GROUP THEORY AND THE PRINCIPLE OF DUALITY

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1. Introduction. In [1] the Principle of Duality of mathematical logic (which states that the duals of equivalent well-formed formulas are also equivalent) is established by first showing that certain transforms related to the duality transform, also possess this property. The transforms involved are $\mathcal{N}$, $\mathcal{M}$, $\mathcal{R}$, and $\mathcal{D}$ which are defined as follows.

Definition of $\mathcal{N}$:

(1) $\mathcal{N}(A) = A$ if A is atomic

(2) $$\mathcal{N}(\sim A) = \begin{cases} \sim A & \text{if } A \text{ is atomic} \\ \mathcal{N}(B) & \text{if } A = \sim B \\ \mathcal{N}(\sim C) \wedge \mathcal{N}(\sim D) & \text{if } A = C \vee D \\ \exists t(\mathcal{N}(\sim E)) & \text{if } A = \forall t E \end{cases}$$

(3) $\mathcal{N}(C \vee D) = \mathcal{N}(C) \vee \mathcal{N}(D)$ whenever $C \vee D$ is a well-formed formula

(4) $\mathcal{N}(\forall t E) = \forall t(\mathcal{N}(E))$ whenever $\forall t E$ is a well-formed formula

Definition of $\mathcal{M}$:

(1) $\mathcal{M}(A) = \sim A$ if A is atomic

(2) $$\mathcal{M}(\sim A) = \begin{cases} A & \text{if } A \text{ is atomic} \\ \mathcal{M}(B) & \text{if } A = \sim B \\ \mathcal{M}(\sim B) \vee \mathcal{M}(\sim C) & \text{if } A = B \vee C \\ \forall t(\mathcal{M}(\sim E)) & \text{if } A = \forall t E \end{cases}$$

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(3) $\mathcal{M}(C \vee D) = \mathcal{M}(C) \wedge \mathcal{M}(D)$ whenever $C \vee D$ is a well-formed formula

(4) $\mathcal{M}(\forall t E) = \exists t(\mathcal{M}(E))$ whenever $\forall t E$ is a well-formed formula.

Definition of $\mathcal{R}$:

(1) $\mathcal{R}(A) = \sim A$ if A is atomic

(2) $$\mathcal{R}(\sim A) = \begin{cases} A & \text{if } A \text{ is atomic} \\ \mathcal{R}(B) & \text{if } A = \sim B \\ \sim\mathcal{R}(A) & \text{otherwise} \end{cases}$$

(3) $\mathcal{R}(C \vee D) = \mathcal{R}(C) \vee \mathcal{R}(D)$ whenever $C \vee D$ is a well-formed formula

(4) $\mathcal{R}(\forall t E) = \forall t(\mathcal{R}(E))$ whenever $\forall t E$ is a well-formed formula.

Definition of $\mathcal{D}$:

(1) $\mathcal{D}(A) = A$ if A is atomic

(2) $$\mathcal{D}(\sim A) = \begin{cases} A & \text{if } A \text{ is atomic} \\ \mathcal{D}(B) & \text{if } A = \sim B \\ \mathcal{D}(\sim B) \vee \mathcal{D}(\sim C) & \text{if } A = B \vee C \\ \forall t(\mathcal{D}(\sim E)) & \text{if } A = \forall t E \end{cases}$$

(3) $\mathcal{D}(C \vee D) = \mathcal{D}(C) \wedge \mathcal{D}(D)$ whenever $C \vee D$ is a well-formed formula

(4) $\mathcal{D}(\forall t E) = \exists t(\mathcal{D}(E))$ whenever $\forall t E$ is a well-formed formula.

The idea is to prove that $\mathcal{N}(A) \equiv A$ and that $\mathcal{M}A \equiv \sim A$ whenever A is a well-formed formula, and that $\vdash \mathcal{R}(A)$ whenever $\vdash A$. The Principle of Duality, namely that $\mathcal{D}(A) \equiv \mathcal{D}(B)$ whenever $A \equiv B$, follows by observing that $\mathcal{D}$ can be factored up to equivalence, i.e. that $\mathcal{R}(\mathcal{M}(A)) \equiv \mathcal{D}(A)$ whenever A is a well-formed formula. This procedure is carried out in [1]. Actually, $\mathcal{D}$ can be factored in a more straightforward manner; namely, $\mathcal{M}\mathcal{R} = \mathcal{D}$, i.e. $\mathcal{M}(\mathcal{R}(A)) = \mathcal{D}(A)$ whenever A is a well-formed formula.

