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Raising Operators of Row Type for Macdonald Polynomials

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Abstract. We construct certain raising operators of row type for Macdonald's symmetric polynomials by an interpolation method.

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1. Introduction

Throughout this paper, we denote by $J_\lambda(x;q,t)$ the integral form of Macdonald's symmetric polynomial in n variables $x=(x_1,\dots,x_n)$ (of type A_{n-1}) associated with a partition λ ([6]). For each $m=0,1,2,\dots$, we consider a q-difference operator B_m which should satisfy the following condition: *For any partition* $\lambda=(\lambda_1,\lambda_2,\dots)$ *whose longest part* λ_1 *has length* $\leqslant m$, *one has*

$$B_m J_\lambda(x;q,t)=\begin{cases} J_{(m,\lambda)}(x;q,t), & \text{if } \ell(\lambda)<n,\\ 0, \quad \text{if } \ell(\lambda)=n, \end{cases} \tag{1.1}$$

where $(m,\lambda)=(m,\lambda_1,\lambda_2,\dots)$ *stands for the partition obtained by adding a row of length* m *to* λ. An operator B_m having this property will be called a *raising operator of row type* for Macdonald polynomials. With such operators, the Macdonald polynomial $J_\lambda(x;q,t)$ for a general partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ can be expressed as

$$B_{\lambda_1}B_{\lambda_2}\dots B_{\lambda_n}.1=J_\lambda(x;q,t),\qquad (\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant 0). \tag{1.2}$$

Namely, one can obtain $J_\lambda(x;q,t)$ by an successive application of the operators B_m starting from $J_\phi(x;q,t)=1$.

The purpose of this paper is to give an explicit construction of such operators B_m $(m=0,1,2,\dots)$. These operators B_m can be considered as a *dual version* of the raising operators *of column type* introduced by A. N. Kirillov and the second author

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[4, 5]. We remark that, as to the Hall–Littlewood polynomials (the case when $q = 0$), such a class of raising operators B_m of row type has been implicitly employed in Macdonald [6], Chapter III, (2.14)

$$B_m = (1-t)\sum_{i=1}^{n} x_i^m \left(\prod_{j\neq i}\frac{x_i - tx_j}{x_i - x_j}\right) T_{0,x_i}, \tag{1.3}$$

for $m = 1, 2, \ldots$, where T_{0,x_i} is the '0-shift operator' in x_i, namely, the substitution of zero for x_i. Our raising operators of row type for Macdonald polynomials can be considered as a generalization of these operators for Hall–Littlewood polynomials.

We will propose first a theorem of unique existence for raising operators of row type. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad T_{q,x}^{\alpha} = T_{q,x_1}^{\alpha_1}\cdots T_{q,x_n}^{\alpha_n}, \tag{1.4}$$

where T_{q,x_i} is the q-shift operator in x_i, defined by

$$T_{q,x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n), \tag{1.5}$$

for $i = 1, \ldots, n$.

THEOREM 1.1. *For each $m = 0, 1, 2, \ldots$, there exists a unique q-difference operator*

$$B_m = \sum_{|\gamma|\leqslant m} b_\gamma^{(m)}(x) T_{q,x}^{\gamma} \tag{1.6}$$

of order $\leqslant m$ satisfying the condition (1.1), *where $b_\gamma^{(m)}(x)$ are rational functions in x with coefficients in $\mathbb{Q}(q, t)$. Furthermore, the operator B_m is invariant under the action of the symmetric group $\mathfrak{S}_n$ of degree n.*

We will also determine the operator B_m explicitly by an interpolation method. In the following, we use the notation $\alpha \leqslant \beta$ for the partial ordering of multi-indices defined by

$$\alpha \leqslant \beta \Leftrightarrow \alpha_i \leqslant \beta_i \quad (i = 1, \ldots, n). \tag{1.7}$$

In order to describe the coefficients of our raising operators, we introduce a variant of *q-binomial coefficients* $C_{\alpha,\beta}(x; q)$ including the variables $x = (x_1, \ldots, x_n)$. For any pair (α, β) of multi-indices such that $\alpha \geqslant \beta$, we set

$$\begin{aligned} C_{\alpha,\beta}(x;q) &= \prod_{1\leqslant i,j\leqslant n} \frac{(q^{\alpha_i-\beta_j+1}x_i/x_j)_{\beta_j}}{(q^{\beta_i-\beta_j+1}x_i/x_j)_{\beta_j}} \\ &= \prod_{j=1}^{n}\frac{(q^{\alpha_j-\beta_j+1})_{\beta_j}}{(q)_{\beta_j}} \prod_{i\neq j}\frac{(q^{\alpha_i-\beta_j+1}x_i/x_j)_{\beta_j}}{(q^{\beta_i-\beta_j+1}x_i/x_j)_{\beta_j}}, \end{aligned} \tag{1.8}$$

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with the notation $(a)_k = (a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$ of the q-shifted factorial. We remark that, if $n=1$, $C_{\alpha,\beta}(x;q)$ reduce to the ordinary q-binomial coefficients $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q$.

THEOREM 1.2. *The q-difference operator B_m of Theorem* 1.1 *can be expressed in the form*

$$B_m = \sum_{|\alpha|=m} b_\alpha^{(m)}(x)\phi_\alpha^{(m)}(x;T_{q,x}), \tag{1.9}$$

where

$$\begin{aligned} b_\alpha^{(m)}(x) \;=\; & (-1)^{|\alpha|} q^{\sum_i \binom{\alpha_i}{2}} x^\alpha \sum_{\beta \leqslant \alpha} (-1)^{|\beta|} q^{\binom{|\beta|}{2}}\, C_{\alpha,\beta}(x;q) \;\times \\ & \times \prod_{i,j=1}^{n} \frac{(tq^{-\beta_j+1}x_i/x_j)_{\beta_j}(q^{-\alpha_j+1}x_i/x_j)_{\alpha_j-\beta_j}}{(q^{\alpha_i-\alpha_j+1}x_i/x_j)_{\alpha_j}} \end{aligned} \tag{1.10}$$

and

$$\phi_\alpha^{(m)}(x;T_{q,x}) = \sum_{\beta \leqslant \alpha} (-1)^{|\alpha|-|\beta|} q^{\binom{|\alpha|-|\beta|+1}{2}} C_{\alpha,\beta}(x;q) T_{q,x}^{\beta}, \tag{1.11}$$

for each α with $|\alpha| = m$.

In the course of the proof of Theorem 1.2, we will make use of a variant of the *q-binomial theorem* for our $C_{\alpha,\beta}(x;q)$, which might also deserve attention (see Proposition 5.2 in Section 5).

