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THE ARITHMETIC OF A SEMIGROUP OF SERIES OF WALSH FUNCTIONS

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Abstract

Let $\mathscr{W} = \{w_k(t)\}_{k=0}^{\infty}$ be the classical system of the Walsh functions, $\mathscr{S}_{\mathscr{W}}$ the multiplicative semigroup of the functions represented by series of functions $w_k(t)$ with non-negative coefficients which sum equals 1. We study the arithmetic of $\mathscr{S}_{\mathscr{W}}$. The analogues of the well-known Khinchin factorization theorems related to the arithmetic of the convolution semigroup of probability measures on the real line are valid in $\mathscr{S}_{\mathscr{W}}$. The classes of idempotent elements, of infinitely divisible elements, of elements without indecomposable factors, and of elements without indecomposable and non-degenerate idempotent factors are completely described. We study also the class of indecomposable elements. Our method is based on the following fact: $\mathscr{S}_{\mathscr{W}}$ is isomorphic to the semigroup of probability measures on the group of characters of the Cantor-Walsh group.

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1. Introduction and statement of results

The arithmetic of the convolution semigroup $\mathscr{P}$ of probability measures on $\mathbb{R}^n$ has been studied intensively since the 1930s (see [9, 10]). Nevertheless, some important problems, for example, the problem of the description of the class $I_0(\mathscr{P})$ of measures without indecomposable components, remain open. In the 1960s, Kendall, Davidson [7, 8], and Urbanik [15] studied semigroups essentially different from $\mathscr{P}$, but with a similar arithmetic. Many other examples of such semigroups were considered later by Bingham, Kennedy, Kingman, Lamperti, Ostrovskii, Ulanovskii (see the expository paper [10]) and by the author [4, 13, 14]. For some of these semigroups, the above mentioned problem has been completely solved.

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The aim of this paper is to study a new example of this kind. Let us denote by $r_i = r_i(t)$, $i = 0, 1, 2, \ldots$, $t \in [0, 1]$, the classical Rademacher functions (see, for example, [5, Chapter 2, Section 2]) defined as

$$r_i(t) = \operatorname{sign}(\sin(2^i \pi t)).$$

Let us agree that r_i is equal to 1 at the discontinuity points. The *Walsh functions* are all finite products of Rademacher functions. We set (see [5, Chapter 4, Section 6])

$$(1)\quad \begin{aligned} &w_0 = \psi_0^1 = r_0 \equiv 1 \\ &w_1 = \psi_1^1 = r_1, \\ &w_2 = \psi_2^1 = r_2, \quad w_3 = \psi_2^2 = r_2 r_1, \\ &w_4 = \psi_3^1 = r_3, \quad w_5 = \psi_3^2 = r_3 r_1, \quad w_6 = \psi_3^3 = r_3 r_2, \quad w_7 = \psi_3^4 = r_3 r_2 r_1, \quad \ldots \end{aligned}$$

For each $s \in \mathbb{N}$, the Walsh functions of the sth series ψ_s^j ($j = 1, 2, \ldots, 2^{s-1}$) are products of the function r_s and all functions of the preceding series. Let us note that $\psi_s^1 = r_s$ for all $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Walsh functions form an orthogonal and normalized system on the interval $[0, 1]$ with respect to the Lebesgue measure ([5, Chapter 4, Section 5]). Obviously, $r_k^2(t) \equiv 1$, $w_k^2(t) \equiv 1$ for all $k \in \mathbb{N}_0$. It is evident that the product of two Walsh functions is a Walsh function as well. The set of all Walsh functions is an abelian group with respect to the multiplication with unity $w_0 = \psi_0^1 \equiv 1$. Every element of this group is inverse to itself. We introduce the following notation:

$\mathscr{W} := \{w_k\}_{k=0}^{\infty}$ is the group of the Walsh functions with the discrete topology,

$\mathscr{S}_{\mathscr{W}}$ is the multiplicative semigroup of all functions $f(t)$, $t \in [0, 1]$, represented in the form

$$(2)\qquad f(t) = \sum_{k=0}^{\infty} a_k w_k(t), \quad a_k \geq 0, \quad \sum_{k=0}^{\infty} a_k = 1.$$

We endow $\mathscr{S}_{\mathscr{W}}$ with the topology of uniform convergence on $[0, 1]$.