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The transforms $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{D}$ follow a simple pattern; the transform $\mathcal{R}$ deviates slightly from this pattern, at a slight gain in simplicity. Sadly, this is more than offset by complications in the theory. Here, we shall replace $\mathcal{R}$ by $\mathcal{R}'$, a transform that achieves the same goal as $\mathcal{R}$, namely of reversing the effect of $\mathcal{M}$ on atomic well-formed formulas, and fits the pattern of $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{D}$. This requires only a slight change in the definition; but the advantages are immediate and impressive. The technique is to consider all the transforms that fit the pattern of $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{D}$, and to show that they form a group under composition. Using elementary group theory we can determine certain relationships between these transforms. In particular, we shall prove that $\mathcal{D} = \mathcal{M}\mathcal{R}' = \mathcal{R}'\mathcal{M}$; moreover, $\mathcal{D}$ can be expressed as the product of two other pairs of transforms in the group.

2. Normal transforms. The first step is to recognize the pattern shared by $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{D}$. Notice that for each of these transforms, say $\mathcal{J}$, $\mathcal{J}(A)$ is either A or $\sim A$ if A is atomic, $\mathcal{J}(C \vee D)$ is either $\mathcal{J}(C) \vee \mathcal{J}(D)$ or $\mathcal{J}(C) \wedge \mathcal{J}(D)$, and $\mathcal{J}(\forall t E)$ is either $\forall t(\mathcal{J}(E))$ or $\exists t(\mathcal{J}(E))$. Moreover, the image of a well-formed formula of the form "$\sim A$" can be reconstructed from the remaining three parts of the definition of $\mathcal{J}$. For example if A is atomic, then $\mathcal{J}(\sim A) = A$ if $\mathcal{J}(A) = \sim A$, whereas $\mathcal{J}(\sim A) = \sim A$ if $\mathcal{J}(A) = A$. So, the transforms $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{D}$ are characterized by three parameters that we shall denote by "n", "d", and "Q". Generalizing this observation, we present the notion of a normal transform.

DEFINITION 2.1: A syntactical transform $\mathcal{J}$ is said to be normal if and only if

(1) $\mathcal{J}(A) = nA$ if A is atomic.

(2) $$\mathcal{J}(\sim A) = \begin{cases} mA & \text{if } A \text{ is atomic} \\ \mathcal{J}(B) & \text{if } A = \sim B \\ \mathcal{J}(\sim B)\, c\, \mathcal{J}(\sim C) & \text{if } A = B \vee C \\ qt(\mathcal{J}(\sim E)) & \text{if } A = \forall t E \end{cases}$$

(3) $\mathcal{J}(C \vee D) = \mathcal{J}(C)\, d\, \mathcal{J}(D)$ whenever $C \vee D$ is a well-formed formula.

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(4) $\mathcal{T}(\forall t E) = Qt(\mathcal{T}(E))$ whenever $\forall t E$ is a well-formed formula.

where n is $\sim$ or is blank, m is $\sim$ or is blank, and $m \neq n$; $\{d, c\} = \{\vee, \wedge\}$; and $\{Q, q\} = \{\forall, \exists\}$.

We are primarily interested in four of the eight normal transforms, namely $\mathcal{N}$, $\mathcal{M}$, $\mathcal{R}'$, and $\mathcal{D}$, whose parameters are given in Table 1.

	n	d	Q
$\mathcal{N}$		$\vee$	$\forall$
$\mathcal{M}$	$\sim$	$\wedge$	$\exists$
$\mathcal{R}'$	$\sim$	$\vee$	$\forall$
$\mathcal{D}$		$\wedge$	$\exists$

TABLE 1

The following lemmas are useful; throughout, $\mathcal{T}$ is a normal transform with parameters n, d, and Q.

LEMMA 2.1. $\mathcal{T}(nA) = A$ whenever A is atomic.

LEMMA 2.2. $\mathcal{T}(mA) = \sim A$ whenever A is atomic.

