LIGHTCURVE INVERSION AND SURFACE REFLECTIVITY

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A simplified lightcurve inversion method is applied for the special case where observations are taken in the equatorial plane of the asteroid. The solution is obtained in terms of a spotted two-surface model using Lambert's law and geometrical reflectivities.

The general problem of interpreting the lightcurve of a rotating body in terms of its shape and surface spottiness has been discussed in detail by Russell (1906). However, in the special case where the rotational axis is perpendicular to the line of sight, the analysis may be greatly simplified. Although the ambiguity between the shape and spot contributions to the light variation remains unresolved, it is possible to examine the type of surface reflectivity law and to set some limits on the range of albedo variation that will be consistent with the observed lightcurve.

Without loss of generality (insofar as being able to reproduce the observed lightcurve is concerned) we can assume the asteroid to be spherical in shape. The surface is taken to consist of bright and dark areas that reflect either geometrically ($\propto \cos\gamma$) or diffusely according to Lambert's law ($\propto \cos^2\gamma$) where γ, defined by

$$\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)$$

is the angle between the outward normal of a surface element and the line of sight in the polar coordinates centered on the asteroid. The polar angle and longitude of the sub-Earth point are designated by θ_0 and ϕ_0, respectively.

In the special case when the observer is in the equatorial plane of the asteroid, the integration over the visible hemisphere is greatly simplified, and we can write the brightness variation of the asteroid as

$$g(\phi_0) = B + \frac{1}{\pi}\int_0^{\pi}\int_{\phi_0-\pi/2}^{\phi_0+\pi/2}[A\sin^2\theta\cos^2(\phi-\phi_0) - B\sin\theta\cos(\phi-\phi_0)]\,h(\phi)\sin\theta\,d\theta d\phi \quad (1)$$

where $g(\phi_0)$ is the ratio of reflected to incident light; B is the normal albedo of the geometrically reflecting surface area; A is the normal albedo of the diffusely reflecting area; and $h(\phi)$ is the spot distribution function that gives, as a function of longitude, the fractional area that reflects diffusely according to the Lambert law. Because no information regarding the latitude distribution of bright and dark areas appears in the lightcurve when the asteroid is viewed from within its equatorial plane, $h(\phi)$ is taken to be constant with latitude. Assuming that $h(\phi)$ can be expressed in the form

$$h(\phi) = \sum_{n=0}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \tag{2}$$

equation (1) can be integrated to obtain a Fourier series in ϕ_0. By comparing the resulting Fourier coefficients with the corresponding terms obtained from a Fourier analysis of the observed lightcurve, we find that the coefficients for the $\cos n\phi_0$ terms are related by

$$C_0 = B + \left(\frac{2}{3}A - B\right) a_0 \tag{3}$$

$$C_1 = \left(\frac{16}{9\pi}A - \frac{\pi}{4}B\right) a_1 \tag{4}$$

$$C_2 = \frac{1}{3}(A - B)\, a_2 \tag{5}$$

$$C_n = (-1)^{(n+1)/2} \frac{16A}{3\pi n(n^2 - 4)} a_n \qquad (n = 3, 5, \ldots) \tag{6}$$

$$C_n = (-1)^{n/2} \frac{B}{n^2 - 1} a_n \qquad (n = 4, 6, \ldots) \tag{7}$$

where the C_n are the Fourier coefficients obtained from the observed lightcurve and the a_n are the coefficients defined in equation (2). The same relationships apply for the $\sin n\phi_0$ terms.

The above set of relations contains the available information regarding the relative proportion and longitude distribution of geometrically and diffusely reflecting surface areas and the range of albedo combinations that are compatible with the observed lightcurve. The limits for the allowed albedo range are imposed by the physical requirement that the spot distribution function $h(\phi)$ must not become negative or exceed unity.

Because of the infinity of possible solutions, it appears best to consider families of solutions for constant A/B ratios. By specifying a ratio for A/B and by setting $a_0 = 0.5$, we define a model for which the surface is evenly divided between geometrically and diffusely reflecting areas. (See fig. 1.) This allows for the greatest amplitude fluctuation for $h(\phi)$ and marks the approximate center of the allowed albedo range for the specified A/B ratio. The locus of these points falls along the broken line shown in figure 2. Then, keeping the A/B ratio fixed, increasing (or decreasing) A and B simultaneously until $h(\phi)$ becomes negative (or greater than unity), establishes the range of albedo combinations that are compatible with the physical restriction imposed on $h(\phi)$.

