^c}(x) = \sum_{i=0}^{k-l+1} \frac{1}{i!(n-l-i)!} [\underbrace{F_t(x)}_i, \underbrace{\bar{F}_t(x)}_{n-l-i}]_{P_l^c},$$

where the summation  $P_l$  with size  $l$  extends over all subsets of  $\{1, \dots, n\}$  and  $P_l^c$  denotes the complement set of  $P_l$ .

*Proof.* For  $1 \leq l < k \leq n$  and  $t \geq 0$ , we have

$$\begin{aligned} & P(X_{k:n} > t + x, X_{l:n} \leq t < X_{l+1:n}) \\ &= \sum_{i=0}^{k-l-1} P(\text{exactly } l \text{ observations in } [0, t], i \text{ observations in } (t, t + x], \\ & \quad \text{and } n - l - i \text{ observations in } (t + x, \infty), \text{ respectively}) \\ &= \sum_{i=0}^{k-l-1} \frac{1}{l! i! (n-l-i)!} \\ & \quad \times \sum_P F_{j_1}(t) \cdots F_{j_l}(t) [F_{j_{l+1}}(t+x) - F_{j_{l+1}}(t)] \cdots [F_{j_{l+i}}(t+x) - F_{j_{l+i}}(t)] \\ & \quad \times \bar{F}_{j_{l+i+1}}(t+x) \cdots \bar{F}_{j_n}(t+x) \\ &= \prod_{i=1}^n \bar{F}_i(t) \sum_{i=0}^{k-l-1} \frac{1}{l! i! (n-l-i)!} \sum_P \phi_{j_1}(t) \cdots \phi_{j_l}(t) F_{j_{l+1},t}(x) \cdots F_{j_{l+i},t}(x) \\ & \quad \times \bar{F}_{j_{l+i+1},t}(x) \cdots \bar{F}_{j_n,t}(x), \end{aligned}$$

where  $P$  denotes the summation over all  $n!$  permutations  $(j_1, \dots, j_n)$  of  $(1, \dots, n)$ . By the definition of permanent and Laplace's expansion, we further have

$$\begin{aligned} &P(X_{k:n} > t + x, X_{l:n} \leq t < X_{l+1:n}) \\ &= \prod_{i=1}^n \bar{F}_i(t) \sum_{i=0}^{k-l-1} \frac{1}{l! i! (n-l-i)!} [\underbrace{\phi(t)}_l, \underbrace{F_t(x)}_i, \underbrace{\bar{F}_t(x)}_{n-l-i}] \\ &= \prod_{i=1}^n \bar{F}_i(t) \sum_{i=0}^{k-l-1} \frac{1}{l! i! (n-l-i)!} \sum_{P_l} [\underbrace{\phi(t)}_l]_{P_l} [\underbrace{F_t(x)}_i]_{P_l} [\underbrace{\bar{F}_t(x)}_{n-l-i}]_{P_l^c} \\ &= \prod_{i=1}^n \bar{F}_i(t) \sum_{P_l} \prod_{m \in P_l} \phi_{j_m} \sum_{i=0}^{k-l-1} \frac{1}{i! (n-l-i)!} [\underbrace{F_t(x)}_i, \underbrace{\bar{F}_t(x)}_{n-l-i}]_{P_l^c} \\ &= \prod_{i=1}^n \bar{F}_i(t) \sum_{P_l} \prod_{m \in P_l} \phi_{j_m} \sum_{i=0}^{k-l-1} \bar{H}_{k-l, n-l, t}^{P_l^c}(x). \end{aligned}$$

On the other hand, we also have (see David and Nagaraja (2003, p. 96))

$$\begin{aligned} &P(X_{l:n} \leq t < X_{l+1:n}) \\ &= P(X_{l+1:n} \geq t) - P(X_{l:n} > t) \\ &= \sum_{i=0}^l \sum_{P_i} \left( \prod_{m \in P_i} F_{j_m}(t) \prod_{m \in P_i^c} \bar{F}_{j_m}(t) \right) - \sum_{i=0}^{l-1} \sum_{P_i} \left( \prod_{m \in P_i} F_{j_m}(t) \prod_{m \in P_i^c} \bar{F}_{j_m}(t) \right) \\ &= \sum_{P_l} \left( \prod_{m \in P_l} F_{j_m}(t) \prod_{m \in P_l^c} \bar{F}_{j_m}(t) \right) \\ &= \prod_{i=1}^n \bar{F}_i(t) \sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t). \end{aligned}$$

By combining the above two equalities, the expression in (2.1) follows immediately.