THEOREM 1.3. *For any $\alpha \in \mathbb{N}^n$, one has*

$$\sum_{\beta \leqslant \alpha} (-1)^{|\beta|} q^{\binom{|\beta|}{2}} C_{\alpha,\beta}(x;q) u^{|\beta|} = (u)_{|\alpha|}. \tag{1.12}$$

We remark that formula (1.12) also implies a generalization of *q-Chu-Vandermonde formulas*

$$\sum_{\beta \leqslant \alpha, |\beta|=r} \prod_{j=1}^{n} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}_q \prod_{i \neq j} \frac{(q^{\alpha_i-\beta_j+1}x_i/x_j)_{\beta_j}}{(q^{\beta_i-\beta_j+1}x_i/x_j)_{\beta_j}} = \begin{bmatrix} n \\ r \end{bmatrix}_q, \tag{1.13}$$

for any α with $|\alpha| = n$ and $0 \leqslant r \leqslant n$.

After recalling some basic facts about Macdonald polynomials in Section 2, we will prove the uniqueness and the existence of raising operators of row type in Section 3 and in Section 4, respectively. Explicit formulas for the q-difference operators $\phi_\alpha^{(m)}(x; T_{q,x})$ and the coefficients $b_\alpha^{(m)}(x)$ ($|\alpha| = m$) of Theorem 1.2 will be given in Section 5 and in Section 6, respectively.

2. Macdonald Polynomials

In order to fix the notation, we recall some basic facts about Macdonald's symmetric polynomials of type A_{n-1}. For the details see [6].

Let $\mathbb{K}[x] = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in n variables $x = (x_1, x_2, \ldots, x_n)$ with coefficients in $\mathbb{K} = \mathbb{Q}(q, t)$, and $\mathbb{K}[x]^{\mathfrak{S}_n}$ the subring of all invariant polynomials under the natural action of the symmetric group $\mathfrak{S}_n$ of degree n.

Macdonald's commuting family of q-difference operators $D_1, D_2, \ldots, D_n$ is defined by the generating function

$$\begin{aligned} D_x(u; q, t) &= \sum_{r=0}^{n} (-u)^r D_r \\ &= \sum_{K \subset \{1,\ldots,n\}} (-u)^{|K|} t^{\binom{|K|}{2}} \prod_{i \in K, j \notin K} \frac{1 - t x_i / x_j}{1 - x_i / x_j} \prod_{i \in K} T_{q,x_i}. \end{aligned} \tag{2.1}$$

Note that $D_x(u; q, t)$ has the determinantal formula

$$\begin{aligned} D_x(u; q, t) &= \frac{1}{\Delta(x)} \det(x_j^{n-i}(1 - u t^{n-i} T_{q,x_i}))_{i,j} \\ &= \frac{1}{\Delta(x)} \sum_{w \in \mathfrak{S}_n} \varepsilon(w) w \left(\prod_{i=1}^{n} x_i^{n-i} (1 - u t^{n-i} T_{q,x_i}) \right), \end{aligned} \tag{2.2}$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$. Macdonald's symmetric polynomials $P_\lambda(x) = P_\lambda(x; q, t)$ are the joint eigenfunctions of the operators $D_1, \ldots, D_n$ on $\mathbb{K}[x]^{\mathfrak{S}_n}$, satisfying the equations

$$D_x(u) P_\lambda(x) = P_\lambda(x) \prod_{i=1}^{n} (1 - u q^{\lambda_i} t^{n-i}), \tag{2.3}$$

each $P_\lambda(x)$ is normalized so that the coefficient of x^λ should be equal to 1. The integral form $J_\lambda(x) = J_\lambda(x; q, t)$ of $P_\lambda(x)$ is defined as

$$J_\lambda(x; q, t) = c_\lambda P_\lambda(x; q, t), \qquad c_\lambda = \prod_{s \in \lambda} (1 - q^{a(s)} t^{l(s)+1}). \tag{2.4}$$

It is known in fact that $J_\lambda(x)$ are linear combinations of monomial symmetric functions with coefficients in $\mathbb{Z}[q, t]$ (see [4] for example).

We recall that the Macdonald polynomials have the generating function

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_i y_j) = \sum_{\lambda} P_{\lambda}(x;q,t)P_{\lambda'}(y;t,q), \tag{2.5}$$

for another set of variables $y = (y_1, \dots, y_m)$, where λ' stands for the conjugate partition of λ, and the summation is taken over all partitions λ such that $l(\lambda') = \lambda_1 \leqslant m$, $l(\lambda) = \lambda'_1 \leqslant n$. This formula will be the key to our study of raising operators of row type. Notice that the dual version of the generation function (2.5) has been employed in [4] for the construction of raising operators of column type.

3. Raising Operators of Row Type and their Uniqueness

Fixing a nonnegative integer m, we will prove in this section the uniqueness of a q-difference operator

$$B_m = \sum_{|\gamma|\leqslant m} b_{\gamma}^{(m)}(x)T_{q,x}^{\gamma}, \qquad (b_{\gamma}^{(m)}(x) \in \mathbb{K}(x)), \tag{3.1}$$

of order $\leqslant m$ such that

$$B_m J_{\lambda}(x;q,t) = \begin{cases} J_{(m,\lambda)}(x;q,t), & \text{if } l(\lambda') \leqslant m, l(\lambda) < n, \\ 0, & \text{if } l(\lambda') \leqslant m, l(\lambda) = n, \end{cases} \tag{3.2}$$

where $(m, \lambda) = (m, \lambda_1, \lambda_2, \dots)$. We remark that the invariance of B_m under the action of $\mathfrak{S}_n$ follows immediately from the uniqueness theorem. Existence of such an operator will be established in the next section.

LEMMA 3.1. *A q-difference operator B_m of order $\leqslant m$ in the form* (3.1) *satisfies the condition* (3.2) *if and only if the following equality holds*

$$B_{m,x}\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_i y_j) = \frac{1}{y_1\dots y_m}D_y(1;t,q)\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_i y_j). \tag{3.3}$$

Proof. Note first that, for each partition $\mu = (\mu_1, \dots, \mu_m)$ of length $\leqslant m$, one has

$$\frac{1}{y_1\dots y_m}D_y(1;t,q)P_{\mu}(y;t,q) = \begin{cases} P_{\mu-(1)^m}(y;t,q)\prod_{i=1}^{m}(1-q^{m-i}t^{\mu_i}), & \text{if } \mu_m > 0, \\ 0, & \text{if } \mu_m = 0. \end{cases} \tag{3.4}$$

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Hence we obtain

$$
\begin{aligned}
&\frac{1}{y_1\dots y_m}D_y(1;t,q)\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j)\\
&\quad=\sum_{l(\nu)\leqslant n,\,l(\nu')=m}P_\nu(x;q,t)P_{\nu'-(1)^m}(y;t,q)\prod_{i=1}^{m}(1-q^{m-i}t^{(\nu')_i})\\
&\quad=\sum_{l(\lambda)\leqslant n-1,\,l'(\lambda)\leqslant m}P_{(m,\lambda)}(x;q,t)P_{\lambda'}(y;t,q)\prod_{i=1}^{m}(1-q^{m-i}t^{(\lambda')_i+1}).
\end{aligned}\tag{3.5}
$$

