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A NOTE ON THE HARRIS-SEVAST'YANOV TRANSFORMATION FOR SUPERCRITICAL BRANCHING PROCESSES

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Abstract

We show that the Harris-Sevast'yanov transformation for supercritical Galton-Watson processes with positive extinction probability q can be modified in such a way that the extinction probability of the new process takes any value between 0 and q. We give a probabilistic interpretation for the new process. This note is closely related to Athreya and Ney (1972), Chapter I.12.

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I. Introduction

In this note we shall deal with the Galton-Watson branching process $(Z_n)_{n=0,1,2,\ldots}$. We refer to the books of Harris (1963) and Athreya and Ney (1972) for the basic theory. As usually we assume that $Z_0 \equiv 1$ and interpret Z_n as the number of individuals alive in the nth generation. We use the same notation as in Athreya and Ney (1972): p_j = probability that an individual has j children, $j = 0, 1, 2, \ldots$; $m = \sum_{j=1}^{\infty} jp_j$, the offspring mean; $f(s) = \sum_{j=0}^{\infty} p_j s^j$, $0 \leqslant s \leqslant 1$, the probability generating function (p.g.f.) of the offspring distribution (or of Z_1); $f_n(s)$ its nth iterate ($=$ p.g.f. of Z_n); $q = P(Z_n = 0 \text{ eventually})$ the extinction probability of $(Z_n)_{n\geqslant 0}$.

We are interested in the supercritical case, that is $q < 1$ (or $1 < m \leqslant \infty$). It is well-known (see Athreya and Ney (1972), Chapter I.10 Theorem 3) that in the

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case $1 < m < \infty$, there always exist positive constants $(C_n)_{n \geqslant 0}$, such that $Z_n/C_n \underset{\text{a.s.}}{\rightarrow} W$ with $P(W = 0) = q$ and $P(0 < W < \infty) = 1 - q$. As q is the only fixed point of f in $[0, 1[$, $q = 0$ if and only if $p_0 = f(0) = 0$. Many results in the supercritical case are easily proved for the case $q = 0$. Harris (1948) and Sevast'yanov found a transformation which reduces the general case to the case $q = 0$: if $f(0) > 0$, consider

$$\hat{f}(s) = [f((1-q)s + q) - q]/(1-q), \qquad 0 \leqslant s \leqslant 1, \tag{1}$$

and the corresponding Galton-Watson process $(\hat{Z}_n)_n$. $\hat{f}(s)$ is a p.g.f. with $\hat{p}_0 = 0$ and $\hat{W}$ has the same distribution as $(1-q)W$ conditioned on the set of non-extinction of $(Z_n)_n$ (see Harris (1948), Theorem 3.2 and Athreya and Ney (1972), Chapter I.12). It can be shown, for example, (see Athreya and Ney (1972), Chapter I.10 Corollary 4 and Lemma 9) that $\hat{W}$ is absolutely continuous, and thus by the transformation above, W is absolutely continuous on the set of non-extinction.

Athreya and Ney (1972), Chapter I.12, give a probabilistic interpretation of the process $(\hat{Z}_n)_n$ (see also Athreya and Karlin (1967), Section 5II). They show that $\hat{Z}_n$ can be thought to be the number of individuals of the nth generation which have an infinite line of descent.

In this note we shall generalize this transformation such that the extinction probability $\hat{q}$ of $(\hat{Z}_n)_n$ can take any value between 0 and q and we shall again interpret $\hat{Z}_n$ in a probabilistic way. We further shall give a detailed proof for the branching property of $(\hat{Z}_n)_n$, which may be also helpful for the study of Athreya and Ney (1972), Chapter I.12 Theorem 1.