It can be readily seen that  $\bar{H}_{k-l, n-l, t}^{P_l^c}(x)$  is the survival function of the life of a  $(k-l)$ -out-of- $(n-l)$  system composed of  $n-l$  used units with residual life  $X_{i,t} = X_i - t \mid (X_i > t)$ ,  $i \in P_l^c$ . In the case with independent and identical components, it is evident that  $\bar{H}_{l, k, n, t}(x) = \bar{H}_{k-l, n-l, t}(x)$ , which is actually the survival function of the life of a  $(k-l)$ -out-of- $(n-l)$  system composed of  $n-l$  commonly used units with residual life  $X_t = X - t \mid (X > t)$ .

We will now focus on stochastic monotone properties of the residual life  $RL_{l, k, n}(t)$  with respect to  $l, k$ , and  $n$ . The following lemma, due to Nanda and Shaked (2001), will be useful in the sequel.

**Lemma 2.1.** *Let  $X_1, X_2, \dots, X_m$  and  $Y_1, Y_2, \dots, Y_n$  be absolutely continuous, independent (not necessarily identical) random variables. For any  $m$  and  $n$ , if  $X_i \leq_{st} Y_i$  for all  $i$  then  $X_{i:m} \leq_{st} Y_{j:n}$  whenever  $i \leq j$  and  $m - i \geq n - j$ .*

The first result below reveals that the residual life  $RL_{l, k, n}(t)$  is stochastically increasing in  $k$  for any fixed  $l, n$ , and  $t$ .

**Theorem 2.2.** For any  $t \geq 0$  and  $1 \leq l < k < n$ ,  $RL_{l,k,n}(t) \leq_{st} RL_{l,k+1,n}(t)$ .

*Proof.* For any  $t \geq 0, x \geq 0$ , and  $k, l$  such that  $1 \leq l < k < n$ , we observe that

$$\bar{H}_{l,k+1,n,t}(x) - \bar{H}_{l,k,n,t}(x) = \frac{\sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t) (\bar{H}_{k-l+1,n-l,t}^{P_l^c}(x) - \bar{H}_{k-l,n-l,t}^{P_l^c}(x))}{\sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t)}.$$

For any permutation  $(j_1, \dots, j_l)$  of  $P_l$ , denote by  $(j_{l+1}, \dots, j_n)$  the corresponding complement permutation of  $P_l^c$ . By Lemma 2.1 we then have

$$\bar{H}_{k-l+1,n-l,t}^{\{j_{l+1}, \dots, j_n\}}(x) - \bar{H}_{k-l,n-l,t}^{\{j_{l+1}, \dots, j_n\}}(x) \geq 0,$$

which implies that  $\bar{H}_{l,k+1,n,t}(x) - \bar{H}_{l,k,n,t}(x) \geq 0$ , and, hence, the required result follows immediately.

The next result states that the residual life  $RL_{l,k,n}(t)$  is stochastically decreasing in  $l$  for any fixed  $k, n$ , and  $t$ .

**Theorem 2.3.** For any  $t \geq 0$  and  $1 < l < k < n$ ,  $RL_{l,k,n}(t) \leq_{st} RL_{l-1,k,n}(t)$ .

*Proof.* For any  $t \geq 0, x \geq 0$ , and  $k, l$  such that  $1 \leq l < k < n$ , we observe that

$$\begin{aligned} & \bar{H}_{l-1,k,n,t}(x) - \bar{H}_{l,k,n,t}(x) \\ & \stackrel{\text{sgn}}{=} \sum_{P_{l-1}} \prod_{m \in P_{l-1}} \phi_{j_m}(t) \bar{H}_{k-l+1,n-l+1,t}^{P_{l-1}^c}(x) \sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t) \\ & \quad - \sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{P_l^c}(x) \sum_{P_{l-1}} \prod_{m \in P_{l-1}} \phi_{j_m}(t) \\ & = \sum_{P_{l-1}} \sum_{P_l} \prod_{m \in P_{l-1} \oplus P_l} \phi_{j_m}(t) \bar{H}_{k-l+1,n-l+1,t}^{P_{l-1}^c}(x) - \sum_{P'_{l-1}} \sum_{P'_l} \prod_{s \in P'_{l-1} \oplus P'_l} \phi_{j_s}(t) \bar{H}_{k-l,n-l,t}^{(P'_l)^c}(x), \end{aligned}$$