This implies that Equation (3.3) is equivalent to the condition

$$
B_mP_\lambda(x;q,t)=\begin{cases}0, & (\text{if } l(\lambda)=n),\\ P_\lambda(x;q,t)\displaystyle\prod_{i=1}^{m}(1-q^{m-i}t^{(\lambda')_i+1}), & (\text{if } l(\lambda)<n),\end{cases}\tag{3.6}
$$

for any λ with $l(\lambda')\leqslant m$. It is easily seen that this coincides with condition (3.2) in terms of the integral forms. □

By making the action of $D_y(1;t,q)$ in (3.3) explicit, we obtain

PROPOSITION 3.1. *A q-difference operator B_m of order $\leqslant m$ is a raising operator of row type for Macdonald polynomials if and only if its coefficients satisfy the following identity of rational functions*

$$
\begin{aligned}
&\sum_{|\gamma|\leqslant m}b_\gamma^{(m)}(x)\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1+q^{\gamma_i}x_iy_j}{1+x_iy_j}\\
&\quad=\frac{1}{y_1\dots y_m}\sum_{K\subset\{1,\dots,m\}}(-1)^{|K|}q^{\binom{|K|}{2}}\times\\
&\qquad\times\prod_{k\in K,l\notin K}\frac{1-qy_k/y_l}{1-y_k/y_l}\prod_{i=1}^{n}\prod_{k\in K}\frac{1+tx_iy_k}{1+x_iy_k}.
\end{aligned}\tag{3.7}
$$

Remark 3.1. By the determinantal representation of $D_y(1;t,q)$, equality (3.7) can also be rewritten in the form

$$
\begin{aligned}
&\sum_{|\gamma|\leqslant m}b_\gamma^{(m)}(x)\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1+q^{\gamma_i}x_iy_j}{1+x_iy_j}\\
&\quad=\frac{1}{y_1\dots y_m\Delta(y)}\det\left(y_j^{m-i}\left(1-q^{m-i}\prod_{r=1}^{n}\frac{1+tx_ry_j}{1+x_ry_j}\right)\right)_{i,j}.
\end{aligned}\tag{3.8}
$$

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Let now B and B' be two q-difference operators of order $\leqslant m$ and suppose that they both satisfy the condition (3.2) of raising operators. Then by Lemma 3.1 one has

$$(B_x - B'_x)\prod_{i=1}^{n}\prod_{j=1}^{m}(1 + x_i y_j) = 0. \tag{3.9}$$

Hence the uniqueness of B_m of Theorem 1.1 follows immediately from the following general proposition on q-difference operators.

PROPOSITION 3.2. *Let $P = \Sigma_{|\gamma|\leqslant m} a_\gamma(x) T^{\gamma}_{q,x}$ be a q-difference operator of order $\leqslant m$ with coefficients in $\mathbb{K}(x)$.*

(a) *If $P_x \prod_{i=1}^{n}\prod_{j=1}^{m}(1 + x_i y_j) = 0$, then $P = 0$ as a q-difference operator.*
(b) *If $Pf(x) = 0$ for any symmetric polynomial $f(x) \in \mathbb{K}[x]^{\mathfrak{S}_n}$ of degree $\leqslant mn$, then $P = 0$ as a q-difference operator.*

Since the statement (b) follows from (a), we give a proof of (a) of Proposition. For each multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, we define a point $p_\alpha(x) \in \mathbb{K}(x)^m$ by

$$\begin{aligned} p_\alpha(x) \ = \ &(-1/x_1, -1/qx_1, \ldots, -1/q^{\alpha_1 - 1}x_1, \ldots, \\ &- 1/x_n, -1/qx_n, \ldots, -1/q^{\alpha_n - 1}x_n). \end{aligned} \tag{3.10}$$

Then we have

LEMMA 3.2. *For any multi-index $\gamma \in \mathbb{N}^n$, one has*

$$\begin{aligned} &\prod_{i=1}^{n}\prod_{j=1}^{m}(1 + q^{\gamma_i} x_i y_j)|_{y=p_\alpha(x)} \\ &\quad = \prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{\nu=0}^{\alpha_j - 1}(1 - q^{\gamma_i - \nu} x_i/x_j) \\ &\quad = \prod_{1\leqslant i,j\leqslant n} (q^{\gamma_i - \alpha_j + 1} x_i/x_j)_{\alpha_j}. \end{aligned} \tag{3.11}$$

In particular, one has $\prod_{i=1}^{n}\prod_{j=1}^{m}(1 + q^{\gamma_i} x_i y_j)|_{y=p_\alpha(x)} = 0$ unless $\gamma \geqslant \alpha$.

Under the assumption of Proposition 3.2(a), we may assume that $a_\alpha(x) \neq 0$ for some $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$ without loosing generality. (If P is of order $l < m$, set $y_{l+1} = \cdots = y_m = 0$ and apply the following argument by replacing m by l.) The assumption on P implies

$$\sum_{|\gamma|\leqslant m} a_\gamma(x)\prod_{i=1}^{n}\prod_{j=1}^{m}(1 + q^{\gamma_i} x_i y_j) = 0. \tag{3.12}$$

Evaluating this equality at $y = p_\alpha(x)$, we have

$$a_\alpha(x) \prod_{1 \leqslant i,j \leqslant n} (q^{\alpha_i - \alpha_j + 1} x_i/x_j)_{\alpha_j} = 0, \tag{3.13}$$

by Lemma 3.2, since, if $|\gamma| \leqslant m$ and $\gamma \geqslant \alpha$, then $\gamma = \alpha$. This contradicts to the assumption $a_\alpha(x) \neq 0$. This completes the proofs of Proposition 3.2 and the uniqueness of B_m in Theorem 1.1.

4. Existence of B_m

In this section, we discuss the existence of a raising operator B_m.

We begin with a lemma which will play an important role in the following argument.