II. Construction of a process with smaller extinction probability

Suppose $q > 0$ and let $0 \leqslant \hat{q} \leqslant q$, then $z = (q - \hat{q})/(1 - \hat{q}) \in [0, q]$. We proceed in analogy to Athreya and Ney (1972), Chapter I.12 and construct the graph of the new p.g.f. $\hat{f}(s)$ out of $f(s)$ by "stretching" the square with opposite corners (z, z) and $(1, 1)$ in Figure 1 into the unit square, mapping (z, z) into $(0, 0)$. The resulting curve will be

$$\hat{f}(s) = [f((1-z)s + z) - z]/[1 - z], \qquad 0 \leqslant s \leqslant 1. \tag{2}$$

As $\hat{f}(0) = (f(z) - z)/(1 - z) \geqslant 0$, it is easily checked that $\hat{f}(s)$ is a powerseries with non-negative coefficients $(\hat{p}_j)_{j \geqslant 0}$ and as $\hat{f}(1) = 1$, $\hat{f}(s)$ is indeed a p.g.f. Furthermore it follows immediately that $\hat{m} = \hat{f}'(1) = f'(1) = m$, that $\hat{f}_n(s) = [f_n((1-z)s + z) - z]/[1 - z]$, $0 \leqslant s \leqslant 1$, $n = 1, 2, \ldots,$ and that $\hat{f}(\hat{q}) = \hat{q}$, that is if $(\hat{Z}_n)_n$ has the offspring distribution $(\hat{p}_0, \hat{p}_1, \ldots)$, then it dies out with probability $\hat{q}$. If $\hat{q} = 0$, then (2) is identical to (1), and if $\hat{q} = q$, then $\hat{f}(s) = f(s)$.

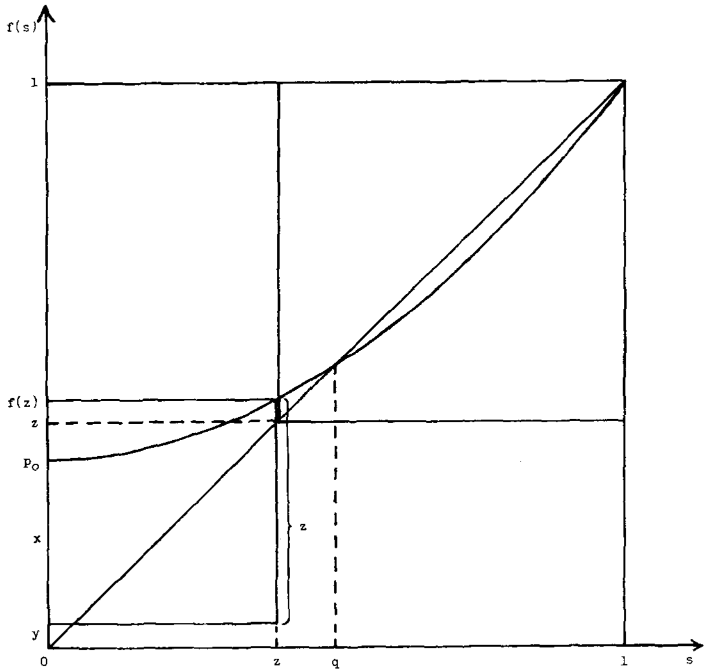

Figure 1

Finally suppose that $1 < m < \infty$ and that $\hat{Z}_n/\hat{C}_n \underset{\text{a.s.}}{\to} \hat{W}$. If $\hat{\phi}(t)$ is the Laplace-transform of $\hat{W}$, then by Athreya and Ney (1972), Chapter I.10 Theorem 3:

$$\hat{\phi}(t) = \hat{f}(\hat{\phi}(t/m)), \quad \text{or}$$

$$(1 - z)\hat{\phi}(t) + z = f((1 - z)\hat{\phi}(t/m) + z).$$

This implies that for $Z_n/C_n \underset{\text{a.s.}}{\to} W$ there exists a constant $0 < c < \infty$ such that $\hat{W}$ conditioned on $\{\hat{Z}_n \nrightarrow 0\}$ and $c \cdot W$ conditioned on $\{Z_n \nrightarrow 0\}$ have the same distribution (see also Harris (1963), Chapter I Theorem 8.2).