LEMMA 2.3. $\mathcal{T}(A d B) = \mathcal{T}(A) \vee \mathcal{T}(B)$ whenever $A d B$ is a well-formed formula

LEMMA 2.4. $\mathcal{T}(A c B) = \mathcal{T}(A) \wedge \mathcal{T}(B)$ whenever $A c B$ is a well-formed formula

LEMMA 2.5. $\mathcal{T}(A \wedge B) = \mathcal{T}(A) c \mathcal{T}(B)$ whenever $A \wedge B$ is a well-formed formula

LEMMA 2.6. $\mathcal{T}(Q t E) = \forall t(\mathcal{T} E)$ whenever $Q t E$ is a well-formed formula

LEMMA 2.7. $\mathcal{T}(q t E) = \exists t(\mathcal{T} E)$ whenever $q t E$ is a well-formed formula

LEMMA 2.8. $\mathcal{T}(\exists t E) = qt(\mathcal{T} E)$ whenever $\exists t E$ is a well-formed formula

We shall show that normal transforms form a group under composition. Let "$\mathcal{J}_1\mathcal{J}_2$" denote the transform that associates $\mathcal{J}_1(\mathcal{J}_2(A))$ with A whenever A is a well-formed formula. First, we point out that $\mathcal{J}_1\mathcal{J}_2$ is normal whenever $\mathcal{J}_1$ and $\mathcal{J}_2$ are normal. Indeed, it is easy to show that the parameters of $\mathcal{J}_1\mathcal{J}_2$ are related to the parameters of $\mathcal{J}_1$ and $\mathcal{J}_2$ as follows. We shall express "n is blank" by writing "n = bl".

THEOREM 2.1. Let $\mathcal{J}_1$ be a normal transform with parameters n_1, d_1, and Q_1, and let $\mathcal{J}_2$ be a normal transform with parameters n_2, d_2, and Q_2. Then $\mathcal{J}_1\mathcal{J}_2$ is a normal transform and its parameters are n, d, and Q where

$$n = \begin{cases} n_1 & \text{if } n_2 = \text{bl} \\ m_1 & \text{if } n_2 = \sim \end{cases}, \quad d = \begin{cases} d_1 & \text{if } d_2 = \vee \\ c_1 & \text{if } d_2 = \wedge \end{cases}, \quad \text{and } Q = \begin{cases} Q_1 & \text{if } Q_2 = \forall \\ q_1 & \text{if } Q_2 = \exists \end{cases}.$$

Using this result, it is easy to verify the following facts.

THEOREM 2.2. $\mathcal{J}\mathcal{J} = \mathcal{N}$ whenever $\mathcal{J}$ is normal.

THEOREM 2.3. $\mathcal{J}\mathcal{N} = \mathcal{J}$ whenever $\mathcal{J}$ is normal.

This establishes that normal transforms form a group under composition; $\mathcal{N}$ is the group identity, and each group element is its own inverse. It is well known that each group of characteristic two is abelian. So, we obtain our next result.

THEOREM 2.4. $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_2\mathcal{J}_1$ whenever $\mathcal{J}_1$ and $\mathcal{J}_2$ are normal.

Next, we work out the parameters of $\mathcal{R}'\mathcal{M}$. By Theorem 2.1, they are bl, $\wedge$, and $\exists$; these are the parameters of $\mathcal{D}$.

LEMMA 2.9. $\mathcal{R}'\mathcal{M} = \mathcal{D}$

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COROLLARY 2.1. $\mathcal{M}\mathcal{R}' = \mathcal{D}$

The next result can be established by applying Theorem 2.1 or algebraically as follows.

COROLLARY 2.2. $\mathcal{D}\mathcal{R}' = \mathcal{M}$

Demonstration. $\mathcal{D}\mathcal{R}' = \mathcal{M}\mathcal{R}'\mathcal{R}' = \mathcal{M}\mathcal{N} = \mathcal{M}$

Notice that the transforms $\mathcal{N}$, $\mathcal{M}$, $\mathcal{R}'$, and $\mathcal{D}$ form a subgroup of our group. Consider the four remaining normal transforms, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{E}$, whose parameters are listed in the following table:

	n	d	Q
$\mathcal{A}$		$\wedge$	$\forall$
$\mathcal{B}$		$\vee$	$\exists$
$\mathcal{C}$	$\sim$	$\wedge$	$\forall$
$\mathcal{E}$	$\sim$	$\vee$	$\exists$

TABLE 2

Applying Theorem 2.1 it is easy to work out the table for our group operation. In particular, we observe that $\mathcal{D}$ can be factored as follows.

THEOREM 2.5. $\mathcal{D} = \mathcal{M}\mathcal{R}' = \mathcal{A}\mathcal{B} = \mathcal{C}\mathcal{E}$

Here is a useful property of normal transforms that can be established by applying the Fundamental Theorem about well formulated formulas.

