5

Background fields and world-volume actions

T-duality is clearly a remarkable phenomenon that is highly indicative of the different view string theory has of spacetime from that of field theories. This heralds a rather rich landscape of possibilities for new physics, and indeed T-duality will govern much of what we will study in the rest of this book, either directly or indirectly. So far, we have uncovered it at the level of the string spectrum, and have used it to discover D-branes and orientifolds. However, we have so far restricted ourselves to flat spacetime backgrounds, with none of the other fields in the string spectrum switched on. In this chapter, we shall study the action of T-duality when the massless fields of the string theory take on non-trivial values, giving us curved backgrounds and/or gauge fields on the world-volume of the D-branes. It is also important to uncover further aspects of the dynamics of D-branes in non-trivial backgrounds, and we shall also uncover an action to describe this here.

5.1 T-duality in background fields

The first thing to notice is that T-duality acts non-trivial on the dilaton, and therefore modifies the string coupling[16, 17]. After dimensional reduction on a circle of radius R, the effective 25-dimensional string coupling read off from the reduced string frame supergravity action is now $g_{\rm s} = e^{\Phi}(2\pi R)^{-1/2}$. Since the resulting 25-dimensional theory is supposed to have the same physics, by T-duality, as a theory with a dilaton $\tilde{\Phi}$, compactified on a circle of radius R', it is required that this coupling is equal to $\tilde{g}_{\rm s} = e^{\tilde{\Phi}}(2\pi R')^{-1/2}$, the string coupling of the dual 25-dimensional theory:

$$e^{\tilde{\Phi}} = e^{\Phi}\frac{\alpha'^{1/2}}{R}. \qquad (5.1)$$

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This is just part of a larger statement about the T-duality transformation properties of background fields in general. Starting with background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, let us first T-dualise in one direction, which we shall label X^{25}, as before. In other words, X^{25} is a direction which is a circle of radius R, and the dual circle X'^{25} is a circle of radius $R' = \alpha'/R$.

We may start with the two dimensional sigma model (2.103) with background fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, and assume that locally, all of the fields are independent of the direction X^{25}. In this case, we may write an equivalent action by introducing a Lagrange multiplier, which we shall call X'^{25}:

$$\begin{aligned} S_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma g^{1/2}\Big\{ & g^{ab}\left[G_{25,25}v_a v_b + 2G_{25,\mu}v_a\partial_b X^\mu + G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right] \\ & + i\epsilon^{ab}\Big[2B_{25,\mu}v_a\partial_b X^\mu + B_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + 2X'^{25}\partial_a v_b\Big] + \alpha' R\Phi\Big\}. \end{aligned} \tag{5.2}$$

Since the equation of motion for the Lagrange multiplier is

$$\frac{\partial \mathcal{L}}{\partial X'^{25}} = i\epsilon^{ab}\partial_a v_b = 0,$$

we can write a solution as $v_b = \partial_b\phi$ for any scalar ϕ, which we might as well call X^{25}, since upon substitution of this solution back into the action, we get our original action in (2.103).

Instead, we can find the equation of motion for the quantity v_a:

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial v_a} - \frac{\partial}{\partial\sigma_b}\left(\frac{\partial\mathcal{L}}{\partial(\partial_b v_a)}\right) = 0 \\ &\quad = g^{ab}\left[G_{25,25}v_b + G_{25,\mu}\partial_b X^\mu\right] + i\epsilon^{ab}\Big[B_{25,\mu}\partial_b X^\mu + \partial_b X'^{25}\Big], \end{aligned} \tag{5.3}$$

which, upon solving it for v_a and substituting back into the equations gives an action of the form (2.103), but with fields $\widetilde{G}_{\mu\nu}$ and $\widetilde{B}_{\mu\nu}$ given by:

$$\begin{aligned} \widetilde{G}_{25,25} &= \frac{1}{G_{25,25}}; \qquad e^{2\widetilde{\Phi}} = \frac{e^{2\Phi}}{G_{25,25}}, \\ \widetilde{G}_{\mu 25} &= \frac{B_{\mu 25}}{G_{25,25}}; \qquad \widetilde{B}_{\mu 25} = \frac{G_{\mu 25}}{G_{25,25}}, \\ \widetilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu 25}G_{\nu 25} - B_{\mu 25}B_{\nu 25}}{G_{25,25}}, \\ \widetilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu 25}G_{\nu 25} - G_{\mu 25}B_{\nu 25}}{G_{25,25}}, \end{aligned} \tag{5.4}$$

where a *one loop* (not tree level) world-sheet computation (e.g. by checking the β-function equations again, or by considering the new path integral

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measure induced by integrating out v_a), gives the new dilaton. This fits with the fact that it couples at the next order in α' (which plays the role of $\hbar$ on the world-sheet) as discussed previously.