III. Probabilistic interpretation

In this section we interpret the branching process $(Z_n)_n$ as a model for the development of the male part of a population, that is Z_n is the number of males in the nth generation and p_j, $j = 0, 1, \ldots,$ is the probability that a male has j sons. Suppose $q > 0$, that is $p_0 > 0$, then decompose $p_0 = x + y$, $x, y > 0$. We construct now an extended version of the process $(Z_n)_n$: every male in the nth

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generation stays either unmarried (and has therefore no children, in particular no sons) with probability x, or gets married and has no sons with probability y, or gets married and has j sons, $j = 1, 2, \ldots,$ with probability p_j, independently of all the other males and of the past of the process. All the sons of the males of the nth generation form the $(n+1)$st generation, and we start the process with one male in the 0th generation. Let $Z_n^{(1)}$ be the number of males in the nth generation and $Z_n^{(0)}$ the number of married males (amongst them) which have no sons. Obviously $(Z_n^{(1)})_n$ and $(Z_n)_n$ describe the same process, and we will therefore not distinguish them. We define $(Z_n')_n = ((Z_n, Z_n^{(0)}))_n$ as the (extended) Galton-Watson process with offspring distribution $p' = (x, y, p_1, p_2, \ldots)$.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space which is large enough to accomodate the process $(Z_n')_n$. (The construction is obvious and can be left to the reader).

Definition. A male (alive in any generation) is called a B-male, if the (extended) Galton-Watson process formed by his male-descendants dies out and all male-descendants without sons are unmarried, that is each line of descent ends with a bachelor. Otherwise he is called an A-male, that is the process of his male-descendants either never dies out (that is infinite line of descent), or at least one of his male-descendants is married but has no sons.

Let I be the male of the 0th generation, and $n_0(\omega) = \max\{n \mid Z_n(\omega) > 0\}$, $\omega \in \Omega$, that is $n_0 = \infty$ on $\{Z_n \to \infty\}$. We define

$A = \{I \text{ is an } A\text{-male}\} = \{n_0 = \infty \text{ or } Z_n^{(0)} > 0 \text{ for some } n\}$;

$B = \{I \text{ is a } B\text{-male}\} = \{n_0 < \infty \text{ and } Z_n^{(0)} = 0 \text{ for all } n\}$;

$A \cup B = \Omega$.

Let $z = P(B)$, then $P(A) = 1 - z$ and $z \leqslant q$. For $j \geqslant 1$: $P(B \cap \{Z_1 = j\}) = P(B \mid Z_1 = j)p_j = z^j p_j$, and for $j = 0$: $P(B \cap \{Z_1 = 0\}) = P(I \text{ stays unmarried}) = x$. Hence $z = P(B) = \sum_{j=0}^{\infty} P(B \cap \{Z_1 = j\}) = x + \sum_{j=1}^{\infty} p_j z^j$, or

$$(3) \qquad x = p_0 - f(z) + z \quad \text{and} \quad y = p_0 - x = f(z) - z.$$

Remark. That $p_0 \geqslant f(z) - z$ follows also from $z \geqslant z\sum_{j=1}^{\infty} p_j \geqslant \sum_{j=1}^{\infty} p_j z^j = f(z) - p_0$.

We define $\hat{Z}_n$ as the number of A-males amongst the Z_n males of the nth generation. Obviously $\hat{Z}_n \equiv 0$ on B.

THEOREM 1. *Conditioned on* A, $(\hat{Z}_n)_{n=0,1,\ldots}$ *is a Galton-Watson process whose offspring distribution has the p.g.f.* $\hat{f}(s)$, *defined in* (2).

REMARK. By (3), x and y can graphically be found as indicated in Figure 1.

Before we prove the theorem we need the following lemma which can be checked easily.

LEMMA. *Suppose* $E_1, \ldots, E_n$ *are mutually exclusive events and for another event* D, $P(D|E_1) = P(D|E_2) = \ldots = P(D|E_n) = p_D$, *then also*

$$P(D|E_1 \cup \cdots \cup E_n) = p_D \tag{4}$$

PROOF OF THEOREM 1. Obviously $\hat{Z}_0 \equiv 1$ on A.