where ‘sgn’ means to have the same sign and  $A \oplus B$  gives the stacking of all elements in two sets  $A$  and  $B$ . Since there are the same number of terms in the above two summations, for any given permutation  $A_{l-1} \subset P_{l-1}$  and  $A_l \subset P_l$ , there must exist corresponding  $A'_{l-1} \subset P'_{l-1}$  and  $A'_l \in P'_l$  such that  $A_{l-1} \subset A'_l$  and  $A_{l-1} \oplus A_l = A'_{l-1} \oplus A'_l$ . That is, for any term in the summation of the first part, we can find a corresponding term in the summation of the second part satisfying the relation

$$\begin{aligned} & \prod_{m \in A_{l-1} \oplus A_l} \phi_{j_m}(t) \bar{H}_{k-l+1,n-l+1,t}^{A_{l-1}^c}(x) - \prod_{s \in A'_{l-1} \oplus A'_l} \phi_{j_s}(t) \bar{H}_{k-l,n-l,t}^{(A'_l)^c}(x) \\ & \stackrel{\text{sgn}}{=} \bar{H}_{k-l+1,n-l+1,t}^{A_{l-1}^c}(x) - \bar{H}_{k-l,n-l,t}^{(A'_l)^c}(x). \end{aligned}$$

Now, by using Lemma 2.1 and the fact that  $(A'_l)^c \subset A_{l-1}^c$ , we have

$$\bar{H}_{k-l+1,n-l+1,t}^{A_{l-1}^c}(x) - \bar{H}_{k-l,n-l,t}^{(A'_l)^c}(x) \geq 0,$$

and so  $\bar{H}_{l-1,k,n,t}(x) - \bar{H}_{l,k,n,t}(x) \geq 0$ . Hence, the result.

**Theorem 2.4.** For any  $t \geq 0$  and  $1 \leq l < k < n$ ,  $RL_{l,k,n}(t) \leq_{st} RL_{l+1,k+1,n}(t)$ .

*Proof.* For any  $t \geq 0, x \geq 0$ , and  $k, l$  such that  $1 < l < k < n$ , we observe that

$$\begin{aligned} & \bar{H}_{l+1,k+1,n,t}(x) - \bar{H}_{l,k,n,t}(x) \\ & \stackrel{\text{sgn}}{=} \sum_{P_{l+1}} \prod_{m \in P_{l+1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l-1,t}^{P_{l+1}^c}(x) \sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t) \\ & \quad - \sum_{P_l} \prod_{m \in P_l} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{P_l^c}(x) \sum_{P_{l+1}} \prod_{m \in P_{l+1}} \phi_{j_m}(t) \\ & = \sum_{P_{l+1}} \sum_{P_l} \prod_{m \in P_{l+1} \oplus P_l} \phi_{j_m}(t) \bar{H}_{k-l,n-l-1,t}^{P_{l+1}}(x) - \sum_{P'_{l+1}} \sum_{P'_l} \prod_{s \in P'_{l+1} \oplus P'_l} \phi_{j_s}(t) \bar{H}_{k-l,n-l,t}^{(P'_l)^c}(x). \end{aligned}$$

By Lemma 2.1, for any  $A_{l+1} \subset P_{l+1}$  and  $A'_l \subset P'_l$ , if  $A'_l \subset A_{l+1}$  then

$$\bar{H}_{k-l,n-l-1,t}^{A_{l+1}^c}(x) - \bar{H}_{k-l,n-l,t}^{(A'_l)^c}(x) \geq 0.$$

By an argument similar to the proof of Theorem 2.3 and based upon the above fact, we can conclude that  $\bar{H}_{l+1,k+1,n,t}(x) - \bar{H}_{l,k,n,t}(x) \geq 0$ . Hence, the result.