Our aim is to study the arithmetic of $\mathscr{S}_{\mathscr{W}}$. Let us give the main definitions. The functions w_k, $k \in \mathbb{N}_0$ are *degenerate* elements of the semigroup $\mathscr{S}_{\mathscr{W}}$. A function $f \in \mathscr{S}_{\mathscr{W}}$ is called *idempotent* if $f^2 = f w_k$ for some $k \in \mathbb{N}_0$. A function $f \in \mathscr{S}_{\mathscr{W}}$ is *infinitely divisible* if for every $n \in \mathbb{N}$ there exist $f_n \in \mathscr{S}_{\mathscr{W}}$ and $k \in \mathbb{N}_0$ such that $f = (f_n)^n w_k$. A function $f_1 \in \mathscr{S}_{\mathscr{W}}$ is called a *factor* of $f \in \mathscr{S}_{\mathscr{W}}$ if there exists $f_2 \in \mathscr{S}_{\mathscr{W}}$ such that $f = f_1 f_2$. A function $f \in \mathscr{S}_{\mathscr{W}}$ is called *indecomposable* if $f \neq w_k$ for all $k \in \mathbb{N}_0$ and if all factors of f are of the forms w_k and $f w_k$ only. We introduce the following notation:

$Id(\mathscr{S}_{\mathscr{W}})$ is the class of all idempotent functions of $\mathscr{S}_{\mathscr{W}}$;

$I(\mathscr{S}_{\mathscr{W}})$ is the class of all infinitely divisible functions of $\mathscr{S}_{\mathscr{W}}$;

$I_0(\mathcal{S}_{\mathcal{W}})$ is the class of functions without indecomposable and non-degenerate idempotent factors;

$\tilde{I}_0(\mathcal{S}_{\mathcal{W}})$ is the class of functions without indecomposable factors;

$N(\mathcal{S}_{\mathcal{W}})$ is the class of indecomposable functions.

In Section 2 we show that $\mathcal{S}_{\mathcal{W}}$ is isomorphic to the semigroup of probability measures on the group of the characters of the Cantor–Walsh group. Therefore we may use results about the factorization of the probability measures on locally compact abelian groups. From the results of Parthasarathy, Rao and Varadhan [11, 12] it follows that three theorems given below are valid in $\mathcal{S}_{\mathcal{W}}$.

THEOREM 1 ([12]). *Every function $f \in \mathcal{S}_{\mathcal{W}}$ can be represented in the form $f = f_1 f_2 f_3$, where f_1 is the maximal idempotent factor of f, $f_2 \in I_0(\mathcal{S}_{\mathcal{W}})$, f_3 is a product of the empty, finite or countable set of indecomposable functions (in the first case $f_3 \equiv 1$, and in the third case the infinite product converges uniformly on* [0, 1]).

THEOREM 2 ([12, 2, Corollary 4.7]). $\tilde{I}_0(\mathcal{S}_{\mathcal{W}}) \subset I(\mathcal{S}_{\mathcal{W}})$.

Theorem 1 and Theorem 2 are analogues of the Khinchin theorems [9, pages 79, 88], related to the arithmetic of the semigroup $\mathcal{P}$.

THEOREM 3 ([11]). *The class $N(\mathcal{S}_{\mathcal{W}})$ is a dense G_δ-set in $\mathcal{S}_{\mathcal{W}}$ with respect to the topology of uniform convergence on* [0, 1].

The main results of our paper are characterizations of the classes $Id(\mathcal{S}_{\mathcal{W}})$, $I(\mathcal{S}_{\mathcal{W}})$, $\tilde{I}_0(\mathcal{S}_{\mathcal{W}})$, $I_0(\mathcal{S}_{\mathcal{W}})$ and a test for the membership of $N(\mathcal{S}_{\mathcal{W}})$.

THEOREM 4. *The class $Id(\mathcal{S}_{\mathcal{W}})$ consists of all functions f representable in the form*

$$f = w_j \prod_{w_i \in K} (0.5 + 0.5 w_i),$$

where $j \in \mathbb{N}_0$ and K is an arbitrary finite subgroup of $\mathcal{W}$.

THEOREM 5. *The class $I(\mathcal{S}_{\mathcal{W}})$ consists of all functions f representable in the form*

$$f = f_1 \exp\left(\sum_{k=1}^{\infty} c_k (w_k - 1)\right),$$

where $c_k \geq 0$, $\sum_{k=1}^{\infty} c_k < \infty$, $f_1 \in Id(\mathcal{S}_{\mathcal{W}})$.

Theorem 5 is an analogue of the well-known Lévy-Khinchin formula [9, page 9] of the characteristic function of an infinitely divisible probability measure on $\mathbb{R}$.

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THEOREM 6. *The class* $\tilde{I}_0(\mathscr{S}_{\mathscr{W}})$ *consists of all functions* f *representable in the form*

$$f = \alpha w_l + (1-\alpha) w_k, \quad \alpha \in [0,1], \quad l, k \in \mathbb{N}_0.$$

COROLLARY 1. $Id(\mathscr{S}_{\mathscr{W}}) \cap \tilde{I}_0(\mathscr{S}_{\mathscr{W}}) = \{0.5 w_l + 0.5 w_k : l, k \in \mathbb{N}_0\}$.