Step 1. We shall show that, conditioned on $A \cap \{\hat{Z}_n = j\}$, $\hat{Z}_{n+1}$ is distributed like the sum of j i.i.d. random variables whose distribution does not depend on n, that is $(\hat{Z}_n)_n$ is a Galton-Watson process.

On $\{Z_n = k\}$, $k \geqslant 0$, let $I_1, \ldots, I_k$ be the k males of the nth generation and M_i the number of sons of I_i, which are A-males ($M_i = 0, 1, 2, \ldots$). Let further

$$\eta_i = \begin{cases} 1 & \text{if } I_i \text{ is married and has no sons,} \\ 0 & \text{otherwise,} \end{cases}$$

$i = 1, \ldots, k$, (that is $\eta_i = 1 \Rightarrow M_i = 0$). I_i is an A-male if and only if $M_i > 0$ or $\eta_i = 1$. The branching property of $(Z_n')_n$ implies that

$$((M_i, \eta_i))_{1<i<k} \text{ are i.i.d., do not depend on } n, \text{ and}$$
$$\hat{Z}_{n+1} = \sum_{i=1}^{k} M_i \quad \text{on } \{Z_n = k\}. \tag{5}$$

Let $1 \leqslant j \leqslant k$. For $l \geqslant 0$ and $1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k$, let $1 \leqslant i_{j+1} < \cdots < i_k \leqslant k$ be such that $\{i_1, \ldots, i_k\} = \{1, \ldots, k\}$ and define

$$D = \{\hat{Z}_{n+1} = l; Z_n = k\} = \left\{ \sum_{i=1}^{k} M_i = l \right\},$$

$$E_{i_1,\ldots,i_j} = \{Z_n = k, I_{i_1}, \ldots, I_{i_j} \text{ are } A\text{-males}, I_{i_{j+1}}, \ldots, I_{i_k} \text{ are } B\text{-males}\},$$
$$= \bigcap_{r=1}^{j} \{M_{i_r} > 0 \text{ or } \eta_{i_r} = 1\} \cap \bigcap_{r=j+1}^{k} \{M_{i_r} = 0 \text{ and } \eta_{i_r} = 0\}.$$

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The $E_{i_1,\ldots,i_j}$'s are mutually exclusive and by (5), $P(D|E_{i_1,\ldots,i_j}) = p_D$ independent of $i_1, \ldots, i_j$. Hence, employing (4) and (5),

$$\begin{aligned}
&P(\hat{Z}_{n+1} = l|\hat{Z}_n = j, Z_n = k)\\
&\quad = P\left(\sum_{i=1}^{k} M_i = l|Z_n = k \text{ and } j \text{ of these males are } A\text{-males}\right)\\
&\quad = P\left(\sum_{i=1}^{k} M_i = l|\bigcup_{\{i_1,\ldots,i_j\}\subset\{1,\ldots,k\}} E_{i_1,\ldots,i_j}\right) = P\left(\sum_{i=1}^{k} M_i = l|E_{1,\ldots,j}\right)\\
&\quad = P\left(\sum_{i=1}^{j} M_i = l|E_{1,\ldots,j}\right) = P\left(\sum_{i=1}^{j} M_i = l|M_r > 0 \text{ or } \eta_r = 1 \text{ for } 1 \leq r \leq j\right)\\
&\quad = P\left(\sum_{i=1}^{j} N_j = l\right), \quad \text{where } N_1, \ldots, N_j \text{ are i.i.d. with distribution}
\end{aligned}$$

$$P(N_i = r) = P(M_i = r|M_i > 0 \text{ or } \eta_i = 1), \quad r = 0, 1, \ldots.$$

As for $j \geq 1$, $\{\hat{Z}_n = j\} \subset A$,

$$\begin{aligned}
P(\hat{Z}_{n+1} = l|(\hat{Z}_n = j) \cap A) &= P(\hat{Z}_{n+1} = l|\hat{Z}_n = j)\\
&= \sum_{k=j}^{\infty} P(\hat{Z}_{n+1} = l|\hat{Z}_n = j, Z_n = k)P(Z_n = k|\hat{Z}_n = j)\\
&= P\left(\sum_{i=1}^{j} N_i = l\right)\sum_{k=j}^{\infty} P(Z_n = k|\hat{Z}_n = j) = P\left(\sum_{i=1}^{j} N_i = l\right).
\end{aligned}$$