Of course, we can T-dualise on many (say d) independent circles, forming a torus T^d. It is not hard to deduce that one can succinctly write the resulting T-dual background as follows. If we define the $D\times D$ metric

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \tag{5.5}$$

and if the circles are in the directions X^i, $i = 1, \dots, d$, with the remaining directions labelled by X^a, then the dual fields are given by

$$\begin{aligned}\widetilde{E}_{ij} &= E^{ij}, \qquad \widetilde{E}_{aj} = E_{ak}E^{kj}, \qquad e^{2\widetilde{\Phi}} = e^{2\Phi}\det(E^{ij}),\\ \widetilde{E}_{ab} &= E_{ab} - E_{ai}E^{ij}E_{jb},\end{aligned} \tag{5.6}$$

where $E_{ik}E^{kj} = \delta_i^{\ j}$ defines E^{ij} as the inverse of E_{ij}. We will find this succinct form of the $O(d,d)$ T-duality transformation very useful later on.

5.2 A first look at the D-brane world-volume action

The D-brane is a dynamical object, and as such, feels the force of gravity. In fact, it must be able to respond to the values of the various background fields in the theory. This is especially obvious if one recalls that the D-branes' location and shaped is controlled (in at least one way of describing them) by the open strings which end on them. These strings respond to the background fields in ways we have already studied (we have written world-sheet actions for them), and so should the D-branes. We must find a world-volume action describing their dynamics.

If we introduce coordinates ξ^a, $a = 0, \dots, p$ on the brane, we can begin to write an action for the dynamics of the brane in terms of fields living on the world-volume in much the same way that we did for the string, in terms of fields living on the world-sheet. The background fields will act as generalised field-dependent couplings. As we discussed before, the fields on the brane are the embedding $X^\mu(\xi)$ and the gauge field $A_a(\xi)$. We shall ignore the latter for now and concentrate just on the embedding part. By direct analogy to the particle and string case studied in chapter 2, the action is

$$S_p = -T_p \int d^{p+1}\xi\, e^{-\Phi} \det{}^{1/2} G_{ab}, \tag{5.7}$$

where G_{ab} is the induced metric on the brane, otherwise known as the

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'pull-back' of the spacetime metric $G_{\mu\nu}$ to the brane:

$$G_{ab} \equiv \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}. \tag{5.8}$$

T_p is the tension of the Dp-brane, which we shall discuss at length later. The dilaton dependence $e^{-\Phi} = g_s^{-1}$ arises because this is an open string tree level action, and so this is the appropriate function of the dilaton to introduce.

N.B. The world-volume reparametrisation invariant action we have just written is in terms of the determinant of the metric. It is a common convention to leave the a, b indices dangling in writing this action and its generalisations, and we shall adopt that somewhat loose notation here. More careful authors sometimes use other symbols, like $\det^{1/2} P[G]$, where the P denotes the pull-back, and G means the metric, now properly thought of as a matrix whose determinant is to be taken. Here, the meaning of what we write using the looser notation should always be clear from the context.

Of course, this cannot be the whole story, and indeed it is clear that we shall need a richer action, since the rules of T-duality action on the background fields mean that T-dualising to a D$(p+1)$– or D$(p-1)$-brane's action will introduce a dependence on $B_{\mu\nu}$, since it mixes with components of the metric. Furthermore, there will be mixing with components of a world-volume gauge field, since some of kinetic terms for the transverse fields, $\partial_a X^m$, $m = p+1, \ldots, D-1$, implicit in the action (5.8), will become derivatives of gauge fields, $2\pi\alpha'\partial_a A_m$ according to the rules of T-duality for open strings deduced in the previous chapter. We shall construct the full T-duality respecting action in the next subsection. Before we do that, let us consider what we can learn about the tension of the D-brane from this simple action, and what we learned about the transformation of the dilaton.