THEOREM 7.
$$\begin{aligned} I_0(\mathscr{S}_{\mathscr{W}}) &= \{\alpha w_l + (1-\alpha) w_k : \alpha \in [0, 0.5) \cup (0.5, 1], l, k \in \mathbb{N}_0\} \\ &= \{w_i \exp(c(w_j - 1)) : c \geq 0, i, j \in \mathbb{N}_0\}. \end{aligned}$$

It should be mentioned that in the semigroups studied earlier and different from $\mathscr{P}$, the class I_0 is rather small. Theorem 7 shows that $I_0(\mathscr{S}_{\mathscr{W}})$ is rather large.

The following theorem gives a test for the membership of $N(\mathscr{S}_{\mathscr{W}})$.

THEOREM 8. *Let*

$$f = \sum_{s=0}^{n} \sum_{j=1}^{2^{s-1}} a_{s,j} \psi_s^j + a_{m,i} \psi_m^i = \varphi + a_{m,i} \psi_m^i, \tag{3}$$

where $n < m$, $a_{m,i} > 0$ *and the sum* $\varphi(t)$ *contains at least two non-zero terms. Then* $f \in N(\mathscr{S}_{\mathscr{W}})$.

2. Probabilistic interpretation of the semigroup $\mathscr{S}_{\mathscr{W}}$

The semigroup $\mathscr{S}_{\mathscr{W}}$ has an interesting probabilistic interpretation. We first introduce (following [1, Section 14.1]) some notation and definitions.

Let us write $\mathscr{C}$ for the set $\{-1, 1\}^{\mathbb{N}}$ of all mappings $\omega : \mathbb{N} \to \{-1, 1\}$. (In [1] $\mathscr{C}$ is defined as the set of all mappings from $\mathbb{Z}$ to $\{-1, 1\}$. It is more convenient for us to consider $\mathbb{N}$ instead of $\mathbb{Z}$.) The set $\mathscr{C}$ is a compact abelian group with respect to the pointwise multiplication and the usual product topology. Every element of $\mathscr{C}$ is inverse to itself. We refer to $\mathscr{C}$ as the *Cantor-Walsh group*.

Let us describe the set $\mathscr{C}^*$ of all characters of $\mathscr{C}$. It is easy to see that for all $n \in \mathbb{N}$ the mapping $\rho_n : \mathscr{C} \to \{-1, 1\}$ defined as

$$\rho_n(\omega) = \omega(n)$$

is a character of the group $\mathscr{C}$, called the nth *Rademacher character*. It is proved in [1, Section 14.1.3], that $\mathscr{C}^*$ is the set of all finite products of Rademacher characters and the function identically equal to 1. The set $\mathscr{C}^*$ with the discrete topology is a (topological) abelian group with respect to multiplication. Let us denote by $\zeta_k(\omega)$, $k \in \mathbb{N}_0$, the elements of $\mathscr{C}^*$. Every function $\zeta_k(\omega)$ is defined by $\rho_i(\omega)$ in the same manner as $w_k(t)$ are defined by $r_i(t)$ in (1). The groups $\mathscr{C}^*$ and $\mathscr{W}$ are isomorphic via the bijection:

$$\rho_n(\omega) \leftrightarrow r_n(t), \qquad \zeta_k(\omega) \leftrightarrow w_k(t).$$

Let $\mathscr{C}^{**}$ be the group of all characters of $\mathscr{C}^*$. The duality theorem of Pontryagin ([3, Section 24]) implies that $\mathscr{C}^{**}$ and $\mathscr{C}$ are isomorphic. Note that for every fixed $k \in \mathbb{N}_0$, $\zeta_k(\omega)$ as a function of $\omega \in \mathscr{C}$ is an element of $\mathscr{C}^*$; and for every fixed $\omega \in \mathscr{C}$, $\zeta_k(\omega)$ as a function of k is an element of $\mathscr{C}^{**} \simeq \mathscr{C}$.

Let us denote by $M^1(\mathscr{C}^*)$ the topological semigroup (with operation of convolution and topology of weak convergence) of probability measures on $\mathscr{C}^*$. We now recall the general definition of characteristic function. Let X be a second countable locally compact abelian group, Y its group of characters, and let (x, y) stand for the value of $y \in Y$ at $x \in X$. Then the characteristic function $\hat{\mu}$ of the probability measure μ on X is defined as follows:

$$\hat{\mu}(y) = \int_X (x, y)\mu(dx).$$

We apply this definition to the group $X = \mathscr{C}^*$. Since $\mathscr{C}^*$ is countable, the characteristic function $\hat{\mu}(\omega)$ of the measure $\mu \in M^1(\mathscr{C}^*)$ is given by

$$\hat{\mu}(\omega) = \sum_{k=0}^{\infty} a_k\zeta_k(\omega), \quad a_k = \mu(\{\zeta_k\}) \geq 0, \quad \sum_{k=0}^{\infty} a_k = 1.$$

Let us denote by $\hat{M}^1(\mathscr{C}^*)$ the multiplicative semigroup of characteristic functions of all measures of $M^1(\mathscr{C}^*)$ with the topology of uniform convergence.