The next result states that  $RL_{l,k,n}(t)$  is stochastically decreasing in  $n$  for any  $l, k$ , and  $t$ .

**Theorem 2.5.** For any  $t \geq 0$  and  $1 \leq l \leq k < n$ ,  $RL_{l,k,n}(t) \leq_{st} RL_{l,k,n-1}(t)$ .

*Proof.* For any  $t \geq 0, x \geq 0$ , and  $k, l$  such that  $1 < l < k < n$ , we observe that

$$\begin{aligned} & \bar{H}_{l,k,n-1,t}(x) - \bar{H}_{l,k,n,t}(x) \\ & \stackrel{\text{sgn}}{=} \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l-1,t}^{(P_l^{n-1})^c}(x) \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \\ & \quad - \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_l^n)^c}(x) \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t), \end{aligned} \tag{2.2}$$

where  $P_l^{n-1}$  extends over all subsets of  $\{1, \dots, n-1\}$  of size  $l$ . Note that, for a vector  $(x_1, \dots, x_n)$  with positive components, we have

$$\sum_{P_l^n} \prod_{m \in P_l^n} x_{j_m} = \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} x_{j_m} + x_n \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} x_{j_m}.$$

From this fact, it follows that

$$\sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) = \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) + \phi_n(t) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t) \tag{2.3}$$

and

$$\begin{aligned} & \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_l^n)^c}(x) \\ & = \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_l^{n-1})^c \cup \{n\}}(x) + \phi_n(t) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_{l-1}^{n-1})^c}(x). \end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we obtain

$$\begin{aligned} & \bar{H}_{l,k,n-1,t}(x) - \bar{H}_{l,k,n,t}(x) \\ & \stackrel{\text{sgn}}{=} \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l-1,t}^{(P_l^{n-1})^c}(x) \left( \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) + \phi_n(t) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t) \right) \\ & - \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \left( \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_l^{n-1})^c \cup n}(x) \right. \\ & \qquad \qquad \qquad \left. + \phi_n(t) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_{l-1}^{n-1})^c}(x) \right) \\ & = A + B, \end{aligned}$$

where

$$\begin{aligned} A &= \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l-1,t}^{(P_l^{n-1})^c}(x) \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \\ & - \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_l^{n-1})^c \cup n}(x) \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) \end{aligned}$$

and

$$\begin{aligned} B &= \phi_n(t) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l-1,t}^{(P_{l-1}^{n-1})^c}(x) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t) \\ & - \phi_n(t) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t) \bar{H}_{k-l,n-l,t}^{(P_{l-1}^{n-1})^c}(x) \sum_{P_{l-1}^{n-1}} \prod_{m \in P_{l-1}^{n-1}} \phi_{j_m}(t). \end{aligned}$$

Now, it suffices to show that  $A \geq 0$  and  $B \geq 0$ . Note that

$$A \stackrel{\text{sgn}}{=} \sum_{P_l^{n-1}} \prod_{m \in P_l^{n-1}} \phi_{j_m}(t) (\bar{H}_{k-l,n-l-1,t}^{(P_l^{n-1})^c}(x) - \bar{H}_{k-l,n-l,t}^{(P_l^{n-1})^c \cup n}(x)),$$

which is indeed nonnegative due to Lemma 2.1. Moreover,

$$\begin{aligned} B & \stackrel{\text{sgn}}{=} \sum_{P_{l-1}^{n-1}} \sum_{m \in (P_{l-1}^{n-1} \oplus P_{l-1}^{n-1})} \prod_{s \in (P_{l-1}^{n-1} \oplus P_{l-1}^{n-1})} \phi_{j_s}(t) \bar{H}_{k-l,n-l-1,t}^{(P_{l-1}^{n-1})^c}(x) \\ & - \sum_{Q_{l-1}^{n-1}} \sum_{s \in (Q_{l-1}^{n-1} \oplus Q_{l-1}^{n-1})} \prod_{s \in (Q_{l-1}^{n-1} \oplus Q_{l-1}^{n-1})} \phi_{j_s}(t) \bar{H}_{k-l,n-l,t}^{(Q_{l-1}^{n-1})^c}(x), \end{aligned}$$

where  $Q_{l-1}^{n-1}$  and  $Q_{l-1}^{n-1}$  are summations defined similarly to  $P_{l-1}^{n-1}$  and  $P_{l-1}^{n-1}$ . Upon using an argument similar to the proof of Theorem 2.3 and using Lemma 2.1 once again, it can be shown that  $B \geq 0$ , which completes the proof of the result.