The tension of the brane controls its response to outside influences which try to make it change its shape, absorb energy, etc., just as we saw for the tension of a string. We shall compute the actual value of the tension in chapter 6. Here, we are going to uncover a useful recursion relation relating the tensions of different D-branes, which follows from T-duality[76, 29]. The mass of a Dp-brane wrapped around a p-torus T^p is

$$T_p e^{-\Phi} \prod_{i=1}^{p} (2\pi R_i). \tag{5.9}$$

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T-dualising on the single direction X^p and recalling the transformation (5.1) of the dilaton, we can rewrite the mass (5.9) in the dual variables:

$$T_p(2\pi\sqrt{\alpha'})e^{-\Phi'}\prod_{i=1}^{p-1}(2\pi R_i) = T_{p-1}e^{-\Phi'}\prod_{i=1}^{p-1}(2\pi R_i). \qquad (5.10)$$

Hence,

$$T_p = T_{p-1}/2\pi\sqrt{\alpha'} \qquad \Rightarrow \qquad T_p = T_{p'}(2\pi\sqrt{\alpha'})^{p'-p}, \qquad (5.11)$$

where we performed the duality recursively to deduce the general relation.

The next step is to take into account new couplings for the embedding coordinates/fields which result of other background spacetime fields like the antisymmetric tensor $B_{\mu\nu}$. This again appears as an induced tensor B_{ab} on the worldvolume, via a formula like (5.8).

It is important to notice that that there is a restriction due to spacetime gauge symmetry on the precise combination of B_{ab} and A^a which can appear in the action. The combination $B_{ab} + 2\pi\alpha' F_{ab}$ can be understood as follows. In the world-sheet sigma model action of the string, we have the usual closed string term (2.103) for B and the boundary action (2.108) for A. So the fields appear in the combination:

$$\frac{1}{2\pi\alpha'}\int_{\mathcal{M}} B + \int_{\partial\mathcal{M}} A. \qquad (5.12)$$

We have written everything in terms of differential forms, since B and A are antisymmetric. For example $\int A \equiv \int A_a d\xi^a$.

This action is invariant under the spacetime gauge transformation $\delta A = d\lambda$. However, the spacetime gauge transformation $\delta B = d\zeta$ will give a surface term which must be cancelled with the following gauge transformation of A: $\delta A = -\zeta/2\pi\alpha'$. So the combination $B + 2\pi\alpha' F$, where $F = dA$ is invariant under both symmetries; this is the combination of A and B which must appear in the action in order for spacetime gauge invariance to be preserved.

5.2.1 *World-volume actions from tilted D-branes*

There are many ways to deduce pieces of the world-volume action. One way is to redo the computation for Weyl invariance of the complete sigma model, including the boundary terms, which will result in the $(p+1)$-dimensional equations of motion for the world-volume fields G_{ab}, B_{ab} and A_a. One can then deduce the $p+1$-dimensional world-volume action from

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which those equations of motion may be derived. We will comment on this below.

Another way, hinted at in the previous subsection, is to use T-duality to build the action piece by piece. For the purposes of learning more about how the branes work, and in view of the various applications to which we will put the branes, this second way is perhaps more instructive.

Consider[38] a D2-brane extended in the X^1 and X^2 directions, and let there be a constant gauge field F_{12}. (We leave the other dimensions unspecified, so the brane could be larger by having extent in other directions. This will not affect our discussion.) We can choose a gauge in which $A_2 = X^1 F_{12}$. Now consider T-dualising along the x^2-direction. The relation (4.68) between the potential and coordinate gives

$$X'^2 = 2\pi\alpha' X^1 F_{12}, \tag{5.13}$$

This says that the resulting D1-brane is tilted at an angle*

$$\theta = \tan^{-1}(2\pi\alpha' F_{12}) \tag{5.14}$$

to the X^2-axis! This gives a geometric factor in the D1-brane world-volume action,

$$S \sim \int_{\mathrm{D1}} ds = \int dX^1 \sqrt{1 + (\partial_1 X'^2)^2} = \int dX^1 \sqrt{1 + (2\pi\alpha' F_{12})^2}. \tag{5.15}$$

