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Three Fixed Point Theorems: Periodic Solutions of a Volterra Type Integral Equation with Infinite Heredity

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Abstract. In this paper we study the existence of periodic solutions of a Volterra type integral equation with infinite heredity. Banach fixed point theorem, Krasnosel'skii's fixed point theorem, and a combination of Krasnosel'skii's and Schaefer's fixed point theorems are employed in the analysis. The combination theorem of Krasnosel'skii and Schaefer requires an *a priori* bound on all solutions. We employ Liapunov's direct method to obtain such an *a priori* bound. In the process, we compare these theorems in terms of assumptions and outcomes.

1 Introduction

To study the qualitative behavior of ordinary or functional differential equations, one normally inverts these into integral equations. The resulting integral equation is frequently a Volterra type equation. The integrals of a Volterra equation can take the form

$$\int_{t-h}^{t}, \quad \text{or} \quad \int_{t_0}^{t}, \quad \text{or} \quad \int_{-\infty}^{t},$$

depending on the duration of "heredity." For example, the neutral functional differential equation $x'(t) = ax(t) + \alpha x'(t-h) - q(x(t), x(t-h)) + r(t)$ can be inverted into the integral equation

$$x(t) = \alpha x(t-h) - \int_{-\infty}^{t} [q(x(s), x(s-h)) - a\alpha x(s-h)]e^{a(t-s)}\, ds + p(t), \tag{1.1}$$

if the integration is carried out from $-\infty$ to t, and we seek a solution function x having the property

$$\lim_{s\to-\infty} [x(s) - \alpha x(s-h)]e^{-as} = 0.$$

Examples of neutral functional differential equations and their applications can be found in [9, 12, 13, 17, 20, 21]. Recently investigators gave heuristic arguments to support their use of these types of equations in describing certain biological phenomena (see [12, 20]).

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In the present paper we consider the following generalization of (1.1).

$$x(t) = f(t, x(t), x(t-h)) - \int_{-\infty}^{t} C(t,s) g(s, x(s), x(s-h))\, ds, \tag{1.2}$$

where $f\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and $C\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are all continuous. We refer to (1.2) as a Volterra type integral equation with infinite heredity. We study the existence of continuous periodic solutions of (1.2) under suitable assumptions on the functions f, g, and C.

Generally, a fixed point theorem is used to study the existence of periodic solutions to this type of equation. We employ the Banach fixed point theorem (also known as the contraction principle), Krasnosel'skii's fixed point theorem (Theorem 1.1) and a fixed point theorem (Theorem 1.3) that combines Krasnosel'skii's theorem and Schaefer's fixed point theorem (Theorem 1.2). This combination theorem was obtained by Burton and Kirk [4]. Statements of these theorems are provided at the end of this section.

In the process of obtaining periodic solutions to (1.2), we compare these theorems in terms of assumptions and outcomes. As we know, the Banach fixed point theorem gives the uniqueness of the solution, but it restricts the sizes of the functions involved in the equation. In particular, we observe that for equation (1.2), the Banach fixed point theorem requires functions C and g to be small for a given f. Likewise, we find that Theorem 1.1 places some size restrictions on functions C and g. On the other hand, Theorem 1.3 does not place any size restrictions on these functions. However, due to Schaefer's fixed point theorem, Theorem 1.3 requires an *a priori* bound on all solutions. Following a technique similar to that of Burton and Kirk [4], we employ Liapunov's direct method to obtain such an *a priori* bound on all periodic solutions of (1.2). We use a Liapunov functional in the analysis and find that functions C and g need to satisfy certain sign conditions. One might be able to obtain the required *a priori* bound without these sign conditions, by employing a different method, or constructing a suitable Liapunov functional different from ours. Our analysis, therefore, indicates that the use of Theorem 1.3 to study periodic solutions of equations like (1.2) has potential for yielding better results than the use of Theorem 1.1 alone.