**Theorem 2.6.** For any  $t \geq 0$  and  $1 \leq l < k \leq n$ ,  $RL_{l,k,n}(t) \leq_{st} RL_{l,k+1,n+1}(t)$ .

*Proof.* For any  $t \geq 0, x \geq 0$ , and  $l, k$  such that  $1 \leq l < k \leq n$ , we can express

$$\begin{aligned} & \tilde{H}_{l,k+1,n+1,t}(x) - \tilde{H}_{l,k,n,t}(x) \\ & \stackrel{\text{sgn}}{=} \sum_{P_l^{n+1}} \prod_{m \in P_l^{n+1}} \phi_{j_m}(t) \tilde{H}_{k-l+1,n-l+1,t}^{(P_l^{n+1})^c}(x) \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \\ & \quad - \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \tilde{H}_{k-l,n-l,t}^{(P_l^n)^c}(x) \sum_{P_l^{n+1}} \prod_{m \in P_l^{n+1}} \phi_{j_m}(t) \\ & = \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \left( \sum_{P_l^{n+1}} \prod_{m \in P_l^{n+1}} \phi_{j_m}(t) \tilde{H}_{k-l+1,n-l+1,t}^{(P_l^n)^c \cup (n+1)}(x) \right. \\ & \quad \left. + \phi_{n+1}(t) \sum_{P_{l-1}^n} \prod_{m \in P_{l-1}^n} \phi_{j_m}(t) \tilde{H}_{k-l+1,n-l+1,t}^{(P_{l-1}^n)^c}(x) \right) \\ & \quad - \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \tilde{H}_{k-l,n-l,t}^{(P_l^n)^c}(x) \left( \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) + \phi_{n+1}(t) \sum_{P_{l-1}^n} \prod_{m \in P_{l-1}^n} \phi_{j_m}(t) \right) \\ & = C + D, \end{aligned}$$

where

$$\begin{aligned} C &= \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \tilde{H}_{k-l+1,n-l+1,t}^{(P_l^n)^c \cup (n+1)}(x) \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \\ & \quad - \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \tilde{H}_{k-l,n-l,t}^{(P_l^n)^c}(x) \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \end{aligned}$$

and

$$\begin{aligned} D &= \phi_{n+1}(t) \sum_{P_{l-1}^n} \prod_{m \in P_{l-1}^n} \phi_{j_m}(t) \tilde{H}_{k-l+1,n-l+1,t}^{(P_{l-1}^n)^c}(x) \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \\ & \quad - \phi_{n+1}(t) \sum_{P_l^n} \prod_{m \in P_l^n} \phi_{j_m}(t) \tilde{H}_{k-l,n-l,t}^{(P_l^n)^c}(x) \sum_{P_{l-1}^n} \prod_{m \in P_{l-1}^n} \phi_{j_m}(t). \end{aligned}$$

Following an argument similar to that used in Theorem 2.5, it follows from Lemma 2.1 that  $C \geq 0$  and  $D \geq 0$ . Hence, the desired result follows.

Combining Theorems 2.2–2.6, we obtain the following corollary, which provides a nice summary of all the results established above.

**Corollary 2.1.** (i) For any  $t \geq 0, 1 \leq l < k \leq n$ , and  $1 \leq s \leq m \leq n$ ,

$$RL_{l,k,n}(t) \leq_{st} RL_{s,m,n}(t) \quad \text{whenever } k \leq m \text{ and } k - l \leq m - s.$$

(ii) For any  $t \geq 0, 1 \leq l < k \leq n$ , and  $1 \leq l \leq p \leq q$ ,

$$RL_{l,k,n}(t) \leq_{st} RL_{l,p,q}(t) \quad \text{whenever } k \leq p \text{ and } n - k \geq q - p.$$

In concluding this section we present an analogous version of Corollary 2.1 for the inactivity time defined in Section 1.