We can always boost the D-brane to be aligned with the coordinate axes and then rotate to bring $F_{\mu\nu}$ to block-diagonal form, and in this way we can reduce the problem to a product of factors like (5.15) giving a determinant:

$$S \sim \int d^D X \, \det{}^{1/2}(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}). \tag{5.16}$$

This is the Born–Infeld action.[42]

In fact, this is the complete action (in a particular 'static' gauge which we will discuss later) for a space-filling D25-brane in flat space, and with the dilaton and antisymmetric tensor field set to zero. In the language of section 2.7, Weyl invariance of the open string sigma model (2.108) amounts to the following analogue of (2.105) for the open string sector:

$$\beta^A_\mu = \alpha' \left(\frac{1}{1 - (2\pi\alpha' F)^2} \right)^{\nu\lambda} \partial_{(\nu} F_{\lambda)\mu} = 0, \tag{5.17}$$

* The reader concerned about achieving irrational angles and hence densely filling the (x^1, x^2) torus should suspend disbelief until chapter 8. There, when we work in the fully consistent quantum theory of superstrings, it will be seen that the fluxes are quantised in just the right units to make this sensible.

these equations of motion follow from the action. In fact, in contrast to the Maxwell action written previously (2.107), and the closed string action (2.106), this action is true to all orders in α', although only for slowly varying field strengths; there are corrections from derivatives of $F_{\mu\nu}$.[32]

5.3 The Dirac–Born–Infeld action

We can uncover a lot of the rest of the action by simply dimensionally reducing. Starting with (5.16), where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual (we will treat the non-Abelian case later) let us assume that $D-p-1$ spatial coordinates are very small circles, small enough that we can neglect all derivatives with respect to those directions, labelled X^m, $m = p+1, \dots, D-1$. (The uncompactified coordinates will be labelled X^a, $a = 0, \dots, p$.) In this case, the matrix whose determinant appears in (5.16) is:

$$\begin{pmatrix} N & -A^T \\ A & M \end{pmatrix}, \tag{5.18}$$

where

$$N = \eta_{ab} + 2\pi\alpha' F_{ab}; \qquad M = \delta_{mn}; \qquad A = 2\pi\alpha' \partial_a A_m. \tag{5.19}$$

Using the fact that its determinant can be written as $|M||N + A^T M^{-1} A|$, our action becomes[56]

$$S \sim -\int d^{p+1}X \, \det{}^{1/2}(\eta_{ab} + \partial_a X^m \partial_b X_m + 2\pi\alpha' F_{ab}), \tag{5.20}$$

up to a numerical factor (coming from the volume of the torus we reduced on. Once again, we used the T-duality rules (4.68) to replace the gauge fields in the T-dual directions by coordinates: $2\pi\alpha' A_m = X^m$.

This is (nearly) the action for a Dp-brane and we have uncovered how to write the action for the collective coordinates X^m representing the fluctuations of the brane transverse to the world-volume. There now remains only the issue of putting in the case of non-trivial metric, $B_{\mu\nu}$ and dilaton. This is easy to guess given that which we have encountered already:

$$S_p = -T_p \int d^{p+1}\xi \, e^{-\Phi} \det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}). \tag{5.21}$$

This is the Dirac–Born–Infeld Lagrangian, for arbitrary background fields. The factor of the dilaton is again a result of the fact that all of this physics arises at open string tree level, hence the factor g_s^{-1}, and the B_{ab} is in the right place because of spacetime gauge invariance. T_p and G_{ab} are in the right place to match onto the discussion we had when we computed the tension. Instead of using T-duality, we could have also deduced this action by a generalisation of the sigma model methods described earlier, and in fact this is how it was first derived in this context[34].

We have re-introduced independent coordinates ξ^a on the world-volume. Note that the actions given in equations (5.15) and (5.20) were written using a choice where we aligned the world-volume with the first $p+1$ spacetime coordinates as $\xi^a = X^a$, leaving the $D-p-1$ transverse coordinates called X^m. We can always do this using world-volume and spacetime diffeomorphism invariance. This choice is called the 'static gauge', and we shall use it quite a bit in these notes. Writing this out (for vanishing dilaton) using the formula (5.8) for the induced metric, for the case of $G_{\mu\nu} = \eta_{\mu\nu}$ we see that we get the action (5.20).

