Note on the equation connecting the mutual distances of four points in a plane.

By Thomas Muir, LL.D.

1. If the distances (12), (13), (14), (23), (24), (34) between four points 1, 2, 3, 4 on the circumference of a circle be denoted by a, b, c, d, e, f respectively, then a certain relation (A) is known to connect a, b, c, d, e, f. The same four points, however, being points in a plane, there subsists between their mutual distances another relation (B). Now, it occurs to one that from these two relations some deduction ought to be possible regarding the mutual distances of four points on a circumference, and the problem is suggested of making the said deduction.

2. Let us first consider the relations (A) and (B) in the forms so elegantly arrived at by Cayley,* viz. :—

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & d^2 & e^2 \\ b^2 & d^2 & 0 & f^2 \\ c^2 & e^2 & f^2 & 0 \end{vmatrix} = 0, \qquad \text{(A)}$$

and

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & d^2 & e^2 \\ 1 & b^2 & d^2 & 0 & f^2 \\ 1 & c^2 & e^2 & f^2 & 0 \end{vmatrix} = 0 ; \qquad \text{(B)}$$

and ask what conclusion can be deduced from them. Observing that the first determinant is the complementary minor of the element in the place (1, 1) of the second determinant, we recall a theorem which gives us a result at once. The theorem* is—*An axisymmetric deter-*

* *Cambridge Mathematical Journal*, II., pp. 267-271.

* This theorem seems to have been first enunciated by Salmon in the 1st edition of his *Lessons Introductory to the Modern Higher Algebra*, p. 124 (Dublin, 1859). Hesse gave it in Crelle's Journal, LXIX., p. 321 (1868), with the foot-note—" Den Satz (8) findet man von Herrn Weierstrass bewiesen in dem " Monatsberichte der Königl. Akademie der Wissenschaften zu Berlin, 4 " März 1858, p. 211. Denn seine von einem Factor abgelöste Function " $\delta_\mu = \Sigma_{\alpha\beta} f(s_\mu)_{\alpha\beta} \Phi_\alpha \Phi_\beta$ welche er als das Quadrat einer. linearen function " darstellt, ist, wenn man $a_o^n = \Phi_o$, $a_1^n \mp \Phi_1$, ..., $a_{n-1}^n = \Phi_{n-4}$ und $a_n^n = 0$ " setzt, gerade die symmetrische Determinante A deren Unterdeterminante " $B = f(s_\mu)$ verschwindet."

minant whose first element has a vanishing complementary minor is, when changed in sign, expressible as an exact second power. By reason of the existence of (A) it follows then that (B) can be put in the form

$$-\left\{\begin{vmatrix} 0 & d^2 & e^2 \\ d^2 & 0 & f^2 \\ e^2 & f^2 & 0 \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} 0 & b^2 & c^2 \\ b^2 & 0 & f^2 \\ c^2 & f^2 & 0 \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} 0 & a^2 & c^2 \\ a^2 & 0 & e^2 \\ c^2 & e^2 & 0 \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} 0 & a^2 & b^2 \\ a^2 & 0 & d^2 \\ b^2 & d^2 & 0 \end{vmatrix}^{\frac{1}{2}}\right\}^2 = 0,$$

whence we have

$$\sqrt{d^2e^2f^2} + \sqrt{b^2c^2f^2} + \sqrt{a^2c^2e^2} + \sqrt{a^2b^2d^2} = 0 \qquad \text{(I)}$$

as the result desired. The ambiguities of sign which occur in it, and which are a little troublesome to disentangle, detract somewhat from the value of this mode of proceeding.

3. The old form of (A) is

$$(af + be + cd)\ (-af + be + cd)\ (af - be + cd)\ (af + be - cd) = 0\ ; \qquad (\mathrm{A}_2)$$

and this, it is manifest, viewed apart from its connection with (B), is much more instructive than the form (A_1). No corresponding non-determinant form of B_1 having like advantages, has, so far as I am aware, been proposed. The following is therefore suggested—

$$\begin{aligned} d^2e^2f^2 + b^2c^2f^2 + a^2c^2e^2 + a^2b^2d^2 - (a^2 + f^2)\ (-a^2f^2 + b^2e^2 + c^2d^2) \\ - (b^2 + e^2)\ (a^2f^2 - b^2e^2 + c^2d^2) \\ - (c^2 + d^2)\ (a^2f^2 + b^2e^2 - c^2d^2) = 0. \quad (\mathrm{B}_2). \end{aligned}$$

By one who remembers (A_2) this also can be easily remembered. Its identity with (B_1) is established by expanding the determinant in (B_1) according to binary products of the elements of the first row and first column, and uniting those of the resulting determinants which have a common factor.

Now, taking these forms (A_2) (B_2), let us see with what readiness an unambiguous result is deducible.

When (A_2) holds,

$$\begin{aligned} \text{either}\quad & af + be + cd = 0, \\ \text{or}\quad & -af + be + cd = 0, \quad \text{(Ptolemy's theorem)} \\ \text{or}\quad & af - be + cd = 0, \\ \text{or}\quad & af + be - cd = 0. \end{aligned}$$

If it be that

$$af + be + cd = 0,$$

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$$\left.\begin{array}{ll}\text{then} & -a^2f^2+b^2e^2+c^2d^2=-2becd\\ \text{and} & a^2f^2-b^2e^2+c^2d^2=-2afcd\\ \text{and} & a^2f^2+b^2e^2-c^2d^2=-2afbe\end{array}\right\},$$

and manifestly we may extract the square root of both sides of (B_2) and obtain the result

$$def+bcf+ace+abd=0.$$

If, secondly, $-af+be+cd=0,$

then $def+bcf-ace-abd=0$ or $\dfrac{a}{f}=\dfrac{bc+de}{bd+ce}.$

If, thirdly, $af-be+cd=0.$

then $def-bcf+ace-abd=0$ or $\dfrac{b}{e}=\dfrac{ac+df}{ad+cf}.$

If, fourthly, $af+be-cd=0,$

then $def-bcf-ace+abd=0$ or $\dfrac{c}{d}=\dfrac{ab+ef}{ae+bf}.$

The "proportion" form of these identities is the familiar one which results from geometrical demonstration.