**Corollary 2.2.** (i) For  $1 \leq l < k \leq n$ ,  $1 \leq s \leq m \leq n$ , and  $t \geq 0$ ,

$$IT_{l,k,n}(t) \leq_{st} IT_{s,m,n}(t) \quad \text{whenever } l \geq s \text{ and } k - l \leq m - s.$$

(ii) For  $1 \leq l < k \leq n$ ,  $1 \leq s < p \leq q$ , and  $t \geq 0$ ,

$$IT_{l,k,n}(t) \leq_{st} IT_{s,p,q}(t) \quad \text{whenever } l \geq s \text{ and } n - l \geq q - s.$$

*Proof.* We only prove part (i), while the proof of part (ii) can be established in a similar manner. Although the random variables discussed here are all taken to be nonnegative, all the results also hold for any random variables on the real line. Let  $(-X)_{r:n}$  ( $1 \leq r \leq n$ ) denote the  $r$ th order statistic among  $-X_1, \dots, -X_n$ . Then, it is evident that  $(-X)_{r:n} = -X_{n-r+1:n}$ . By Corollary 2.1(i) we have, for  $1 \leq l < k \leq n$  and  $1 \leq s < m \leq n$ ,

$$[(-X)_{k:n} - y \mid (-X)_{l:n} \leq y < (-X)_{l+1:n}] \leq_{st} [(-X)_{m:n} - y \mid (-X)_{s:n} \leq y < (-X)_{s+1:n}]$$

whenever  $k \leq m$ ,  $k - l \leq m - s$ , and  $y \in \mathbb{R}$ , which is equivalent to

$$[-X_{n-k+1:n} \mid -X_{n-l+1:n} \leq y < -X_{n-l:n}] \leq_{st} [-X_{n-m+1:n} \mid -X_{n-s+1:n} \leq y < -X_{n-s:n}]$$

whenever  $k \leq m$ ,  $k - l \leq m - s$ , and  $y \in \mathbb{R}$ . That is,

$$[t - X_{n-k+1:n} \mid X_{n-l:n} \leq t < X_{n-l+1:n}] \leq_{st} [t - X_{n-m+1:n} \mid X_{n-s:n} \leq t < X_{n-s+1:n}]$$

whenever  $k \leq m$ ,  $k - l \leq m - s$ , and  $t \in \mathbb{R}$ . Moreover,

$$[t - X_{l:n} \mid X_{k:n} \leq t < X_{k+1:n}] \leq_{st} [t - X_{s:n} \mid X_{m:n} \leq t < X_{m+1:n}]$$

whenever  $l \geq s$ ,  $n - l \geq q - s$ , and  $t \in \mathbb{R}$  for  $1 \leq l < k \leq n$  and  $1 \leq s \leq m \leq n$ , which is just the desired result.

### 3. Stochastic comparisons

It is important in reliability theory to compare the variability of coherent systems so as to design more reliable systems. We now turn our attention to stochastic comparisons based on the RLs of two  $k$ -out-of- $n$  systems, both of which consist of independent (not necessarily identical) components. Before stating our main result, we first present a useful lemma due to Lillo *et al.* (2001) and Boland *et al.* (2002).

**Lemma 3.1.** Let  $X_1, X_2, \dots, X_m$  and  $Y_1, Y_2, \dots, Y_n$  be absolutely continuous, independent (not necessarily identical) random variables. For any  $m$  and  $n$ ,

(a) if  $X_i \leq_{lr} Y_j$  for all  $i$  and  $j$  then  $X_{i:m} \leq_{lr} Y_{j:n}$  whenever  $i \leq j$  and  $m - i \geq n - j$ ;

(b) if  $X_i \leq_{hr} Y_j$  for all  $i$  and  $j$  then  $X_{i:m} \leq_{hr} Y_{j:n}$  whenever  $i \leq j$  and  $m - i \geq n - j$ .