Since $\mathscr{C}^*$ and $\mathscr{W}$ are isomorphic, we see that the semigroups $\hat{M}^1(\mathscr{C}^*)$ and $\mathscr{S}_{\mathscr{W}}$ are isomorphic. Since $M^1(\mathscr{C}^*)$ and $\hat{M}^1(\mathscr{C}^*)$ are isomorphic, we infer that

$\mathscr{S}_{\mathscr{W}}$ is isomorphic to the semigroup $M^1(\mathscr{C}^*)$ of probability measures on the group of characters of the Cantor-Walsh group.

Therefore, we can study the arithmetic of $M^1(\mathscr{C}^*)$ or $\hat{M}^1(\mathscr{C}^*)$. In what follows we use facts related to the arithmetic of probability measures on groups as detailed in [2].

3. Infinitely divisible and idempotent elements. Proof of Theorem 4 and Theorem 5

Let us prove Theorem 4. It is known ([2, Section 2.14]) that the set of all idempotent measures on a locally compact abelian group X coincides with the set of shifts of Haar distributions on compact subgroups K of X. The characteristic function $\hat{m}_K(y)$ of the

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Haar measure m_K is given by

$$\hat{m}_K(y) = \begin{cases} 1 & y \in A(Y, K); \\ 0 & \text{otherwise,} \end{cases}$$

where $A(Y, K) = \{y \in Y : (x, y) = 1 \text{ for all } x \in K\}$. Therefore, the characteristic functions of idempotent measures have the form $(x_0, y)\hat{m}_K(y)$, where $x_0 \in X$. In the case $X = \mathscr{C}^*$ we see that the characteristic functions of idempotent elements in $M^1(\mathscr{C}^*)$ have the form $\zeta_j(\omega)\hat{m}_K(\omega)$, where $j \in \mathbb{N}_0$, K is an arbitrary compact subgroup of $\mathscr{C}$, and

$$\hat{m}_K(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in A(\mathscr{C}^*, K); \\ 0 & \text{otherwise.} \end{cases}$$

Since the groups $\mathscr{S}_{\mathscr{W}}$ and $\hat{M}^1(\mathscr{C}^*)$ are isomorphic, every element of the class $Id(\mathscr{S}_{\mathscr{W}})$ has the form $f(t) = w_j(t)\tau_K(t)$, where $j \in \mathbb{N}_0$, K is an arbitrary compact subgroup of $\mathscr{W}$, and

$$\tau_K(t) = \begin{cases} 1 & \text{if} \quad w_i(t) = 1 \text{ for all } w_i \in K; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the function $\tau_K(t)$ can be represented in the form

$$\tau_K = \prod_{w_i \in K} (0.5 + 0.5 w_i)\,.$$

To complete the proof we observe that the compact subgroups of the discrete group $\mathscr{W}$ are exactly all *finite* subgroups of $\mathscr{W}$. □

For the proof of Theorem 5 we need the following theorem which was proved by Parthasarathy, Rao, Varadhan, Sazonov, and which gives the form of the characteristic function of an infinitely divisible measure on a group X.

THEOREM 9 ([12], see also [2, Theorem 2.21]). *The characteristic function $\hat{\mu}(y)$ of an infinitely divisible measure μ on X can be represented in the form*

$$\hat{\mu}(y) = (x_0, y)\hat{m}_K(y) \exp\left(\int_{X \setminus \{0\}} ((x, y) - 1 - ig(x, y))\Phi(dx) - \varphi(y) \right),$$

where $x_0 \in X$, m_K is the Haar measure of a compact subgroup K of X, Φ is a measure on X such that $\Phi(X \setminus V) < \infty$ for all neighbourhoods V of zero of X, and for all $y \in Y$

$$\int_{X \setminus \{0\}} (1 - \Re(x, y))\Phi(dx) < \infty, \tag{4}$$

$\varphi(y)$ is a continuous nonnegative quadratic form on Y, that is, a continuous nonnegative function on Y such that

$$(5) \qquad \varphi(y_1 + y_2) + \varphi(y_1 - y_2) = 2(\varphi(y_1) + \varphi(y_2)) \quad \text{for all} \quad y_1, y_2 \in Y,$$

$g(x, y)$ is a function on $X \times Y$ (g is independent of μ) such that the following conditions are valid:

(a) *$g(x, y)$ is continuous with respect to the variables x and y;*
(b) $\sup_{x \in X} \sup_{y \in A} |g(x, y)| < \infty$ *for all compact sets $A \subset Y$;*
(c) *$g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2)$ for all $x \in X$ and $y_1, y_2 \in Y$, $g(-x, y) = -g(x, y)$ for all $x \in X$, $y \in Y$;*
(d) *for every compact subset A of the group Y there exists a neighbourhood V_A of the zero element of X such that $(x, y) = \exp(ig(x, y))$ for all $x \in V_A$ and $y \in A$;*
(e) *for every compact subset A of Y, $g(x, y) \to 0$ as $x \to 0 \in X$ uniformly with respect to $y \in A$.*

REMARK 1. If every element of X is inverse to itself, then $g(x, y) \equiv 0$. Indeed, if $-x = x$ for all $x \in X$, then the second condition in (c) gives $g(x, y) = -g(x, y)$.

REMARK 2. If every element of Y is inverse to itself, then $\varphi(y) \equiv 0$. Taking $y_1 = y_2 = 0$ in (5) we have $\varphi(0) = 0$. Taking $y_1 = y_2 = y$ in (5) we have $\varphi(y + y) + \varphi(0) = 4\varphi(y)$. Since $y = -y$ we conclude that $4\varphi(y) = 2\varphi(0) = 0$.

We apply Theorem 9 to the case $X = \mathscr{C}^*$. It is easy to see that $g(x, y) \equiv 0$ satisfies (a)–(e). Indeed, (d) follows if we take for V_A the set consisting of one function $\zeta_0(\omega) \equiv 1$. This set is open since the topology is discrete, and the other conditions are trivial. By Remark 1 the function $g(x, y) \equiv 0$ is unique. Since $Y = \mathscr{C}^{**}$ is isomorphic to $\mathscr{C}$ and using Remark 2, we see that (5) is valid only for the $\varphi(y) \equiv 0$. We note now that if Φ is a measure on $\mathscr{C}^*$ and $\Phi(\{\zeta_k\}) = c_k$, $k \in \mathbb{N}_0$, then the condition $\Phi(X \setminus V) < \infty$ for every neighbourhood V of zero holds if and only if $\sum_{k=1}^{\infty} c_k < \infty$. The latter condition implies (4).

Therefore, the characteristic function $\hat{\mu}$ of an infinitely divisible measure $\mu \in M^1(\mathscr{C}^*)$ has the form

$$\hat{\mu}(\omega) = \hat{\mu}_1(\omega) \exp\left(\sum_{k=1}^{\infty} c_k(\zeta_k(\omega) - 1)\right),$$

where $c_k \geq 0$, $\sum_{k=1}^{\infty} c_k < \infty$ and $\hat{\mu}_1(\omega)$ is the characteristic function of an idempotent measure $\mu_1 \in M^1(\mathscr{C}^*)$. Since $M^1(\mathscr{C}^*)$ and $\mathscr{S}_{\mathscr{W}}$ are isomorphic, we obtain Theorem 5. □

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4. Functions without indecomposable factors. Proof of Theorem 6 and Theorem 7

We need some lemmas.

LEMMA 1. *For all $i \in \mathbb{N}$ and $c \in \mathbb{R}$,*

$$\exp(c(w_i - 1)) = \alpha + \beta w_i,$$

where $\alpha = e^{-c}\cosh c$, $\beta = e^{-c}\sinh c$.

PROOF. Since $w_i^2(t) \equiv 1$, we have

$$\begin{aligned}\exp(c(w_i - 1)) &= e^{-c}\sum_{m=0}^{\infty}\frac{c^{2m}}{(2m)!} + e^{-c}\sum_{m=0}^{\infty}\frac{c^{2m+1}}{(2m+1)!}w_i \\ &= e^{-c}\cosh c + e^{-c}\sinh c\, w_i = \alpha + \beta w_i\,.\end{aligned}$$ □

LEMMA 2. *Let*

$$\varphi := \exp(c_i(w_i - 1) + c_j(w_j - 1) - \varepsilon(w_i w_j - 1)), \quad c_i, c_j, \varepsilon > 0\,.$$

Then there exists $\varepsilon_0 = \varepsilon_0(c_i, c_j)$ such that $\varphi \in \mathscr{S}_{\mathscr{W}}$ for all $\varepsilon \in (0, \varepsilon_0)$.