4. It is interesting to note that that when e and f are both put equal to d, the sides of the equation (B) become divisible by d^2, and the equation takes the form

$$\Sigma a^4=\Sigma a^2b^2. \qquad \text{(II.)}$$

What is the import of this algebraic symmetry, when there is no symmetry apparent in the geometry? The answer to this is best seen by looking upon $\Sigma a^4=\Sigma a^2b^2$ as the equation to which we are led in trying to solve the geometrical problem—

Given the distances a, b, c *of a point from the vertices of an equilateral triangle: find the side* d *of the said triangle.*

The symmetry then shows that we should be led to the same equation if the problem were—

Given an equilateral triangle whose side is a *(or* b, *or* c*) and a point whose distances from two of the vertices of the triangle are* b, c *(or* c, a*; or* a, b*): find the distance of the point from the third vertex.* Hence if we can solve the one problem we can solve the other. But the latter viewed as a geometrical problem requires *no* solution. Consequently the solution of the former is merely the construction indicated in the data of the latter. That is to say, the construction is—*Form a triangle whose sides are* a, b, c*: describe an equilateral triangle on* *any one of them; and join the vertices which are not common to the two triangles.*

The correctness of the construction is at once established by describing an equilateral triangle on the line found and using Euc. I., 4.

5. As the equilateral triangle of the construction has two possible positions the problem has two solutions, the point from which the distances are measured being in the one case inside the triangle found, and in the other outside. Let us see what light the algebraical equation $\Sigma a^4 = \Sigma a^2 b^2$ throws on this dualism.

Arranging the equation as a quadratic is d^2 we have

$$d^4 - (a^2 + b^2 + c^2)d^2 + (a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2) = 0,$$

whence we find

$$2d^2 = a^2 + b^2 + c^2 \pm \sqrt{3(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}.$$

But $2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$ is $16\Delta^2_{abc}$, if we denote by Δ_{abc} the area of the triangle whose sides are a, b, c. Hence

$$2d^2 = a^2 + b^2 + c^2 \pm 4\Delta_{abc}\sqrt{3},$$

and therefore

$$\tfrac{1}{2}d^2\sqrt{3} = \tfrac{1}{4}a^2\sqrt{3} + \tfrac{1}{4}b^2\sqrt{3} + \tfrac{1}{4}c^2\sqrt{3} \pm 3\Delta_{abc}.$$

Using now d_1 and d_2 to stand for the two values of d, and writing Δ_a for the area of the equilateral triangle whose side is a, *i.e.*, for $\frac{1}{4}a^2\sqrt{3}$ we have

$$2\Delta_{d_1} = \Delta_a + \Delta_b + \Delta_c + 3\Delta_{abc},$$

and

$$2\Delta_{d_2} = \Delta_a + \Delta_b + \Delta_c - 3\Delta_{abc};$$

and hence, by addition and subtraction, the theorems

$$\Delta_{d_1} + \Delta_{d_2} = \Delta_a + \Delta_b + \Delta_c, \qquad \text{(III)}.$$

$$\Delta_{d_1} - \Delta_{d_2} = 3\Delta_{abc}; \qquad \text{(IV)}.$$

that is to say, *The sum of the two equilateral triangles, each of which has its vertices at three given distances from a fixed point is equal to the sum of the equilateral triangles described on the distances; and the difference of the said pair of equilateral triangles is equal to three times the triangle whose sides are the given distances.*

6. When in (B) we put $e = d$ and $f = \rho d$ (which of course is a more general case than that of §§ 4, 5) the equation is also symmetrical, but now only with respect to two of the letters involved. Consequently with proper restrictive changes we can solve a more general geometrical problem than that of § 4, viz., where the triangle instead of being equilateral has two sides equal and enclosing a definite angle. We must now know which of the three given distances is the one drawn to the meeting point of the equal sides of the triangle; and taking this distance we make it one of the equal sides of an isosceles triangle similar to that specified, and on the third side describe a triangle with its two other sides equal to the remaining two given distances, and as before join the vertices not common to the two triangles.*

BISHOPTON, GLASGOW,
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Geometrical Notes.

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I. A straight line KK′ meets the circumference of a circle at two real or two imaginary points K, K′, and H is the middle point of the real or imaginary chord KK′. If A, B, C, D be any four points on the circumference, and the pairs of straight lines AB,DC, AC,BD, AD,CB meet KK′ at the pairs of points E,E′, F,F′, G,G′; then if any one pair of points be equidistant from H, the two other pairs will also be equidistant.

To prove that E,E′ are equidistant from H, if F,F′ are.

First Demonstration. (*Figures* 14, 15.)

Through A draw AA′ parallel to KK′; join A′E′, A′F′, and since E′, F′ are on DC, DB, join A′D.

Then AFF′A′ may be proved to be a convex or crossed isosceles and therefore cyclic trapezium, having A′F′ = AF, and angle A′F′E′ = angle AFE.

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* Professor Chrystal pointed out that a particular case of this, viz., where the triangle is isosceles *right-angled* is dealt with in the *Annals of Mathematics*, I., p. 24, and Mr Fraser has since received from Dr Rennet, of Aberdeen, a reference to Thomas Simpson's Algebra, 2nd edition, (1755) p. 369, where a very general problem of this nature is stated and solved.

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