LEMMA 4.1. *Let $F(y) \in \mathbb{K}(x)[y]^{\mathfrak{S}_m}$ be a symmetric polynomial in $y = (y_1, \ldots, y_m)$ with coefficients in $\mathbb{K}(x)$, and suppose that $F(y)$ is of degree $\leqslant n-1$ in y_j for each $j = 1, \ldots, m$. If $F(p_\alpha(x)) = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, then $F(y)$ is identically zero as a polynomial in y.*

Proof. We prove Lemma by the induction on m. The case when $m = 1$ is obvious since $F(y)$ is of degree $\leqslant n-1$ and has n distinct zeros $-1/x_1, \ldots, -1/x_n$. For $m \geqslant 2$, we first expand $F(y)$ in terms of y_m as follows

$$F(y) = F(y_1, \ldots, y_m) = \sum_{i=0}^{n-1} F_i(y_1, \ldots, y_{m-1}) y_m^i, \tag{4.1}$$

where each coefficient $F_i(y_1, \ldots, y_{m-1})$ has degree $\leqslant n-1$ in all y_j ($j = 1, \ldots, m-1$). Let $\beta \in \mathbb{N}^n$ a multi-index with $|\beta| = m-1$ and consider the polynomial

$$f(y_m) = F(p_\beta(x), y_m) = \sum_{i=0}^{n-1} F_i(p_\beta(x)) y_m^i, \tag{4.2}$$

by evaluating $F(y)$ at $(y_1, \ldots, y_{m-1}) = p_\beta(x)$. From the assumption on $F(y)$, it follows that the polynomial $f(y_m)$ has n distinct zeros $y_m = -1/q^{\beta_i} x_i$ ($i = 1, \ldots, n$). Hence $f(y_m)$ is identically 0 as a polynomial in y_m. This implies that $F_i(p_\beta(x)) = 0$ for each $i = 0, \ldots, m-1$ and for any $\beta \in \mathbb{N}^n$ with $|\beta| = m-1$. By the induction hypothesis, we conclude that the coefficients $F_i(y_1, \ldots, y_{m-1})$ are identically zero as polynomials in $(y_1, \ldots, y_{m-1})$, namely, $F(y)$ is identically zero as a polynomial in $y = (y_1, \ldots, y_m)$. □

In view of Lemma 3.1, we propose to construct a q-difference operator

$$B = \sum_{|\alpha| \leqslant m} b_\alpha(x) T_{q,x}^\alpha \tag{4.3}$$

of order $\leqslant m$ such that

$$B_x \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + x_i y_j) = \frac{1}{y_1 \ldots y_m} D_y(1; t, q) \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + x_i y_j). \tag{4.4}$$

In the following, we denote the left-hand side and the right-hand side of this equality by $\Phi(x; y)$ and by $\Psi(x; y)$, respectively. In terms of the coefficients $b_\alpha(x)$, $\Phi(x; y)$ is expressed as

$$\Phi(x; y) = \sum_{|\alpha| \leqslant m} b_\alpha(x) \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + q^{\alpha_i} x_i y_j). \tag{4.5}$$

Note also that $\Psi(x; y)$ is a polynomial in $y = (y_1, \ldots, y_m)$ and has degree $\leqslant n - 1$ in each y_j $(j = 1, \ldots, m)$ as can be seen from (3.4). Hence, by Lemma 4.1, we see that B satisfies the desired equality if and only if

(1) $\Phi(x; y)$ is of degree $\leqslant n - 1$ in each y_j for $j = 1, \ldots, m$.
(2) $\Phi(x; p_\alpha(x)) = \Psi(x; p_\alpha(x))$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$.

Suppose now that the operator B has the property (1) mentioned above. Since the degree of $\Phi(x; y)$ in y_j is less than n for each $j = 1, \ldots, m$, we have

$$\Phi(x; y) \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + x_i y_j)^{-1} |_{y_1 \to \infty, \ldots, y_m \to \infty} = 0. \tag{4.6}$$

Hence by (4.5) we obtain

$$\sum_{|\alpha| \leqslant m} b_\alpha(x) q^{|\alpha| m} = 0, \quad \text{i.e.,} \quad b_0(x) = - \sum_{0 < |\alpha| \leqslant m} b_\alpha(x) q^{|\alpha| m}. \tag{4.7}$$

This implies that B can be represent as

$$B = \sum_{1 \leqslant |\alpha| \leqslant m} b_\alpha(x) (T_{q,x}^\alpha - q^{|\alpha| m}). \tag{4.8}$$

Note that a general B of order $\leqslant m$ has an expression of this form if and only if

$$F_1(x; y_1) = \Phi(x; y) \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + x_i y_j)^{-1} |_{y_2 \to \infty, \ldots, y_m \to \infty} \tag{4.9}$$

is of degree $\leqslant n - 1$ in y_1. We now show inductively that, for $l = 0, 1, \ldots, m$, B can be represented as follows

$$B = \sum_{l \leqslant |\alpha| \leqslant m} b_\alpha(x) \phi_{l;\alpha}(x, T_{q,x}), \tag{4.10}$$

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where

$$\phi_{l;\alpha}(x,T_{q,x})=T_{q,x}^{\alpha}+\sum_{\beta<\alpha,|\beta|<l}\phi_{l;\alpha,\beta}(x)T_{q,x}^{\beta}. \tag{4.11}$$

Assume that we have constructed such an expression for l with $l<m$. Note that

$$\Phi(x;y) = \sum_{l\leqslant|\alpha|\leqslant m} b_\alpha(x)\Bigg(\prod_{i=1}^{n}\prod_{j=1}^{m}(1+q^{\alpha_i}x_iy_j)+ + \sum_{\beta\leqslant\alpha,\,|\beta|<l}\phi_{l;\alpha,\beta}(x)\prod_{i=1}^{n}\prod_{j=1}^{m}(1+q^{\beta_i}x_iy_j)\Bigg). \tag{4.12}$$

Since property (1) of $\Phi(x;y)$ implies

$$\Phi(x;y)\prod_{i=1}^{n}\prod_{j=l+1}^{m}(1+x_iy_j)^{-1}|_{y_{l+1}\to\infty,\dots,y_m\to\infty}=0, \tag{4.13}$$

we obtain the relation

$$\sum_{l\leqslant|\alpha|\leqslant m} b_\alpha(x)\Bigg(q^{|\alpha|(m-l)}\prod_{i=1}^{n}\prod_{j=1}^{l}(1+q^{\alpha_i}x_iy_j)+ + \sum_{\beta\leqslant\alpha,\,|\beta|<l}\phi_{l;\alpha,\beta}(x)q^{|\beta|(m-l)}\prod_{i=1}^{n}\prod_{j=1}^{l}(1+q^{\beta_i}x_iy_j)\Bigg)=0. \tag{4.14}$$

In this formula we consider to specialize $y'=(y_1,\dots,y_l)$ at $p_\gamma(x)$, with the notation of (3.10), for each γ with $|\gamma|=l$. By Lemma 3.2

$$\prod_{i=1}^{n}\prod_{j=1}^{l}(1+q^{\beta_i}x_iy_j)|_{y'=p_\gamma(x)}=0,$$

unless $\beta\geqslant\gamma$. Hence formula (4.14) with $y'=p_\gamma(x)$ gives rise to

$$b_\gamma(x)q^{l(m-l)}\prod_{1\leqslant i,j\leqslant n}(q^{\gamma_i-\gamma_j+1}x_i/x_j)_{\gamma_j}+ + \sum_{|\alpha|>l} b_\alpha(x)q^{|\alpha|(m-l)}\prod_{1\leqslant i,j\leqslant n}(q^{\alpha_i-\gamma_j+1}x_i/x_j)_{\gamma_j}=0 \tag{4.15}$$