For convenience, let us introduce some notation now. For  $1 \leq l \leq k \leq m$  and  $t \geq 0$ , let  $RL_{l,k,m}^X(t)$  be the residual life of a  $k$ -out-of- $m$  system with  $m$  independent (not necessarily identical) components having  $X_i$  ( $1 \leq i \leq m$ ) as their respective lifetimes. Similarly, for  $1 \leq s \leq u \leq n$  and  $t \geq 0$ , let  $RL_{s,u,n}^Y(t)$  be the residual life of another  $u$ -out-of- $n$

system with  $n$  independent (not necessarily identical) components having  $Y_j (1 \leq j \leq n)$  as their respective lifetimes. Let  $\bar{F}_i$  and  $\bar{G}_j$  denote the corresponding survival functions of  $X_i$  and  $Y_j$ , respectively, and define  $\phi_i^X = F_i/\bar{F}_i$  and  $\phi_j^Y = G_j/\bar{G}_j$ . Also, let  $\bar{H}_{l,k,m,t}^X(x)$  and  $\bar{H}_{s,u,n,t}^Y(x)$  be the corresponding survival functions of  $RL_{l,k,m}^X(t)$  and  $RL_{s,u,n}^Y(t)$ , let  $H_{l,k,m,t}^X(x)$  and  $H_{s,u,n,t}^Y(x)$  be the corresponding distribution functions, and let  $f_{l,k,m,t}^X(x)$  and  $f_{s,u,n,t}^Y(x)$  be the corresponding density functions, respectively.

**Theorem 3.1.** *For two systems with independent (not necessarily identical) components  $X_1, X_2, \dots, X_m$  and  $Y_1, Y_2, \dots, Y_n$ ,*

- (a) *if  $X_i \leq_{lr} Y_j$  for all  $i$  and  $j$  then  $RL_{l,k,m}^X(t) \leq_{lr} RL_{s,u,n}^Y(t)$  whenever  $k - l \leq u - s$  and  $m - k \geq n - u$ ;*
- (b) *if  $X_i \leq_{hr} Y_j$  for all  $i$  and  $j$  then  $RL_{l,k,m}^X(t) \leq_{hr} RL_{s,u,n}^Y(t)$  whenever  $k - l \leq u - s$  and  $m - k \geq n - u$ .*

*Proof.* (a) From (2.1), the density functions of  $RL_{l,k,m}^X(t)$  and  $RL_{s,u,n}^Y(t)$  are given by

$$f_{l,k,m,t}^X(x) = \frac{\sum_{P_l^m} \prod_{d \in P_l^m} \phi_{j_d}^X(t) f_{k-l,m-l,t}^{(P_l^m)^c}(x)}{\sum_{P_l^m} \prod_{d \in P_l^m} \phi_{j_d}^X(t)}$$

and

$$f_{s,u,n,t}^Y(x) = \frac{\sum_{P_s^n} \prod_{d \in P_s^n} \phi_{j_d}^Y(t) g_{u-s,n-s,t}^{(P_s^n)^c}(x)}{\sum_{P_s^n} \prod_{d \in P_s^n} \phi_{j_d}^Y(t)},$$

where  $f_{k-l,m-l,t}^{(P_l^m)^c}(x)$  is the density function of the  $(k - l)$ th order statistic from independent (not necessarily identical) sample  $X_{i,t}, i \in (P_l^m)^c$ , and  $g_{u-s,n-s,t}^{(P_s^n)^c}(x)$  is the density function of the  $(u - s)$ th order statistic from independent (not necessarily identical) sample  $Y_{j,t}, j \in (P_s^n)^c$ . Their ratio can then be written as

$$\frac{f_{k-l,m-l,t}^{(P_l^m)^c}(x)}{g_{u-s,n-s,t}^{(P_s^n)^c}(x)} \propto \frac{\sum_{P_l^m} \prod_{d \in P_l^m} \phi_{j_d}^X(t) f_{k-l,m-l,t}^{(P_l^m)^c}(x)}{\sum_{P_s^n} \prod_{d \in P_s^n} \phi_{j_d}^Y(t) g_{u-s,n-s,t}^{(P_s^n)^c}(x)}. \tag{3.1}$$