THEOREM 2.6. Let $\mathcal{J}$ be any normal transform; then $\mathcal{J}(\sim A) \equiv \sim\mathcal{J}(A)$ whenever A is a well-formed formula.

3. Principle of Duality. By applying the Fundamental Theorem about well-formed formulas it is easy to verify the following statements.

LEMMA 3.1. $\mathcal{N}(A) \equiv A$ whenever A is a well-formed formula.

LEMMA 3.2. $\mathcal{M}(A) \equiv \sim A$ whenever A is a well-formed formula.

Of course, the next two lemmas follow from these results.

LEMMA 3.3. $\mathcal{N}(A) \equiv \mathcal{N}(B)$ if $A \equiv B$.

LEMMA 3.4. $\mathcal{M}(A) \equiv \mathcal{M}(B)$ if $A \equiv B$.

Moreover, using the Fundamental Theorem about Provable well formulated formulas we can verify this fact about $\mathcal{R}'$.

LEMMA 3.5. $\vdash \mathcal{R}' A$ if $\vdash A$.

To see the value of Theorem 2.6, we can prove

LEMMA 3.6. $\mathcal{R}'(A \to B) \equiv \mathcal{R}'(A) \to \mathcal{R}' B$ whenever $A \to B$ is a well formulated formula.

More generally, we can establish

LEMMA 3.7. Let $\mathcal{T}$ be a normal transform such that $d = \vee$; then $\mathcal{T}(A \to B) \equiv \mathcal{T}(A) \to \mathcal{T}(B)$ whenever $A \to B$ is a well form

Demonstration. $\mathcal{T}(\sim A \vee B) = \mathcal{T}(\sim A) \vee \mathcal{T}(B) \equiv \sim\mathcal{T}(A) \vee \mathcal{T}(B)$ by Theorem 2.6.

Returning to $\mathcal{R}'$, we point out that

LEMMA 3.8. $\mathcal{R}'(A) \equiv \mathcal{R}'(B)$ whenever $A \equiv B$.

Using these results it is easy to establish the Principle of Duality.

Principle of Duality. $\mathcal{D}(A) \equiv \mathcal{D}(B)$ whenever $A \equiv B$.

Demonstration. Let $A \equiv B$; then $\mathcal{R}'(A) \equiv \mathcal{R}'(B)$, so $\mathcal{M}(\mathcal{R}'(A)) \equiv \mathcal{M}(\mathcal{R}'(B))$. Thus, by Corollary 2.1, $\mathcal{D}(A) \equiv \mathcal{D}(B)$.

Here is a well known property of $\mathcal{D}$ that follows from our results.

THEOREM 3.1. $\vdash A$ if and only if $\vdash \sim\mathcal{D}(A)$.

COROLLARY 3.1. $\vdash \mathcal{D}(A)$ if and only if $\vdash \sim A$.

Using this result it is easy to verify

THEOREM 3.2. $\vdash \mathcal{D}(B) \to \mathcal{L}(A)$ if $\vdash A \to B$.

Of course, the group involving all normal transforms has many four-member subgroups; indeed $\{\mathcal{N}, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_1\mathcal{J}_2\}$ is a subgroup whenever $\mathcal{J}_1 \neq \mathcal{J}_2$, $\mathcal{J}_1 \neq \mathcal{N}$, and $\mathcal{J}_2 \neq \mathcal{N}$. It is easy to verify that $S = \{\mathcal{J} \mid \mathcal{J}$ is normal and $\mathcal{J}(A) \equiv \mathcal{J}(B)$ whenever $A \equiv B\}$ is a subgroup of our group. We have already established that $\{\mathcal{N}, \mathcal{M}, \mathcal{R}', \mathcal{D}\} \subset S$; so either $S = \{\mathcal{N}, \mathcal{M}, \mathcal{R}', \mathcal{D}\}$ or S is the set of all normal transforms. We shall show that $\mathcal{A} \bar{\in} S$. Let $A = Fx \vee \sim Fx$ and let $B = \forall y(Gy) \to Gz$. Clearly, $A \equiv B$; however, it is easy to verify that "$\mathcal{A}(A) \equiv \mathcal{A}(B)$" is false. This proves that $\mathcal{A} \bar{\in} S$; we conclude that $S = \{\mathcal{N}, \mathcal{M}, \mathcal{R}, \mathcal{D}\}$.

REFERENCES

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