From this we have

$$b_\gamma(x)=-\sum_{\alpha>\gamma}b_\alpha(x)\psi_{\alpha,\gamma}(x), \tag{4.16}$$

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where

$$\begin{aligned}\psi_{\alpha,\gamma}(x) &= q^{(|\alpha|-|\gamma|)(m-|\gamma|)} \prod_{1\leqslant i,j\leqslant n} \frac{(q^{\alpha_i-\gamma_j+1}x_i/x_j)_{\gamma_j}}{(q^{\gamma_i-\gamma_j+1}x_i/x_j)_{\gamma_j}} \\ &= q^{(|\alpha|-|\gamma|)(m-|\gamma|)} C_{\alpha,\gamma}(x;q),\end{aligned} \tag{4.17}$$

with the notation of (1.8). Note that $\psi_{\alpha,\gamma}(x)$ depends on m but does *not* on B. Thus we obtain

$$\begin{aligned}B &= \sum_{|\gamma|=l} b_\gamma(x)\phi_{l;\gamma}(x,T_{q,x}) + \sum_{l<|\alpha|\leqslant m} b_\alpha(x)\phi_{l;\alpha}(x,T_{q,x}) \\ &= \sum_{l+1\leqslant|\alpha|\leqslant m} b_\alpha(x)\phi_{l+1;\alpha}(x,T_{q,x}).\end{aligned} \tag{4.18}$$

where $\phi_{l+1;\alpha}(x,T_{q,x})$ $(l+1\leqslant|\alpha|\leqslant m)$ are determined by

$$\phi_{l+1;\alpha}(x,T_{q,x}) = \phi_{l;\alpha}(x,T_{q,x}) - \sum_{\gamma<\alpha,|\gamma|=l} \psi_{\alpha,\gamma}(x)\phi_{l;\gamma}(x,T_{q,x}). \tag{4.19}$$

In other words, the coefficients of $\phi_{l+1;\alpha}(x;T_{q,x})$ are determined by the recurrence formula

$$\phi_{l+1;\alpha,\beta}(x) = \phi_{l;\alpha,\beta}(x) - \sum_{\beta<\gamma<\alpha,\,|\gamma|=l} \psi_{\alpha,\gamma}(x)\phi_{l;\gamma,\beta}(x) \tag{4.20}$$

for all β such that $\beta<\alpha$ and $|\beta|<l$. In this induction procedure, it is also seen by Lemma 4.1 that a general B of order $\leqslant m$ has an expression of this form (4.10) with (4.11) if and only if

$$F_l(x;y_1,\ldots,y_m) = \Phi(x;y) \prod_{i=1}^{n} \prod_{j=1}^{m} (1+x_iy_j)^{-1}|_{y_{l+1}\to\infty,\ldots,y_m\to\infty} \tag{4.21}$$

is of degree $\leqslant n-1$ in y_j for each $j=1,\ldots,l$.

In this way, we can define the q-difference operators $\phi_{l;\alpha}(x;T_{q,x})$ $(l\leqslant|\alpha|\leqslant m)$ for $l=0,\ldots,m$, inductively on l by (4.19). Note that these operators depend on the m that we have fixed in advance, but do *not* on the operator B. By using the operators we obtained at the final step $l=m$, we have the expression

$$B = \sum_{|\alpha|=m} b_\alpha(x)\phi_\alpha^{(m)}(x;T_{q,x}), \tag{4.22}$$

for B, where $\phi_\alpha^{(m)}(x;T_{q,x}) = \phi_{m;\alpha}(x;T_{q,x})$.

From this construction, we obtain the following proposition.

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PROPOSITION 4.1. *For each $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, define the q-difference operator $\phi_\alpha^{(m)}(x; T_{q,x})$ as above. Then, for any q-difference operator B of order $\leqslant m$ with coefficients in $\mathbb{K}(x)$, the following two conditions are equivalent.*

(a) $\Phi(x; y) = B_x \prod_{i=1}^n \prod_{j=1}^m (1 + x_i y_j)$ *is of degree* $\leqslant n - 1$ *in* y_j *for each* $j = 1, \dots, m$.

(b) B *is represented as*

$$B = \sum_{|\alpha|=m} b_\alpha(x) \phi_\alpha^{(m)}(x; T_{q,x}), \tag{4.23}$$

for some $b_\alpha(x) \in \mathbb{K}(x)$.

We now consider a q-difference operator B of the form Proposition 4.1, (b), so that $\Phi(x; y)$ is of degree $\leqslant n - 1$ in each y_j $(j = 1, \dots, m)$. With $\Psi(x; y)$ being the right-hand side of (4.4), the equality $\Phi(x; y) = \Psi(x; y)$ holds if and only if $\Phi(x; p_\alpha(x)) = \Psi(x; p_\alpha(x))$ for any α with $|\alpha| = m$, as we remarked before. Since

$$\Phi(x; p_\alpha(x)) = b_\alpha(x) \prod_{1 \leqslant i,j \leqslant n} (q^{\alpha_i - \alpha_j + 1} x_i / x_j)_{\alpha_j}, \tag{4.24}$$

by Lemma 3.2, the coefficients $b_\alpha(x)$ are determined as

$$b_\alpha(x) = \Psi(x; p_\alpha(x)) \prod_{1 \leqslant i,j \leqslant n} (q^{\alpha_i - \alpha_j + 1} x_i / x_j)_{\alpha_j}^{-1}, \tag{4.25}$$

for all α with $|\alpha| = m$. This completes the proof of existence of a raising operator B_m.

From the recurrence formula (4.20) we see that, for any α with $l \leqslant |\alpha| \leqslant m$, the coefficients $\phi_{l;\alpha,\beta}(x)$ of $\phi_{l;\alpha}(x; T_{q,x})$ are expressed as

$$\begin{aligned} &\phi_{l;\alpha,\beta}(x) \\ &\quad = \sum_{r=1}^{l} (-1)^r \sum_{\alpha > \gamma_1 > \cdots > \gamma_r = \beta; |\gamma_1| < l} \psi_{\alpha,\gamma_1}(x) \psi_{\gamma_1,\gamma_2}(x) \cdots \psi_{\gamma_{r-1},\gamma_r}(x), \end{aligned} \tag{4.26}$$

for all β with $\beta < \alpha$, $|\beta| < l$. In particular, we have

PROPOSITION 4.2. *For any pair (α, β) of multi-indices with $\beta \leqslant \alpha$, define a rational function $\psi_{\alpha,\beta}^{(m)}(x)$ by*

$$\begin{aligned} \psi_{\alpha,\beta}^{(m)}(x) &= q^{(|\alpha|-|\beta|)(m-|\beta|)} C_{\alpha,\beta}(x; q) \\ &= q^{(|\alpha|-|\beta|)(m-|\beta|)} \prod_{1 \leqslant i,j \leqslant n} \frac{(q^{\alpha_i - \beta_j + 1} x_i / x_j)_{\beta_j}}{(q^{\beta_i - \beta_j + 1} x_i / x_j)_{\beta_j}}. \end{aligned} \tag{4.27}$$