Related to Schaefer's theorem, the degree-theoretic work of Granas [14], which also requires an *a priori* bound on all solutions, has been used by many researchers to study the existence of bounded and/or periodic solutions of certain equations (see [5–7, 10, 11, 15]). Recently, some researchers have studied these existence results for functional equations using fixed point theorems on time scales. We refer readers interested in time scales to [1, 19], and the references therein.

We remark that in this paper we use Liapunov's method for the integral equation (1.2). Although Liapunov's direct method has been used extensively for ordinary and functional differential equations, its use on integral equations is relatively new and somewhat limited. Readers interested in Liapunov's method for integral equations will find [3] a very useful resource. In a parallel article [18], the author has studied periodic solutions of an integral equation with finite delay employing the same fixed point theorems used in this paper. For results on basic existence theory for Volterra type integral equations, we refer readers to [8, 16, 23].

Theorem 1.1 (Krasnosel'skii) *Let M be a closed convex subset of a Banach space S. Suppose A and B map M into S such that*

(i) *$x, y \in M$, implies $Ax + By \in M$,*
(ii) *A is continuous and AM is contained in a compact subset of S,*
(iii) *B is a contraction mapping.*

Then there exists $z \in M$ with $z = Az + Bz$.

Theorem 1.2 (Schaefer) *Let S be a normed space and H a continuous mapping of S into S that is compact on each bounded subset X of S. Then either* (i) *the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or* (ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded. See* [24].

Theorem 1.3 (Krasnosel'skii–Schaefer) *Let S be a Banach space. Suppose A and B map S into S, where B is a contraction, and A is continuous with A mapping bounded sets into compact sets. Then either* (i) *$x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ has a solution in S for $\lambda = 1$, or* (ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded. See* [4].

2 Solution by Banach Fixed Point Theorem.

In this section, we employ the Banach fixed point theorem on (1.2), and obtain a unique periodic solution. In addition to the basic continuity conditions on functions f, g, and C, we assume the following:

(A1) there exists a constant $T > 0$ such that

$$f(t+T, x, y) = f(t, x, y), \quad g(t+T, x, y) = g(t, x, y),$$
$$C(t+T, s+T) = C(t, s);$$

(A2) there exist positive constants a, b, c, and d such that

(i) $|f(t, x_1, y_1) - f(t, x_2, y_2)| \le a|x_1 - x_2| + b|y_1 - y_2|, \quad a + b < 1,$
(ii) $|g(t, x_1, y_1) - g(t, x_2, y_2)| \le c|x_1 - x_2| + d|y_1 - y_2|;$

(A3) there exists a constant $C^* > 0$ such that

$$\sup_{t \in R} \int_{-\infty}^{t} |C(t, s)|\, ds \le C^*;$$

(A4) there exists a constant $Q > 0$ such that for $0 \le u \le v \le T$,

$$\int_{-\infty}^{u} |C(u, s) - C(v, s)|\, ds \le Q|u - v|.$$

Let P_T be the Banach space of all continuous T-periodic real-valued functions with the supremum norm $\|\cdot\|$.

Theorem 2.1 *Suppose* (A1)–(A4) *hold, and suppose $(a + b) + (c + d)C^* < 1$. Then* (1.2) *has a unique continuous T-periodic solution.*

Proof For $\varphi \in P_T$, let

$$(P\varphi)(t) = (B\varphi)(t) + (A\varphi)(t), \tag{2.1}$$

where

$$(B\varphi)(t) = f(t, \varphi(t), \varphi(t-h)), \tag{2.2}$$

$$(A\varphi)(t) = -\int_{-\infty}^{t} C(t,s) g(s, \varphi(s), \varphi(s-h))\, ds. \tag{2.3}$$