PROOF. By Lemma 1

$$\varphi = (\alpha + \beta w_i)(\gamma + \delta w_j)(\nu - \sigma w_i w_j),$$

where $\alpha, \beta, \gamma, \delta, \nu = e^{\varepsilon}\cosh\varepsilon, \sigma = e^{\varepsilon}\sinh\varepsilon > 0$, and hence

$$\begin{aligned}\varphi = (\alpha\gamma\nu - \beta\delta\sigma) &+ (\beta\gamma\nu - \alpha\delta\sigma)w_i \\ &+ (\alpha\delta\nu - \beta\gamma\sigma)w_j + (\beta\delta\nu - \alpha\gamma\sigma)w_i w_j\,.\end{aligned}$$

Taking $\varepsilon_0 = \varepsilon_0(c_i, c_j)$ so small that all coefficients in parentheses in the last formula are positive for $0 < \varepsilon < \varepsilon_0$, we have $\varphi \in \mathscr{S}_{\mathscr{W}}$. □

LEMMA 3. *Let*

$$\lambda := \exp(c_i(w_i - 1) + c_j(w_j - 1)), \quad c_i, c_j > 0,\ i, j \in \mathbb{N},\ i \neq j\,.$$

Then $\lambda \notin \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$.

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PROOF. Let us write $\lambda = \varphi\psi$, where

$$\varphi = \exp(c_i(w_i - 1) + c_j(w_j - 1) - \varepsilon(w_i w_j - 1)),$$
$$\psi = \exp(\varepsilon(w_i w_j - 1)), \quad \varepsilon > 0.$$

Since $w_i w_j \in \mathscr{W}$ and applying Theorem 5, we have $\psi \in I(\mathscr{S}_{\mathscr{W}})$. So, $\psi \in \mathscr{S}_{\mathscr{W}}$. According to Lemma 2, $\varphi \in \mathscr{S}_{\mathscr{W}}$ for $\varepsilon > 0$ small enough. Therefore, φ is a factor of λ. Let us prove that $\varphi \notin I(\mathscr{S}_{\mathscr{W}})$. If this is not the case, then φ can be represented as

$$\exp\left(\sum_{i=1}^{\infty} c_i(w_i - 1)\right), \quad c_i \geq 0, \quad \sum_{i=1}^{\infty} c_i < \infty.$$

This is a consequence of positivity of φ, Theorem 4 and Theorem 5. But $w_i w_j = w_k$ for some $k \neq 0, i, j$, and we have a contradiction to the uniqueness of the decomposition as a series of the Walsh functions. By Theorem 2, $\varphi \notin \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$, and hence $\lambda \notin \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$. □

LEMMA 4. *Let*

$$\xi := (0.5 + 0.5w_i)\left(0.5 + 0.5w_j\right), \quad i, j \in \mathbb{N}, \quad i \neq j.$$

Then $\xi \notin \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$.

PROOF. We note that $\xi = 1$ for $w_i = 1$, $w_j = 1$ and $\xi = 0$ otherwise. We also note that the function λ in Lemma 3 satisfies the condition: $\lambda = 1$ for $w_i = 1$, $w_j = 1$. Therefore, $\xi = \xi\lambda$. Lemma 4 follows now from Lemma 3. □

LEMMA 5. *Let*

$$\eta := (0.5 + 0.5w_i)\exp(c_j(w_j - 1)), \quad c_j > 0, \quad i, j \in \mathbb{N}, \quad i \neq j.$$

Then $\eta \notin \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$.

PROOF. We have

$$0.5 + 0.5w_i = (0.5 + 0.5w_i)\exp(c_i(w_i - 1))$$

for all $c_i > 0$. To prove this identity, it is sufficient to substitute $+1$ and -1 for w_i. Therefore, we can write η in the form

$$\eta = (0.5 + 0.5w_i)\,\lambda,$$

where λ is the function of Lemma 3. Lemma 5 follows now from Lemma 3. □

LEMMA 6. *Let*

$$\zeta := \alpha w_m + (1-\alpha) w_k,$$

where $0 \le \alpha \le 1$, $m, k \in \mathbb{N}_0$, $m \ne k$. *Then* $\zeta \in \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$. *In addition,*

(1) *if* $\alpha \ne 0.5$, *then* $\zeta \in I_0(\mathscr{S}_{\mathscr{W}})$;
(2) *if* $\alpha = 0.5$, *then* $\zeta \in \tilde{I}_0(\mathscr{S}_{\mathscr{W}}) \setminus I_0(\mathscr{S}_{\mathscr{W}})$.

PROOF. The case $\alpha = 0$ or $\alpha = 1$ is trivial. Let us consider $0 < \alpha < 1$. Let $\zeta = \zeta_1\zeta_2$, where $\zeta_1, \zeta_2 \in \mathscr{S}_{\mathscr{W}}$, $\zeta_1 \ne w_k$. First we note that the expansion of ζ_1 into the series of the Walsh functions contains exactly two nonzero terms. Indeed, if there are three or more terms, then the series expansion of ζ also contains three or more terms. This gives a contradiction. We have used here that if $i \ne j$, then $w_i w_k \ne w_j w_k$. We have proved that

$$\zeta_1 = a w_p + (1-a) w_q, \quad 0 < a < 1, \quad p \ne q.$$

Later on we use the following fact. A convex linear combination of two distinct Walsh functions is equal to zero at some point if and only if the coefficients of this linear combination are both 0.5.