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Then, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, the coefficients of the q-difference operator

$$\phi_\alpha^{(m)}(x; T_{q,x}) = \sum_{\beta \leqslant \alpha} \phi_{\alpha,\beta}^{(m)}(x) T_{q,x}^\beta \tag{4.28}$$

are determined by the formula

$$\phi_{\alpha,\beta}^{(m)}(x) = \sum_{r=0}^{m} (-1)^r \sum_{\alpha=\gamma_0>\gamma_1>\ldots>\gamma_r=\beta} \psi_{\gamma_0,\gamma_1}^{(m)}(x) \cdots \psi_{\gamma_{r-1},\gamma_r}^{(m)}(x), \tag{4.29}$$

where the summation is taken over all paths in the lattice $\mathbb{N}^n$ connecting α and β.

In the next section, we will give explicit formulas for these coefficients $\phi_{\alpha,\beta}^{(m)}(x)$.

5. Explicit Formulas for $\phi_\alpha^{(m)}(x; T_{q,x})$

The goal of this section is to give the explicit formula

$$\phi_\alpha^{(m)}(x; T_{q,x}) = \sum_{\beta \leqslant \alpha} (-1)^{|\alpha|-|\beta|} q^{\binom{|\alpha|-|\beta|+1}{2}} C_{\alpha,\beta}(x; q) T_{q,x}^\beta \tag{5.1}$$

for $\phi_\alpha^{(m)}(x, T_{q,x})$ $(|\alpha| = m)$ as in Theorem 1.2. With the notation of Proposition 4.2, this formula is equivalent to

$$\begin{aligned} \phi_{\alpha,\beta}^{(m)}(x) &= (-1)^{|\alpha|-|\beta|} q^{\binom{|\alpha|-|\beta|+1}{2}} C_{\alpha,\beta}(x; q) \\ &= (-1)^{|\alpha|-|\beta|} q^{\binom{|\alpha|-|\beta|+1}{2}} \prod_{1 \leqslant i,j \leqslant n} \frac{(q^{\alpha_i-\beta_j+1} x_i/x_j)_{\beta_j}}{(q^{\beta_i-\beta_j+1} x_i/x_j)_{\beta_j}}. \end{aligned} \tag{5.2}$$

for $\beta \leqslant \alpha$.

In view of the dependence of $\psi_{\alpha,\beta}^{(m)}(x)$ on m (see Proposition 4.2), we define a function $g_{\alpha,\beta}(x)$ by

$$g_{\alpha,\beta}(x) = q^{-(|\alpha|-|\beta|)|\beta|} C_{\alpha,\beta}(x; q), \tag{5.3}$$

for any $\alpha, \beta \in \mathbb{N}^n$ with $\beta \leqslant \alpha$, so that $\psi_{\alpha,\beta}^{(m)}(x) = q^{(|\alpha|-|\beta|)m} g_{\alpha,\beta}(x)$. With these $g_{\alpha,\beta}(x)$, we also define a function $f_{\alpha,\beta}(x)$, by

$$f_{\alpha,\beta}(x) = \sum_{r=0}^{|\alpha|-|\beta|} (-1)^r \sum_{\alpha=\gamma_0>\gamma_1>\ldots>\gamma_r=\beta} g_{\gamma_0,\gamma_1}(x) \cdots g_{\gamma_{r-1},\gamma_r}(x), \tag{5.4}$$

for any $\alpha, \beta \in \mathbb{N}^n$ with $\beta \leqslant \alpha$. Then by Proposition 4.2 we have

$$\phi_{\alpha,\beta}^{(m)}(x) = q^{(|\alpha|-|\beta|)m} f_{\alpha,\beta}(x) \tag{5.5}$$

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if $|\alpha| = m$ and $\beta \leqslant \alpha$. Hence, the formula (5.2) follows from the following proposition.

PROPOSITION 5.1. *Define the rational functions $f_{\alpha,\beta}(x)$ ($\beta \leqslant \alpha$) by the formulas* (5.4) *together with* (5.3). *Then they can be determined as*

$$f_{\alpha,\beta}(x) = (-1)^{|\alpha|-|\beta|} q^{-\binom{|\alpha|-|\beta|}{2}-(|\alpha|-|\beta|)|\beta|} C_{\alpha,\beta}(x;q) \tag{5.6}$$

for any α, β with $\beta \leqslant \alpha$.

For the proof of Proposition 5.1, notice that the functions $f_{\alpha,\beta}(x)$ are defined as the matrix elements of the inverse matrix of the lower unitriangular matrix $G = (g_{\alpha,\beta}(x))_{\alpha,\beta}$. Hence we have only to show the inverse matrix of G is given by $G^{-1} = (\widetilde{f}_{\alpha,\beta}(x))_{\alpha,\beta}$ with

$$\widetilde{f}_{\alpha,\beta}(x) = (-1)^{|\alpha|-|\beta|} q^{-\binom{|\alpha|-|\beta|}{2}-(|\alpha|-|\beta|)|\beta|} C_{\alpha,\beta}(x;q). \tag{5.7}$$

Proposition 5.1 thus reduces to

LEMMA 5.1. *For any α, β with $\alpha > \beta$, one has*

$$\sum_{\alpha \geqslant \gamma \geqslant \beta} \widetilde{f}_{\alpha,\gamma}(x) g_{\gamma,\beta}(x) = 0. \tag{5.8}$$

By the definition of $g_{\alpha,\beta}(x)$ and $\widetilde{f}_{\alpha,\beta}(x)$, we have

$$\begin{aligned}
&\sum_{\alpha \geqslant \gamma \geqslant \beta} \widetilde{f_{\alpha,\gamma}}(x) g_{\gamma,\beta}(x) \\
&\quad = \sum_{\alpha \geqslant \gamma \geqslant \beta} (-1)^{|\alpha|-|\gamma|} q^{-\binom{|\alpha|-|\gamma|}{2}-(|\alpha|-|\gamma|)|\gamma|-(|\gamma|-|\beta|)|\beta|} \times \\
&\qquad \times C_{\alpha,\gamma}(x;q) C_{\gamma,\beta}(x;q).
\end{aligned} \tag{5.9}$$