Clearly, $(B\varphi)(t)$ is continuous and T-periodic in t because f is continuous and T-periodic in t. We want to show that $(A\varphi)(t)$ is also T-periodic and continuous in t. It is an easy exercise to see that $(A\varphi)(t+T) = (A\varphi)(t)$. Now we show that $(A\varphi)(t)$ is continuous in t. For any $\varphi \in P_T$, $\|\varphi\| \leq m$ for some m. Since g is a continuous function and g is T-periodic in t, for $\varphi \in P_T$ with $\|\varphi\| \leq m$ there exists a $\bar{G}$ such that $|g(t, \varphi(t), \varphi(t-h))| \leq \bar{G}$ for $t \in R$. Since C is continuous, there exists a constant $\bar{C} > 0$ such that if $0 \leq t \leq T$, $0 \leq s \leq T$, then $|C(t,s)| \leq \bar{C}$.

Now using (A4), one obtains for $0 \leq u \leq v \leq T$,

$$\begin{aligned} |(A\varphi)(u) - (A\varphi)(v)| &\leq \int_{-\infty}^{u} |C(u,s) - C(v,s)||g(s, \varphi(s), \varphi(s-h))|\, ds \\ &\quad + \int_{u}^{v} |C(v,s)||g(s, \psi(s), \psi(s-h))|\, ds \\ &\leq (Q + \bar{C})\bar{G}|u - v|. \end{aligned}$$

This shows that $(A\varphi)(t)$ is continuous in t. Therefore, it follows from (2.1) that $(P\varphi)(t)$ is continuous and T-periodic, *i.e.,* $P\varphi \in P_T$ for each $\varphi \in P_T$. Now we show that P is a contraction mapping on P_T. Let $\varphi, \psi \in P_T$. Then

$$\begin{aligned} &|(P\varphi)(t) - (P\psi)(t)| \\ &\quad \leq |(B\varphi)(t) - (B\psi)(t)| + |(A\varphi)(t) - (A\psi)(t)| \\ &\quad \leq |f(t, \varphi(t), \varphi(t-h)) - f(t, \psi(t), \psi(t-h))| \\ &\qquad + \Big| -\int_{-\infty}^{t} C(t,s) g(s, \varphi(s), \varphi(s-h))\, ds \\ &\qquad\quad + \int_{-\infty}^{t} C(t,s) g(s, \psi(s), \psi(s-h))\, ds \Big| \\ &\quad \leq a|\varphi(t) - \psi(t)| + b|\varphi(t-h) - \psi(t-h)| \\ &\qquad + \int_{-\infty}^{t} |C(t,s)||g(s, \varphi(s), \varphi(s-h)) - g(s, \psi(s), \psi(s-h))|\, ds \\ &\quad \leq (a+b)\|\varphi - \psi\| + \int_{-\infty}^{t} |C(t,s)|(c+d)\|\varphi - \psi\|\, ds \\ &\quad \leq [(a+b) + (c+d)C^*]\|\varphi - \psi\|. \end{aligned}$$

This shows that P is a contraction on P_T. Therefore, by the Banach fixed point theorem, there exists a unique function $\varphi \in P_T$ such that $P\varphi = \varphi$; then φ is the unique solution of (1.2). ■

We remark that to satisfy the condition $(a+b)+(c+d)C^* < 1$, if $(a+b)$ is close to 1, then $(c+d)C^*$ needs to be small. This means that to apply the Banach fixed point theorem on (1.2), one may have to choose functions C and g small for a given f.

3 Solution by Krasnosel'skii's Fixed Point Theorem.

In this section, we apply Theorem 1.1 to (1.2) to obtain a periodic solution. Let P_T be the Banach space defined in Section 2. We continue to assume that assumptions (A1)–(A4) hold. Let $m > 0$ be any constant. Then

$$M = \{\varphi \in P_T : \|\varphi\| \le m\} \tag{3.1}$$

is a closed convex subset of P_T.

Assume that

(A5) the function g is bounded. *i.e.,* there exists a positive constant G such that $|g(t,x,y)| \le G$ for all $t,x,y \in R$.