It is easy to verify that  $X_i \leq_{lr} Y_j$  implies that  $X_{i,t} \leq_{lr} Y_{j,t}$  for all  $i$  and  $j$ . For any choice  $(i_1, \dots, i_l)$  of  $P_l^m$ , and any choice  $(j_1, \dots, j_s)$  of  $P_s^n$ , according to part (a) of Lemma 3.1, it follows that

$$\frac{\prod_{d=1}^l \phi_{i_d}^X(t) f_{k-l,m-l,t}^{\{i_{l+1}, \dots, i_m\}}(x)}{\prod_{d=1}^s \phi_{j_d}^Y(t) g_{u-s,n-s,t}^{\{j_{s+1}, \dots, j_n\}}(x)}$$

is decreasing in  $x$  whenever  $k - l \leq u - s$  and  $m - k \geq n - u$ . On the other hand, if  $a_1, \dots, a_m$  and  $b_1, \dots, b_n$  are nonnegative univariate functions such that  $a_i(x)/b_j(x)$  is decreasing in  $x$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , then  $\sum_i a_i(x)/\sum_j b_j(x)$  is decreasing in  $x$ . Based on this fact, we now can conclude that the ratio in (3.1) is decreasing in  $x$ .

Since the assumption  $X_i \leq_{hr} Y_j$  implies that  $X_{i,t} \leq_{hr} Y_{j,t}$  for all  $i$  and  $j$ , part (b) can be readily established by following a similar argument and then using part (b) of Lemma 3.1.

Finally, we present the corresponding result on the inactivity time. For  $1 \leq l < k \leq m, 1 \leq s < u \leq n$ , and  $t \geq 0$ , let  $IT_{l,k,m}^X(t)$  and  $IT_{s,u,n}^Y(t)$  be the corresponding inactivity times.

**Theorem 3.2.** For two systems with independent (not necessarily identical) components  $X_1, X_2, \dots, X_m$  and  $Y_1, Y_2, \dots, Y_n$ ,

- (a) if  $X_i \geq_{lr} Y_j$  for all  $i$  and  $j$  then  $IT_{l,k,m}^X(t) \leq_{lr} IT_{s,u,n}^Y(t)$  whenever  $l \geq s$  and  $k-l \leq u-s$ ;
- (b) if  $X_i \geq_{rh} Y_j$  for all  $i$  and  $j$  then  $IT_{l,k,m}^X(t) \leq_{hr} IT_{s,u,n}^Y(t)$  whenever  $l \geq s$  and  $k-l \leq u-s$ .

*Proof.* (a) For two random variables  $X$  and  $Y$ , it is well known that  $X \geq_{lr} Y$  implies that  $-X \leq_{lr} -Y$ . Using this fact and part (a) of Theorem 3.1, it follows that, for  $1 \leq l < k \leq m$  and  $1 \leq s < u \leq n$ ,

$$[(-X)_{k:m} - y \mid (-X)_{l:m} \leq y < (-X)_{l+1:m}] \leq_{lr} [(-Y)_{u:n} - y \mid (-Y)_{s:n} \leq y < (-Y)_{s+1:n}]$$

whenever  $k-l \leq u-s, m-k \geq n-u$ , and  $y \in \mathbb{R}$ , which is equivalent to

$$[-X_{m-k+1:m} \mid -X_{m-l+1:m} \leq y < -X_{m-l:m}] \leq_{lr} [-Y_{n-u+1:n} \mid -Y_{n-s+1:n} \leq y < -Y_{n-s:n}]$$

whenever  $k-l \leq u-s, m-k \geq n-u$ , and  $y \in \mathbb{R}$ . That is,

$$[t - X_{m-k+1:m} \mid X_{m-l:m} \leq t < X_{m-l+1:m}] \leq_{lr} [t - Y_{n-u+1:n} \mid Y_{n-s:n} \leq t < Y_{n-s+1:n}]$$

whenever  $k-l \leq u-s, m-k \geq n-u$ , and  $t \in \mathbb{R}$ , which means that

$$[t - X_{l:m} \mid X_{k:m} \leq t < X_{k+1:m}] \leq_{lr} [t - Y_{s:n} \mid Y_{u:n} \leq t < Y_{u+1:n}]$$

whenever  $l \geq s, k-l \leq u-s$  for  $1 \leq l < k \leq m$  and  $1 \leq s \leq u \leq n$ . This is the desired result.

(b) By part (b) of Theorem 3.1 and the fact that  $X \geq_{rh} Y$  implies that  $-X \leq_{hr} -Y$ , the proof can be completed using an argument similar to that of part (a).

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