Just as in the case of binomial coefficients, it is directly shown that our $C_{\alpha,\beta}(x;q)$ satisfy the following identity

$$\begin{aligned}
C_{\alpha,\gamma}(x;q) C_{\gamma,\beta}(x;q) &= C_{\alpha,\beta}(x;q) \prod_{i,j} \frac{(q^{\gamma_i-\beta_j+1} x_i/x_j)_{\alpha_i-\gamma_i}}{(q^{\gamma_i-\gamma_j+1} x_i/x_j)_{\alpha_i-\gamma_i}} \\
&= C_{\alpha,\beta}(x;q)\, C_{\alpha-\beta,\alpha-\gamma}(1/q^{\alpha} x;q),
\end{aligned} \tag{5.10}$$

where $1/q^{\alpha} x = (1/q^{\alpha_1} x_1, \ldots, 1/q^{\alpha_n} x_n)$. Hence we obtain

$$\begin{aligned}
&\sum_{\alpha \geqslant \gamma \geqslant \beta} \widetilde{f_{\alpha,\gamma}}(x) g_{\gamma,\beta}(x) \\
&\quad = q^{-(|\alpha|-|\beta|)|\beta|} C_{\alpha,\beta}(x;q) \times \\
&\qquad \times \sum_{\alpha \geqslant \gamma \geqslant \beta} (-1)^{|\alpha|-|\gamma|} q^{-\binom{|\alpha|-|\gamma|}{2}-(|\alpha|-|\gamma|)(|\gamma|-|\beta|)} C_{\alpha-\beta,\alpha-\gamma}(1/q^{\alpha} x;q).
\end{aligned} \tag{5.11}$$

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Setting $\alpha-\beta=\lambda$ and $\alpha-\gamma=\mu$, the last summation can be rewritten in the form

$$\sum_{0\leqslant\mu\leqslant\lambda}(-1)^{|\mu|}q^{|\mu|(1-|\lambda|)}q^{\binom{|\mu|}{2}}C_{\lambda,\mu}(1/q^{\alpha}x). \tag{5.12}$$

Hence Lemma 5.1 is reduced to proving that this formula becomes zero. It is in fact a special case of the following analogue of the q-binomial theorem. (Replace x by $1/q^{\alpha}x$ and set $u=q^{1-|\lambda|}$ in (5.13) below, to see that (5.12) becomes zero.)

PROPOSITION 5.2. *For any $\lambda\in\mathbb{N}^n$, one has*

$$\sum_{0\leqslant\mu\leqslant\lambda}(-u)^{|\mu|}q^{\binom{|\mu|}{2}}C_{\lambda,\mu}(x;q)=(u)_{|\lambda|}, \tag{5.13}$$

where u is an indeterminate.

Proof. This 'q-binomial theorem' follows from an identity for Macdonald's q-difference operator $D_z(u;t,q)$ in N variables $z=(z_1,\ldots,z_N)$ with $N=|\lambda|$. Since $D_z(u;t,q).1=(u)_N$, we have

$$\sum_{K\subset\{1,\ldots,N\}}(-u)^{|K|}q^{\binom{|K|}{2}}\prod_{k\in K;l\notin K}\frac{1-qz_k/z_l}{1-z_k/z_l}=(u)_N. \tag{5.14}$$

For a multi-index $\lambda\in\mathbb{N}^n$ with $|\lambda|=N$, let us specialize (5.14) at $z=p_\lambda(x)$ with the notation of (3.10). Note that, when we specialize z at $p_\lambda(x)$, the indexing set $\{1,\ldots,N\}$ is divided into n blocks with cardinality $\lambda_1,\ldots,\lambda_n$, respectively. Furthermore, for a configuration K of points in $\{1,\ldots,N\}$, the product $\prod_{k\in K;l\notin K}(1-qz_k/z_l)/(1-z_k/z_l)$ becomes zero unless the elements of K should be packed to the left in each block. Such configurations K are parameterized by multi-indices $\mu\leqslant\lambda$ such that $|\mu|=|K|$ and that μ_i denotes the number of points of K sitting in the ith block for $i=1,\ldots,n$. For such a K, one has

$$\begin{aligned}\prod_{k\in K;l\notin K}\left.\frac{1-qz_k/z_l}{1-z_k/z_l}\right|_{z=p_\lambda(x)}&=\prod_{1\leqslant i,j\leqslant n}\ \prod_{\mu_i\leqslant a<\lambda_i;0\leqslant b<\mu_j}\frac{1-q^{a-b+1}x_i/x_j}{1-q^{a-b}x_i/x_j}\\&=\prod_{1\leqslant i,j\leqslant n}\frac{(q^{\lambda_i-\mu_j+1}x_i/x_j)_{\mu_j}}{(q^{\mu_i-\mu_j+1}x_i/x_j)_{\mu_j}}=C_{\lambda,\mu}(x;q).\end{aligned} \tag{5.15}$$

(The indices are renamed by $k\to(j,b)$, $l\to(i,a)$.) Hence we obtain (5.13). □

This completes the proof of formula (5.1).

Remark. 5.1. In the case of one variable, Equation (5.13) reduces the ordinary q-binomial theorem

$$\sum_{k=0}^{l}(-1)^k q^{\binom{k}{2}}u^k\begin{bmatrix}l\\k\end{bmatrix}_q=(u)_l. \tag{5.16}$$

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If we take the coefficient of u^k in formula (5.13), we obtain

$$\sum_{\mu\leqslant\lambda,|\mu|=k}\prod_{j=1}^{n}\begin{bmatrix}\lambda_j\\ \mu_j\end{bmatrix}_q\prod_{i\neq j}\frac{(q^{\lambda_i-\mu_j+1}x_i/x_j)_{\mu_j}}{(q^{\mu_i-\mu_j+1}x_i/x_j)_{\mu_j}}=\begin{bmatrix}|\lambda|\\ k\end{bmatrix}_q, \tag{5.17}$$

for $k=0,1,\ldots,|\lambda|$. This gives a generalization of the q-Chu–Vandermonde formula. From (5.13), we also obtain another type of q-Chu–Vandermonde formula for our $C_{\alpha,\beta}(x;q)$

$$\sum_{\mu\leqslant\alpha,\nu\leqslant\beta,|\mu|+|\nu|=k} q^{(|\alpha|+|\mu|)|\nu|}C_{\alpha,\mu}(x;q)C_{\beta,\nu}(x;q)=\begin{bmatrix}|\alpha|+|\beta|\\ k\end{bmatrix}_q. \tag{5.18}$$

6. Determination of $b_{\boldsymbol{\alpha}}^{(\boldsymbol{m})}(x)$

We have already proved that our raising operator

$$B_m=\sum_{|\gamma|\leqslant m} b_\gamma^{(m)}(x)T_{q,x}^{\gamma}, \tag{6.1}$$

of row type for Macdonald polynomials has an expression

$$B_m=\sum_{|\alpha|=m} b_\alpha^{(m)}(x)\phi_\alpha^{(m)}(x;T_{q,x}), \tag{6.2}$$

with the q-difference operators $\phi_\alpha^{(m)}(x;T_{q,x})$ of (5.1). In this section, we give explicit formulas for $b_\alpha^{(m)}(x)$ for all α with $|\alpha|=m$.