Using (2.3), we define a mapping $A\colon M \to P_T$, *i.e.,* for $\varphi \in M$,

$$(A\varphi)(t) = -\int_{-\infty}^{t} C(t,s)g(s,\varphi(s),\varphi(s-h))\,ds. \tag{3.2}$$

Again it is easy to see that $(A\varphi)(t)$ is T-periodic and continuous in t, and hence $A\varphi \in P_T$ for $\varphi \in M$. Also, for each $\varphi \in M$,

$$|(A\varphi)(t)| \le \int_{-\infty}^{t} |C(t,s)||g(s,\varphi(s),\varphi(s-h))|\,ds \le C^*G, \tag{3.3}$$

where C^* is the constant in (A3), and G is the constant in (A5). This proves that the set $\{A\varphi : \varphi \in M\}$ is (uniformly) bounded. The arguments used earlier to show the continuity of $(A\varphi)(t)$ in t, will in fact prove that the set $\{A\varphi : \varphi \in M\}$ is equicontinuous. Therefore, by the Arzela–Ascoli theorem A maps M into a compact set. Now we show that mapping A of (3.2) is continuous. For that, pick $\varphi,\psi \in M$. Then we have for $0 \le u \le T$

$$\begin{aligned}|(A\varphi)(u) - (A\psi)(u)| &\le \int_{-\infty}^{u} |C(u,s)||g(s,\varphi(s),\varphi(s-h)) - g(s,\psi(s),\psi(s-h))|\,ds\\ &\le (c+d)C^*\|\varphi-\psi\|.\end{aligned}$$

This proves that mapping A is continuous. Thus, condition (ii) of Theorem 1.1 is satisfied.

Using (2.2), we define $B\colon M \to P_T$, *i.e.,* $\varphi \in M$, $(B\varphi)(t) = f(t,\varphi(t),\varphi(t-h))$. We know that $(B\varphi)(t)$ is continuous and T-periodic in t and hence $B\varphi \in P_T$ for each $\varphi \in M$.

It follows from (A2)(i) that B is a contraction mapping, which satisfies condition (iii) of Theorem 1.1. Let

$$\bar{m} = \max\{|f(t,0,0)| : 0 \le t \le T\}. \tag{3.4}$$

Choose a constant m such that

$$(a+b)m + \bar{m} + C^*G \le m, \tag{3.5}$$

where C^* is defined in (A3) and G in (A5). Now for the m of (3.5) consider the set M defined in (3.1). For $\varphi, \psi \in M$, we have

$$\begin{aligned}|(A\varphi)(t) + (B\psi)(t)| \le |f(t,\psi(t),\psi(t-h))| \\ + \left| \int_{-\infty}^{t} C(t,s)g(s,\varphi(s),\varphi(s-h))\,ds \right|.\end{aligned} \tag{3.6}$$

Notice that

$$\begin{aligned}|f(t,\psi(t),\psi(t-h))| &\le |f(t,\psi(t),\psi(t-h)) - f(t,0,0)| + |f(t,0,0)| \\ &\le a|\psi(t)| + b|\psi(t-h)| + \bar{m} \\ &\le (a+b)\|\psi\| + \bar{m}.\end{aligned} \tag{3.7}$$

Then, for $\varphi, \psi \in M$, it follows from (3.3), (3.5), (3.6), and (3.7), that

$$|(A\varphi)(t) + (B\psi)(t)| \le (a+b)m + \bar{m} + C^*G \le m.$$

This proves that for $\varphi, \psi \in M$ we have $A\varphi + B\psi \in M$, which establishes condition (i) of Theorem 1.1.

Now we obtain the existence of a periodic solution of (1.2) in the next theorem.

Theorem 3.1 *Suppose assumptions* (A1)-(A5) *hold. Then there exists a continuous T-periodic solution of* (1.2).