This procedure defines two separate albedo regions—one corresponding to bright spots, the other to dark spots. Because all albedo combinations in a given enclosure are equally capable of reproducing the observed lightcurve to the same degree of accuracy, it is clearly impossible to differentiate between bright spot and dark spot models on the basis of the observed lightcurve alone.

The size and location of the limiting enclosures depends both on the size of the lightcurve coefficients and on the proportionality factors appearing in relations (3) through (7). Generally speaking, each additional Fourier term that is included to approximate the observed lightcurve tends to diminish the size of the allowed albedo region. However, because of observational scatter, the higher order Fourier terms become increasingly unreliable. This is an important factor because the $n = 1$ and $n = 2$ terms contain contributions from both geometrically and diffusely reflecting areas. It is only on the basis of terms

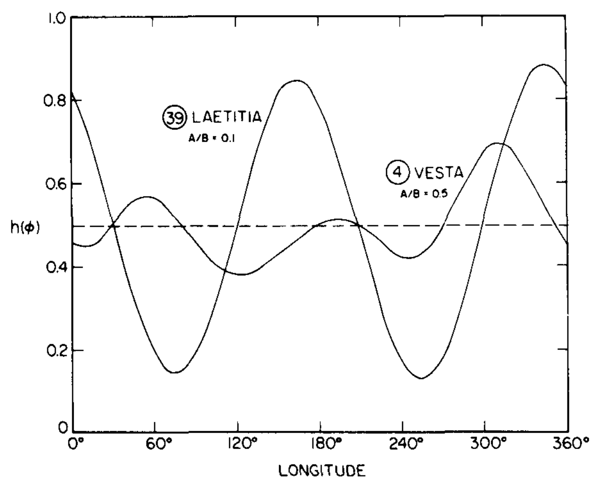

Figure 1.—The fractional area that reflects diffusely according to Lambert's law. For 4 Vesta, $h(\phi)$ includes Fourier terms up to $n = 4$. For 39 Laetitia, terms up to $n = 2$ are included. In both cases $a_0 = 0.5$.

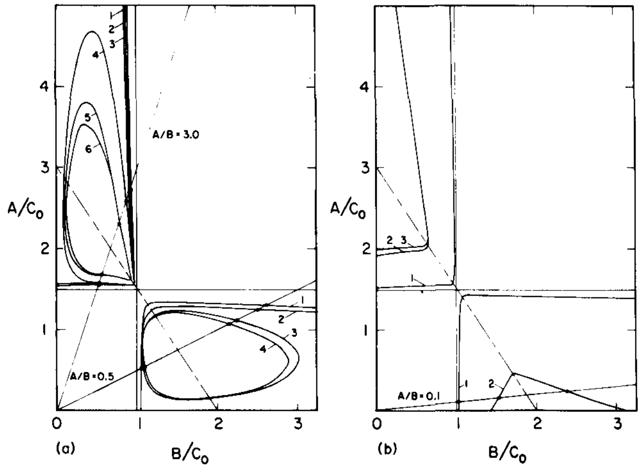

Figure 2.–The most probable albedo combinations for diffuse A and geometrical B reflectivities. Both scales have been normalized in terms of the average normal albedo of the asteroid. The broken line is the locus of points for $a_0 = 0.5$. The numbered curves enclose the allowed albedo combinations as defined by the number of Fourier terms used to approximate the observed lightcurve. (*a*) 4 Vesta, April 10, 1967. (*b*) 39 Laetitia, December 1955.

$n = 3$ and greater that we can verify the presence of diffusely or geometrically reflecting areas.

The lightcurves used in the present analysis are those taken by Gehrels (1967) for 4 Vesta and by van Houten-Groeneveld and van Houten (1958) for 39 Laetitia. These are typical examples of single maximum and double maximum lightcurves that also satisfy the requirement of being observed within the equatorial plane of the asteroid. This is indeed true for 4 Vesta, which is observed in the equatorial plane, although for 39 Laetitia the inclination may be as large as 20°. Also, it should be noted that the phase angle for 4 Vesta was −18° and that of 39 Laetitia was about the same. Although the present calculations are specifically applicable for a zero phase angle, the derived albedo limits should not be significantly affected unless the form of the assumed reflectivity laws changes significantly with phase.

The results of Fourier analyzing the 4 Vesta and 39 Laetitia lightcurves are summarized in table I. For convenience of comparison, the Fourier coefficients of the sin $n\phi_0$ and cos $n\phi_0$ terms have been combined and are normalized with respect to C_0, the mean normal albedo of the asteroid. Also included is the root-mean-square difference between the observed points of the lightcurve and the nth Fourier representation.