(1) Assume $\alpha \ne 0.5$. Then we have $\zeta(t) \ne 0$ for all $t \in [0, 1]$. Therefore, $\zeta_1(t) \ne 0$, and hence $a \ne 0.5$. We can assume without loss of generality that $a > 0.5$. By Lemma 1 we conclude that

$$\begin{aligned}\zeta_1 &= w_p(a + (1-a) w_p w_q) \\ &= w_p(a + (1-a) w_i) = w_p \exp(c(w_i - 1)), \quad c > 0.\end{aligned}$$

By Theorem 5, $\zeta_1 \in I(\mathscr{S}_{\mathscr{W}})$. Therefore, ζ_1 is decomposable. Since $\zeta_1(t) \ne 0$ for $t \in [0, 1]$, ζ_1 is not a non-degenerate idempotent function, and hence $\zeta \in I_0(\mathscr{S}_{\mathscr{W}})$.

(2) Assume $\alpha = 0.5$. Then the function ζ has zeros on $[0, 1]$ and the coefficient a in the definition of ζ_1 can be equal to 0.5. In this case also the function $\zeta_1 = 0.5 w_p + 0.5 w_q$ is decomposable. This follows from the representation

$$\zeta_1 = 0.5 w_p + 0.5 w_q = w_p\,(0.5 + 0.5 w_i) = w_p\,(0.5 + 0.5 w_i)^2.$$

Therefore, $\zeta \in \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$. But ζ is a non-degenerate idempotent element for $\alpha = 0.5$, and hence $\zeta \notin I_0(\mathscr{S}_{\mathscr{W}})$. □

PROOF (of Theorem 6 and Theorem 7). Let $f \in \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$. According to Theorem 2, Theorem 4 and Theorem 5 we have

$$f = w_j \prod_{w_i \in K} (0.5 + 0.5 w_i) \exp\left(\sum_{k=1}^{\infty} c_k (w_k - 1)\right),$$

where $j \in \mathbb{N}_0$, $c_k \geq 0$, $\sum_{k=1}^{\infty} c_k < \infty$, and K is a finite subgroup of W. It follows from Lemma 3 that only one coefficient c_k can be non-zero, for otherwise f has an indecomposable factor. Lemma 4 implies that either $K = \{w_0\}$ or $K = \{w_0, w_i\}$, for otherwise f has an indecomposable factor. It follows from Lemma 5 that the case $K = \{w_0, w_i\}$ is possible only if $c_k = 0$ for all $k \in \mathbb{N}$, for otherwise f has an indecomposable factor. Therefore, if $f \in \tilde{I}_0(\mathscr{S}_{\mathscr{W}})$, then either $f = w_j(0.5 + 0.5w_i)$ or $f = w_j \exp(c_i(w_i - 1))$. According to Lemma 1 $f = \alpha w_m + (1-\alpha) w_k$ where $0 \leq \alpha \leq 1$, $m, k \in \mathbb{N}_0$, $m \neq k$. Theorem 6 and Theorem 7 now follow from Lemma 6. □

REMARK 3. Theorem 7 can also be deduced from known general theorems on decomposition of the generalized Poisson distribution on groups. Let E_x be the probability measure on X concentrated at the point $x \in X$. For every measure Φ on X the generalized Poisson distribution is defined by the formula

$$e(\Phi) := \exp(-\Phi(X)) \left(E_0 + \Phi + \frac{\Phi^{2*}}{2!} + \cdots + \frac{\Phi^{n*}}{n!} + \cdots \right).$$

THEOREM 10 (Rukhin, see [2, Section 6, Proposition 6.6]). *Let* $\mu = e(\Phi)$, *where* $\Phi = \psi E_x$, $\psi > 0$, $x \in X$.

Then $\mu \in I_0(M(X))$ *if* x *is either element of infinite order or order* 2 *and* $\mu \notin I_0(M(X))$ *if* x *is element of order* $p > 2$.

THEOREM 11 (Fel'dman, see [2, Section 6, Proposition 6.11]). *Let* $\Phi = \psi_1 E_{x_1} + \psi_2 E_{x_2}$, *where* $\psi_j > 0$, $x_j \neq 0$, $2x_j = 0, j = 1, 2$, $x_1 \neq x_2$. *Then* $\mu = e(\Phi) \notin I_0(M(X))$.

Since all elements of $\mathscr{C}^*$, except zero, have order 2 it follows from Theorem 10 that

$$\exp(c(w_i - 1)) \in I_0(\mathscr{S}_{\mathscr{W}}), \quad i \in \mathbb{N}, \quad c \geq 0.$$

It follows from Theorem 11 that

$$\exp(c_i(w_i - 1) + c_j(w_j - 1)) \notin I_0(\mathscr{S}_{\mathscr{W}}), \quad c_i, c_j > 0, \quad i \neq j, \quad i, j \in \mathbb{N}.$$

5. Indecomposable elements of $\mathscr{S}_{\mathscr{W}}$. Proof of Theorem 8

We use the following notation.