On $\{\hat{Z}_n = 0\} \cap A$, Z_n consists only of B-males and hence Z_{n+1} consists only of B-males, that is

$$P(\hat{Z}_{n+1} = 0|(\hat{Z}_n = 0) \cap A) = 1 = P\left(\sum_{i=1}^{0} N_i = 0\right).$$

Hence on A, $\hat{Z}_{n+1}$ is distributed like $\sum_{i=1}^{\hat{Z}_n} N_i$, that is $(\hat{Z}_n)_n$ is a Galton-Watson process.

Step 2. It is left to show that $(\hat{Z}_n)_n$ conditioned on A has the offspring distribution which corresponds to $\hat{f}(s)$. It is enough to calculate the p.g.f. of $\hat{Z}_1$ conditioned on A.

$$\begin{aligned}
P(A) = 1 - z;\ P((\hat{Z}_1 = 0) \cap A) &= P(I \text{ is married but has no sons})\\
&= y = f(z) - z \text{ by (3).}
\end{aligned}$$

For $1 \leqslant j \leqslant k$: $P(\hat{Z}_1 = j | Z_1 = k) = P(j$ of the k males are A-males) $= \binom{k}{j}(1-z)^j z^{k-j}$ by the branching property of $(Z'_n)_n$. As $(\hat{Z}_1 = j) \subset A$,

$$P((\hat{Z}_1 = j) \cap A) = P(\hat{Z}_1 = j) = \sum_{k=j}^{\infty} P(\hat{Z}_1 = j | Z_1 = k) P(Z_1 = k)$$
$$= \sum_{k=j}^{\infty} \binom{k}{j} (1-z)^j z^{k-j} p_k.$$

Hence

$$\sum_{j=0}^{\infty} P(\hat{Z}_1 = j | A) s^j = (1-z)^{-1} \left(f(z) - z + \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \binom{k}{j} (1-z)^j s^j z^{k-j} p_k \right)$$
$$= (1-z)^{-1} \left(f(z) - z + \sum_{k=1}^{\infty} p_k \sum_{j=1}^{k} \binom{k}{j} (1-z)^j s^j z^{k-j} \right)$$
$$= (1-z)^{-1} \left(f(z) - z + \sum_{k=1}^{\infty} p_k (((1-z)s + z)^k - z^k) \right)$$
$$= (1-z)^{-1} (f(z) - z + f((1-z)s + z) - f(z)) = \hat{f}(s).$$

Remark. In the case $\hat{q} = 0$ we have the following simplifications: $z = q$, $x = p_0$, $y = 0$: an A-male is a male with an infinite line of descent, and $\eta_i \equiv 0$.

The following two results can be shown in a similar way as the Theorems 2 and 3 of Athreya and Ney (1972), Chapter I.12.

Theorem 2. *On* $\{Z_n \to \infty\}$,

$$\hat{Z}_n \Big/ Z_n \underset{a.s.}{\to} (1-z).$$

If $\hat{Z}_n = 0$ *for some* n, *then also* $Z_{n'} = 0$ *for some* n'.

Theorem 3. *Conditioned on* B, *the process* $(Z_n)_n$ *is a subcritical Galton-Watson process* (*that is* $E(Z_1|B) < 1$) *whose offspring distribution has the p.g.f.*

$$\tilde{f}(s) = [f(z \cdot s) - (f(z) - z)]/z, \qquad 0 \leqslant s \leqslant 1.$$

$(\Rightarrow \tilde{m} = \tilde{f}'(1) = f'(z) < 1.)$

Remark. (a) On B, Z_n is the number of B-males in the nth generation.

(b) The graph of $\tilde{f}(s)$ can be constructed out of the graph of $f(s)$ by "stretching" the square with opposite corners $(0, y)$ and $(z, f(z))$ in Figure 1 into the unit square, mapping $(0, y)$ into $(0, 0)$ and $(z, f(z))$ into $(1, 1)$.

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