Proof From the preceding work it follows from Theorem 1.1 that there exists a function $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. This function φ is a solution of (1.2). ■

Remark One can see from condition (3.5) that if $(a+b)$ is close to 1, then constants C^* and G need to be small, which means functions C and g need to be small. Also observe that if $(a+b)$ is close to 1, then $|f(t,0,0)|$ needs to be small.

4 Solution by Theorem 1.3

In this section, we employ Theorem 1.3 stated in the introduction. We continue to assume that functions f, g, and C are all continuous, and assumptions (A1)–(A4) hold.

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Let P_T be the Banach space defined in Section 2. Define mappings B and A from P_T into P_T by (2.2) and (2.3), respectively. Clearly, B is a contraction with contraction constant $(a+b)$. Using the arguments similar to those of the previous sections, one can easily verify that A is a continuous mapping from P_T into P_T and that A maps bounded sets into compact sets. To show that A maps bounded sets into compact sets, assumption (A5) is not needed.

Next, notice that if mapping B is defined by $(Bx)(t) = f(t, x(t), x(t-h))$, then for any scalar λ,

$$\Big(\lambda B\Big(\frac{x}{\lambda}\Big)\Big)(t) = \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big). \tag{4.1}$$

Lemma 4.1 ([3, Proposition 6.1.1]) *The mapping* $(\lambda B(\frac{x}{\lambda}))$ *defined in* (4.1) *is a contraction on* P_T.

Proof First notice that for $x \in P_T, (x/\lambda) \in P_T$. So $B(x/\lambda) \in P_T$ because B is a mapping on P_T. Therefore, $\lambda B(x/\lambda) \in P_T$. Now, for any $x, y \in P_T$, it follows from (A5) that

$$\begin{aligned}|\lambda B(x/\lambda)(t) - \lambda B(y/\lambda)(t)| &= \Big|\lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h}{\lambda}\Big) - \lambda f\Big(t, \frac{y(t)}{\lambda}, \frac{y(t-h)}{\lambda}\Big)\Big| \\ &= \lambda\Big[a\Big|\frac{x(t)}{\lambda} - \frac{y(t)}{\lambda}\Big| + b\Big|\frac{x(t-h)}{\lambda} - \frac{y(t-h)}{\lambda}\Big|\Big] \\ &\le (a+b)\|x-y\|.\end{aligned}$$

Therefore, $(\lambda B(x/\lambda))$ is a contraction with contraction constant $a+b<1$. ■

We already know that A defined in (2.3) (also in (3.2)) is a continuous mapping on P_T and it maps bounded sets into compact sets. So for any λ, $0 < \lambda \le 1$, the same properties hold for mapping λA. Therefore, by Theorem 1.3 the equation $x = \lambda B(x/\lambda) + \lambda Ax$ has a solution $x \in P_T$ provided the set of all solutions x, $0<\lambda<1$, is bounded. This means if we can show that all solutions of

$$x(t) = \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big) - \lambda \int_{-\infty}^{t} C(t,s) g(s, x(s), x(s-h))\, ds. \tag{4.2}$$

for all λ, $0<\lambda<1$, are bounded by a fixed constant, independent of λ, then (1.2) has a continuous T-periodic solution. In the next lemma, we show the existence of such a fixed bound on all T-periodic solutions of (4.2) for all λ, $0<\lambda<1$.

Lemma 4.2 *Assume* (A1)–(A4) *hold. Also, assume*

(A6) $xg(t,x,y) \ge 0$, *and there exists* $\beta > 0$ *and* $L > 0$ *such that*

$$\big[-(1-a)xg(t,x,y) + b|y||g(t,x,y)| + \bar{m}|g(t,x,y)|\big] \le L - \beta|g(t,x,y)|,$$

where $\bar{m}$ *is the constant defined in* (3.4);

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(A7) $C_s(t,s) \geq 0,\ C_{st}(t,s) \leq 0,\ C_s,\ C_{st}$ *continuous,*

$$\sup_{t\in R} \int_{-\infty}^{t} C_s(t,s)\,ds \leq M^* < \infty,$$

$$\sup_{t\in R} \int_{-\infty}^{t} C_s(t,s)(t-s)^2\,ds \leq K^* < \infty;$$

(A8) *for each* t, $\lim_{s\to-\infty}(t-s)C(t,s) = 0$.