5.4 The action of T-duality

It is amusing[41, 51] to note that our full action obeys (as it should) the rules of T-duality which we already wrote down for our background fields. The action for the Dp-brane is built out of the determinant $|E_{ab} + 2\pi\alpha' F_{ab}|$, where the $(a, b = 0, \dots, p)$ indices on E_{ab} mean that we have performed the pullback of $E_{\mu\nu}$ (defined in (5.5)) to the world-volume. This matrix becomes, if we T-dualise on n directions labelled by X^i and use the rules we wrote in (5.6):

$$\left| \begin{array}{cc} E_{ab} - E_{ai}E^{ij}E_{jb} + 2\pi\alpha' F_{ab} & -E_{ak}E^{kj} - \partial_a X^i \\ E^{ik}E_{kb} + \partial_b X^i & E^{ij} \end{array} \right|, \tag{5.22}$$

which has determinant $|E^{ij}||E_{ab} + 2\pi\alpha' F_{ab}|$. In forming the square root, we get again the determinant needed for the definition of a T-dual DBI action, as the extra determinant $|E^{ij}|$ precisely cancels the determinant factor picked up by the dilaton under T-duality. (Recall, E^{ij} is the inverse of E_{ij}.)

Furthermore, the tension $T_{p'}$ comes out correctly, because there is a factor of $\Pi_i^n(2\pi R_i)$ from integrating over the torus directions, and a factor $\Pi_i^n(R_i/\sqrt{\alpha'})$ from converting the factor $e^{-\Phi}$, (see equation (5.1)), which fits nicely with the recursion formula (5.11) relating T_p and $T_{p'}$.

The above was done as though the directions on which we dualised were all Neumann or all Dirichlet. Clearly, we can also extrapolate to the more general case.

5.5 Non-Abelian extensions

For N D-branes the story is more complicated. The various fields on the brane representing the collective motions, A_a and X^m, become matrices

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valued in the adjoint. In the Abelian case, the various spacetime background fields (here denoted F_μ for the sake of argument) which can appear on the world-volume typically depend on the transverse coordinates X^m in some (possibly) non-trivial way. In the non-Abelian case, with N D-branes, the transverse coordinates are really $N \times N$ matrices, $2\pi\alpha'\Phi^m$, since they are T-dual to non-Abelian gauge fields as we learned in previous sections, and so inherit the behaviour of gauge fields (see equation (4.68)). We write them as $\Phi^m = X^m/(2\pi\alpha')$. So not only should the background fields F_μ depend on the Abelian part, but they ought to possibly depend (implicitly or explicitly) on the full non-Abelian part as $F(\Phi)_\mu$ in the action.

Furthermore, in (5.21) we have used the partial derivatives $\partial_a X^\mu$ to pull back spacetime indices μ to the world-volume indices, a, e.g. $F_a = F_\mu \partial_a X^\mu$, and so on. To make this gauge covariant in the non-Abelian case, we should pull back with the covariant derivative: $F_a = F_\mu \mathcal{D}_a X^\mu = F_\mu(\partial_a X^\mu + [A_a, X^\mu])$.

With the introduction of non-Abelian quantities in all of these places, we need to consider just how to perform a trace, in order to get a gauge invariant quantity to use for the action. Starting with the fully Neumann case (5.16), a first guess is that things generalise by performing a trace (in the fundamental of $U(N)$) of the square rooted expression. The meaning of the Tr needs to be stated, It is proposed that is means the 'symmetric' trace, denoted 'STr' which is to say that one symmetrises over gauge indices, consequently ignoring all commutators of the field strengths encountered while expanding the action[43]. (This suggestion is consistent with various studies of scattering amplitudes and also the BPS nature of various non-Abelian soliton solutions. There is still apparently some ambiguity in the definition which results in problems beyond fifth order in the field strength[44, 343].)