As we already remarked in Section 4, the coefficients $b_\alpha^{(m)}(x)$ ($|\alpha|=m$) are determined by

$$b_\alpha(x)=\Psi(x;p_\alpha(x))\prod_{1\leqslant i,j\leqslant n}(q^{\alpha_i-\alpha_j+1}x_i/x_j)_{\alpha_j}^{-1}, \tag{6.3}$$

where

$$\Psi(x;y)=\frac{1}{y_1\ldots y_n}D_y(1;t,q)\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j). \tag{6.4}$$

(See (4.25).) Recall that

$$\begin{aligned}\Psi(x;y) &= \frac{1}{y_1\cdots y_m}\sum_{K\in\{1,\ldots m\}}(-1)^{|K|}q^{\binom{|K|}{2}}\prod_{k\in K,l\notin K}\frac{1-qy_k/y_l}{1-y_k/y_l}\times\\ &\quad\times\prod_{i=1}^{n}\left\{\prod_{k\in K}(1+tx_iy_k)\prod_{l\notin K}(1+x_iy_l)\right\}.\end{aligned}$$

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We specialize this formula at $y = p_\alpha(x)$ for each α with $|\alpha| = m$, in the same way as we did in the proof of Proposition 5.2. All the subsets K that give rise to nonzero summands after the specialization $y = p_\alpha(x)$ are parameterized by the multi-indices β such that $\beta \leqslant \alpha$ and $|\beta| = K$. With this parameterization, we already showed that

$$\prod_{k\in K, l\notin K} \frac{1 - q y_k/y_l}{1 - y_k/y_l}\bigg|_{y=p_\alpha(x)} = C_{\alpha,\beta}(x;q). \tag{6.5}$$

Renaming the indices by $k \to (j, b)$, we have

$$\begin{aligned}
&\prod_{i=1}^{n}\left\{\prod_{k\in K}(1 + t x_i y_k)\prod_{l\notin K}(1 + x_i y_l)\right\}\\
&\quad= \prod_{1\leqslant i,j\leqslant n}\prod_{b=0}^{\beta_j - 1}(1 - tq^{-b}x_i/x_j)\prod_{b=\beta_j}^{\alpha_j - 1}(1 - q^{-b}x_i/x_j)\\
&\quad= \prod_{1\leqslant i,j\leqslant n}(tq^{-\beta_j+1}x_i/x_j)_{\beta_j}(q^{-\alpha_j+1}x_i/x_j)_{\alpha_j-\beta_j}.
\end{aligned} \tag{6.6}$$

Hence we have

$$\begin{aligned}
\Psi(x; p_\alpha(x)) &= (-1)^m q^{\sum_i \binom{\alpha_i}{2}} x^\alpha \sum_{\beta\leqslant\alpha}(-1)^{|\beta|} q^{\binom{|\beta|}{2}} C_{\alpha,\beta}(x;q) \times\\
&\qquad\times \prod_{1\leqslant i,j\leqslant n}(tq^{-\beta_j+1}x_i/x_j)_{\beta_j}(q^{-\alpha_j+1}x_i/x_j)_{\alpha_j-\beta_j}.
\end{aligned}$$

By (6.3), we finally obtain

$$\begin{aligned}
b_\alpha^{(m)}(x) &= q^{\sum_i \binom{\alpha_i}{2}} x^\alpha \sum_{\beta\leqslant\alpha}(-1)^{|\alpha|-|\beta|} q^{\binom{|\beta|}{2}} C_{\alpha,\beta}(x;q) \times\\
&\qquad\times \prod_{1\leqslant i,j\leqslant n}\frac{(tq^{-\beta_j+1}x_i/x_j)_{\beta_j}(q^{-\alpha_j+1}x_i/x_j)_{\alpha_j-\beta_j}}{(q^{\alpha_i-\alpha_j+1}x_i/x_j)_{\alpha_j}}\\
&= q^{\sum_i \binom{\alpha_i}{2}} x^\alpha \sum_{\beta\leqslant\alpha}(-1)^{|\alpha|-|\beta|} q^{\binom{|\beta|}{2}} \times\\
&\qquad\times \prod_{1\leqslant i,j\leqslant n}\frac{(tq^{-\beta_j+1}x_i/x_j)_{\beta_j}(q^{-\alpha_j+1}x_i/x_j)_{\alpha_j-\beta_j}}{(q^{\beta_i-\beta_j+1}x_i/x_j)_{\beta_j}(q^{\alpha_i\alpha_j+1}x_i/x_j)_{\alpha_j-\beta_j}}
\end{aligned}$$

for any α with $|\alpha| = m$. This completes the proof of Theorem 1.2.

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Notes added

After finishing this work, Prof. C. Krattenthaller kindly gave us a comment that formula (1.12) is a special case of the ${}_6\phi_6$ Bailey summation theorem in $\mathrm{SU}(n)$ due to R. A. Gustafson [1]. Dr. M. Schlosser also pointed out that (1.13) is precisely Theorem 5.44 in S. C. Milne [7]. We are grateful for their comments. We also remark that lowering operators of row type can be constructed by the method of this paper. For each $m = 1, 2, \ldots$, we consider the following q-difference operator $A_m = \sum_{|\alpha|=m} a_\alpha^{(m)}(x)\psi_\alpha^{(m)}(x; T_{q,x})$, with

$$\psi_\alpha^{(m)}(x; T_{q,x}) = \sum_{\beta\leqslant\alpha} q^{-(|\alpha|-|\beta|)}\phi_{\alpha,\beta}^{(m)}(x)T_{q,x}^{\beta},$$

where the coefficients $a_\alpha^{(m)}(x)$ are defined by

$$\begin{aligned} a_\alpha^{(m)}(x) &= (-1)^m q^{\sum_i \binom{\alpha_i}{2}} x^{-\alpha} \sum_{\beta\leqslant\alpha} (-q^{-n}t^{1-m})^{|\beta|} q^{\binom{|\beta|}{2}} C_{\alpha,\beta}(x; q) \times \\ &\qquad \times \prod_{i,j} \frac{(tq^{-\beta_j+1}x_i/x_j)_{\beta_j}(q^{-\alpha_i+1}x_i/x_j)_{\alpha_j-\beta_j}}{(q^{\alpha_i-\alpha_j+1}x_i/x_j)_{\alpha_j}}. \end{aligned}$$

Then we have

$$A_m J_\lambda(x; q, t) = a_\lambda^m J_{\lambda-(m)}(x; q, t),$$

$$a_\lambda^m = \prod_{i=1}^{m}(1 - t^{(\lambda')_i}q^{m-i})(1 - t^{(\lambda')_i-n-1}q^{1-i}),$$

for any λ with $\lambda_1 \leqslant m$, where $\lambda - (m) = (\lambda_2, \lambda_3, \ldots)$.

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