The difference between the single maximum and double maximum lightcurves is illustrated by the prominence of the even-order terms in the 39 Laetitia lightcurve and the odd-order terms in the 4 Vesta lightcurve.

TABLE I.—*Fourier Analysis of Lightcurves*

n	4 Vesta		39 Laetitia	
	$(C_n^2 + D_n^2)^{1/2}/C_0$	rms/C_0	$(C_n^2 + D_n^2)^{1/2}/C_0$	rms/C_0
0	1.000	0.042	1.000	0.139
1	.056	.011	.025	.137
2	.005	.011	.203	.022
3	.007	.009	.009	.020
4	.002	.009	.022	.016
5	.005	.008	.012	.015
6	.000	.008	.018	.010

Clearly, the even-order terms for 39 Laetitia may be interpreted as being due to the projected area of an irregular object. This leaves a 5 to 10 percent light variation due to the odd-order terms, which may be associated with either a spotted surface or Lambert law reflectivity. However, it appears likely that if 39 Laetitia is observed closer to its equatorial plane, the size of these terms will be somewhat reduced.

For 4 Vesta, interpreting the 1.4 and 1.0 percent light variation contributed by the $n = 3$ and $n = 5$ terms as significant would indicate the definite presence of a Lambert law contribution to the surface reflectivity. The $n = 5$ term seems to suggest bright diffusely reflecting spots on a darker geometrically reflecting surface. However, the limited accuracy of such high-order terms does not necessarily preclude a model with dark spots on a bright geometrically reflecting surface.

REFERENCES

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Houten-Groeneveld, I. van, and Houten, C. J. van. 1958, Photometric Studies of Asteroids. VII. Astrophys. J. **127**, 253-273.

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DISCUSSION

VEVERKA: In your analysis you assume a surface composed of areas which scatter either "geometrically" ($\propto \cos \gamma$) or "diffusely" ($\propto \cos^2 \gamma$) according to Lambert's law. For asteroids, this is an invalid assumption. For intricate surfaces, in which multiple scattering is not dominant, the first part of the assumption is not too bad at small phase angles, but still, strictly speaking, you can only have "geometric" scattering at opposition and nowhere else. (See, for example, Irvine, 1966.)

However, you can never have Lambert scattering on such a surface at visible wavelengths. At visible wavelengths, even quasi-Lambert scatterers are rare and usually consist of extremely bright patches in which multiple scattering is dominant (for example, snow or MgO). There is no evidence that such areas occur in asteroids, and much evidence that they do not occur (for example, deep negative branches in the polarization curves).

You are therefore trying to force a fit using two generally inappropriate scattering laws, and a probably incorrect shape (a spherical asteroid). Thus the calculation, although interesting, cannot have much application to asteroids.

JOHNSON: The lack of color variation seems to indicate that in many cases the asteroid variation is due to shape rather than spots. Is it possible to make the model yield shape information as well as spot distribution?

LACIS: The even terms of the spot distribution function $h(\phi)$ can be directly associated with the shape of the object. By assuming a constant albedo over the surface, a rough estimate for the shape is given by $R(\phi) = 1 - h(\phi)$. For 39 Laetitia, this relation indicates an oblong object with a length-to-width ratio of approximately 3:2.

KUIPER: When we started our precision photometry of asteroids at McDonald about 1949, we found that the rule was to have two maxima and two minima in the lightcurve. It was concluded that the light variation was primarily due to shape. Variation in surface reflectivity could contribute something, but when the variation is 0.3 mag or more, the main effect must be due to shape.

LACIS: Inverting the lightcurve in terms of a spotted sphere gives us little more than a geometrical model that is capable of reproducing the observed light variation. However, in the case where the observations are made in the equatorial plane of the asteroid, we can infer the type of reflectivity law from the strength of the different Fourier terms present in the observed lightcurve. At opposition, the geometrical reflectivity is a special case of the Lommel-Seeliger law. The assumed Lambert law reflectivity could conceivably refer to a more specular type of reflection law. It is just that the presence of odd Fourier terms ($n = 3, 5, \ldots$) cannot be accounted for in terms of geometrical reflectivity alone.

Also, it may be of interest to note that there is a systematic increase in the odd Fourier terms and a decrease in the even-order terms as the observing point moves away from the equatorial plane of the asteroid. This may be helpful in locating the orientation of the rotational axis.

DISCUSSION REFERENCE

Irvine, W. M. 1966, The Shadow Effect in Diffuse Radiation. J. Geophys. Res. 71, 2931.