Once we have this action, we can then again use T-duality to deduce the form for the lower dimensional, Dp-brane actions. The point is that we can reproduce the steps of the previous analysis, but keeping commutator terms such as $[A_a, \Phi^m]$ and $[\Phi^m, \Phi^n]$. We will not reproduce those steps here, as they are similar in spirit to that which we have already done (for a complete discussion, the reader is invited to consult some of the literature[45].) The resulting action is:

$$
\begin{aligned}
S_p &= -T_p \int d^{p+1}\xi e^{-\Phi}\mathcal{L}, \quad \text{where} \\
\mathcal{L} &= \mathrm{STr}\left\{\det{}^{1/2}[E_{ab} + E_{ai}(Q^{-1}-\delta)^{ij}E_{jb} + 2\pi\alpha' F_{ab}]\det{}^{1/2}[Q^i{}_j]\right\}, \qquad (5.23)
\end{aligned}
$$

where $Q^i{}_j = \delta^i{}_j + i2\pi\alpha'[\Phi^i, \Phi^k]E_{kj}$, and we have raised indices with E^{ij}.

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5.6 D-branes and gauge theory

In fact, we are now in a position to compute the constant C in equation (2.107), by considering N D25-branes, which is the same as an ordinary (fully Neumann) N-valued Chan–Paton factor. Expanding the D25-brane Lagrangian (5.16) to second order in the gauge field, we get

$$-\frac{T_{25}}{4}(2\pi\alpha')^2 e^{-\Phi}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}, \tag{5.24}$$

with the trace in the fundamental representation of $U(N)$. This gives the precise numerical relation between the open and closed string couplings.

Actually, with Dirichlet and Neumann directions, performing the same expansion, and in addition noting that

$$\det[Q^i{}_j] = 1 - \frac{(2\pi\alpha')^2}{4}[\Phi^i,\Phi^j][\Phi^i,\Phi^j] + \cdots, \tag{5.25}$$

one can write the leading order action (5.23) as

$$S_p = -\frac{T_p(2\pi\alpha')^2}{4}\int d^{p+1}\xi\, e^{-\Phi}\mathrm{Tr}\left[F_{ab}F^{ab} + 2\mathcal{D}_a\Phi^i\mathcal{D}_a\Phi^i + [\Phi^i,\Phi^j]^2\right]\Big\}. \tag{5.26}$$

This is the dimensional reduction of the D-dimensional Yang–Mills term, displaying the non-trivial commutator for the adjoint scalars. This is an important term in many modern applications, as we shall see. Note that the $(p+1)$-dimensional Yang–Mills coupling for the theory on the branes is

$$g^2_{\mathrm{YM},p} = g_{\mathrm{s}}T_p^{-1}(2\pi\alpha')^{-2}. \tag{5.27}$$

This is worth noting[70]. With the superstring value of T_p which we will compute later, it is used in many applications to give the correct relation between gauge theory couplings and string quantities.

5.7 BPS lumps on the world-volume

We can of course treat the Dirac–Born–Infeld action as an interesting theory in its own right, and seek for interesting solutions of it. These solutions will have both a $(p+1)$-dimensional interpretation and a D-dimensional one.

We shall not dwell on this in great detail, but include a brief discussion here to illustrate an important point, and refer to the literature for more complete discussions.[55] More details will appear when we get to the supersymmetric case. One can derive an expression for the energy density contained in the fields on the world-volume:

$$\mathcal{E}^2 = E^a E^b F_{ca} F_{cb} + E^a E^b G_{ab} + \det(G + 2\pi\alpha' F), \tag{5.28}$$

where here the matrix F_{ab} contains only the magnetic components (i.e. no time derivatives) and E^a are the electric components, subject to the Gauss Law constraint $\vec{\nabla} \cdot \vec{E} = 0$. Also, as before

$$G_{ab} = \eta_{ab} + \partial_a X^m \partial_b X^m, \qquad m = p+1, \ldots, D-1. \tag{5.29}$$

Let us consider the case where we have no magnetic components and only one of the transverse fields, say X^{25}, switched on. In this case, we have

$$\mathcal{E}^2 = (1 \pm \vec{E} \cdot \vec{\nabla} X^{25})^2 + (\vec{E} \mp \vec{\nabla} X^{25})^2, \tag{5.30}$$

and so we see that we have the Bogomol'nyi condition

$$\mathcal{E} \geq |\vec{E} \cdot \vec{\nabla} X^{25}| + 1. \tag{5.31}$$

This condition is saturated if $\vec{E} = \pm\vec{\nabla} X^{25}$. In such a case, we have

$$\nabla^2 X^{25} = 0 \qquad \Rightarrow \qquad X^{25} = \frac{c_p}{r^{p-2}}, \tag{5.32}$$

a harmonic solution, where c_p is a constant to be determined. The total energy (beyond that of the brane itself) is, integrating over the world-volume:

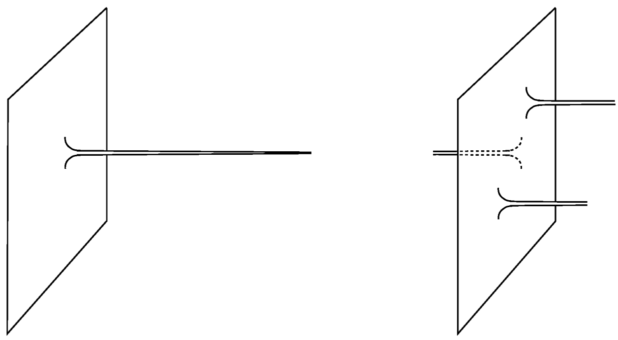
$$\begin{aligned} E_{\rm tot} &= \lim_{\epsilon \to \infty} T_p \int_\epsilon^\infty r^{p-1} dr d\Omega_{p-1} (\vec{\nabla} X^{25})^2 = \lim_{\epsilon \to \infty} T_p \frac{c_p^2 (p-2) \Omega_{p-1}}{\epsilon^{p-2}} \\ &= \lim_{\epsilon \to \infty} T_p c_p (p-2) \Omega_{p-1} X^{25}(\epsilon), \end{aligned} \tag{5.33}$$

where Ω_{p-1} is the volume of the sphere S^{p-1} surrounding our point charge source, and we have cut off the divergent integral by integrating down to $r = \epsilon$. (We will save the case of $p = 1$ for later[140, 60].) Now we can choose† a value of the electric flux such that we get $(p-2)c_p\Omega_{p-1}T_p = (2\pi\alpha')^{-1}$. Putting this into our equation for the total energy, we see that the (divergent) energy of our configuration is:

$$E_{\rm tot} = \frac{1}{2\pi\alpha'} X^{25}(\epsilon). \tag{5.34}$$

What does this mean? Well, recall that $X^{25}(\xi)$ gives the transverse position of the brane in the X^{25} direction. So we see that the brane

† In the supersymmetric case, this has a physical meaning, since overall consistency of the D-brane charges set a minimum electric flux. Here, it is more arbitrary, and so we choose a value by hand to make the point we wish to illustrate.



(a) (b)

Fig. 5.1. The D-dimensional interpretation of the BIon solution. (a) It is an infinitely long spike representing a fundamental string ending on the D-brane. (b) BIons are BPS and therefore can be added together at no cost to make a multi-BIon solution.

has grown a semi-infinite spike at $r = 0$, and the base of this spike is our point charge. The interpretation of the divergent energy is simply the (infinite) length of the spike multiplied by a mass per unit length. But this mass per unit length is precisely the fundamental string tension $T = (2\pi\alpha')^{-1}$! In other words, the spike solution is the fundamental string stretched perpendicular to the brane and ending on it, forming a point electric charge, known as a 'BIon'; see figure 5.1(a). In fact, a general BIon includes the non-linear corrections to this spike solution, which we have neglected here, having only written the linearised solution.

It is a worthwhile computation to show that if test source with the same charges is placed on the brane, there is no force of attraction or repulsion between it and the source just constructed, as would happen with pure Maxwell charges. This is because our sources have in addition to electric charge, some scalar (X^{25}) charge, which can also be attractive or repulsive. In fact, the scalar charges are such that the force due to electromagnetic charges is cancelled by the force of the scalar charge, another characteristic property of these solutions, which are said to be 'Bogomol'nyi–Prasad–Sommerfield' (BPS)-saturated[61, 62]. We shall encounter solutions with this sort of behaviour a number of times in what is to follow.

Because of this property, the solution is easily generalised to include any number of BIons, at arbitrary positions, with positive and negative charges. The two choices of charge simply represents strings either leaving from, or arriving on the brane; see figure 5.1(b).