$W_m := \{\psi_m^i\}_{i=1}^{2^{m-1}}$ is the set of Walsh functions of the mth series,

$$W_{(<m)} := \bigcup_{j=0}^{m-1} W_j; \quad W_{(>m)} := \bigcup_{j=m+1}^{\infty} W_j.$$

The definition of the Walsh functions implies the following statement.

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REMARK 4. If $w_i, w_j \in W_m$, then $w_i w_j \in W_{(<m)}$. If $w_i \in W_m, w_j \in W_k$, and $k < m$ then $w_i w_j \in W_m$.

PROOF (of Theorem 8). Let f has the form (3). Assuming $f \notin N(\mathscr{S}_{\Psi})$ we have $f = f_1 f_2$, where $f_1, f_2 \in \mathscr{S}_{\Psi}$, $f_1, f_2 \neq w_k$, $k \in \mathbb{N}_0$. Without loss of generality we may assume that the coefficient $c_{0,1}$ of the function $\psi_0^1(t) \equiv 1$ in the series expansion of f_2 is non-zero. We divide the proof of Theorem 8 into several steps.

(a) We note that f_1 does not contain any term from $W_{(>m)}$, for otherwise condition $c_{0,1} \neq 0$ implies that f contains terms from $W_{(>m)}$. This is a contradiction.

(b) f_1 contains exactly one term from W_m, namely ψ_m^i. If there are two such terms, then the condition $c_{0,1} \neq 0$ implies that f contains two terms from W_m. This is a contradiction. If f_1 does not contain any term from W_m, then it contains at least two terms from $W_{(<m)}$. Hence, according to the condition $a_{m,i} > 0$, f_2 contains at least one term from W_m (see Remark 4). Then f contains at least two terms from W_m. This is a contradiction.

(c) f_1 contains at least one term from $W_{(<m)}$ because f_1 contains at least two terms in general.

(d) f_2 does not contain any term from $W_{(>m)}$. Indeed, if there is such a term, then it is contained also in f (see (c) and Remark 4).

(e) f_2 contains exactly one term from W_m. If there are two such terms then according to (c) f contains two such terms, but this is not the case. If there is no such term, then f_2 contains at least two terms from $W_{(<m)}$. Consequently, according to (b), f contains two terms from W_m, which is not the case.

(f) f_2 contains exactly one term from $W_{(<m)}$, namely $\psi_0^1 \equiv 1$. If there are two such terms, then according to (b) f has two terms from W_m.

(g) f_1 has exactly one term from $W_{(<m)}$. If there are two such terms, then according to (e) f has two terms from W_m. This is not the case.

It follows from (a)–(g) that

$$f_1 = a\psi_n^k + b\psi_m^i, n < m, \quad \text{and} \quad f_2 = c + d\psi_m^l.$$

Then

$$f = ac\psi_n^k + bc\psi_m^i + ad\psi_n^k\psi_m^l + bd\psi_m^i\psi_m^l.$$

Since f has only one term from W_m and since $\psi_n^k\psi_m^l \in W_m$ (Remark 4), we have $\psi_n^k = \psi_0^1 \equiv 1$ and $l = i$. Therefore,

$$f = (ac + bd) + (bc + ad)\psi_m^i,$$

that is, f contains exactly two terms. This is a contradiction.

We now present a sequence of functions from $N(\mathscr{S}_{\mathscr{W}})$ which is uniformly convergent to a given function $f \in \mathscr{S}_{\mathscr{W}}$. Let

$$f = \sum_k \sum_j a_{k,j} \psi_k^j.$$

We consider two cases: (i) there are at least two terms in f, (ii) $f = \psi_k^j$.

In the first case we take

$$f_n = \sum_{k=0}^n \sum_j \frac{a_{k,j}}{S_n} \psi_k^j + \frac{1}{n} \psi_{n+1}^1 \quad \left(S_n = \frac{n}{n-1} \sum_{k=0}^n \sum_j a_{k,j} \right).$$

Let n be so large that the sum in the definition of f_n contains at least two terms. According to Theorem 8, $f_n \in N(\mathscr{S}_{\mathscr{W}})$. It is evident that $f_n \to f$ as $n \to \infty$ uniformly on [0, 1].

In the second case, we consider

$$f_n = \left(1 - \frac{2}{n}\right) \psi_k^j + \frac{1}{n} \psi_{k+1}^1 + \frac{1}{n} \psi_{k+2}^1.$$

It follows from Theorem 8 that $f_n \in N(\mathscr{S}_{\mathscr{W}})$. Evidently, $f_n \to f$ as $n \to \infty$ uniformly. □

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