Then for any λ, $0 < \lambda \leq 1$, *if* x *satisfies* (4.2), *then there exists a positive constant* K, *independent of* λ, *such that* $\|x\| < K$.

Proof Let x be a T-periodic solution of (4.2). Define a Liapunov functional

$$V(t) := V(t,x(\cdot)) = \lambda^2 \int_{-\infty}^{t} C_s(t,s)\Big(\int_s^t g(u,x(u),x(u-h))du\Big)^2\,ds. \tag{4.3}$$

One can easily verify that $V(t)$ is T-periodic in t. Differentiating (4.3),

$$V'(t) = \lambda^2 \int_{-\infty}^{t} C_{st}(t,s)\Big(\int_s^t g(u,x(u),x(u-h))du\Big)^2\,ds$$
$$+ 2\lambda^2 g(t,x(t),x(t-h)) \int_{-\infty}^{t} C_s(t,s) \int_s^t g(u,x(u),x(u-h))\,duds.$$

Integrating the second term by parts, one gets

$$2\lambda^2 g(t,x(t),x(t-h))\Big[C(t,s)\int_s^t g(u,x(u),x(u-h),du\Big|_{s=-\infty}^{t}$$
$$+ \int_{-\infty}^{t} C(t,s)g(s,x(s),x(s-h))\,ds\Big].$$

Here x is a T-periodic solution function of (4.2), and hence x is bounded. Since g is bounded, for each bounded x the first term of the above integral vanishes at both limits by (A8). Since $C_{st} \leq 0$, the first term of V' is not positive. So we can write, using (4.2),

$$V'(t) \leq 2\lambda^2 g(t,x(t),x(t-h))\Big[\int_{-\infty}^{t} C(t,s)g(s,x(s),x(s-h))\,ds\Big]$$
$$= 2\lambda g(t,x(t),x(t-h))\Big[\lambda f\Big(t,\frac{x(t)}{\lambda},\frac{x(t-h)}{\lambda}\Big) - x(t)\Big].$$

It follows from (3.7) that

$$\Big|\lambda f\Big(t,\frac{x(t)}{\lambda},\frac{x(t-h)}{\lambda}\Big)\Big| \leq a|x(t)| + b|x(t-h)| + \bar{m}, \tag{4.4}$$

where $\bar{m}$ is the constant defined in (3.4). So

$$\begin{aligned} V'(t) &\le 2\lambda\Big[|g(t,x(t),x(t-h))| \big\{ a|x(t)| + b|x(t-h)| + \bar{m}\} \\ &\qquad - x(t)g(t,x(t),x(t-h))\Big] \\ &= 2\lambda\Big[-(1-a)x(t)g(t,x(t),x(t-h)) + b|x(t-h)||g(t,x(t),x(t-h))| \\ &\qquad + \bar{m}|g(t,x(t),x(t-h))|\Big] \\ &\le \lambda[2L - 2\beta|g(t,x(t),x(t-h))|]. \end{aligned}$$

Assumption (A6) is used in the last step of the above inequality.

Since V is T-periodic, there is a sequence $\{t_n\} \uparrow \infty$ with $V(t_n) \ge V(s)$ for $s \le t_n$. Thus, intergating the above inequality from s to t_n, we get

$$0 \le V(t_n) - V(s) \le -2\lambda\beta \int_s^{t_n} |g(v,x(v),x(v-h))|\,dv + 2\lambda L(t_n - s).$$

This implies

$$\beta \int_s^{t_n} |g(v,x(v),x(v-h))|\,dv \le L(t_n - s).$$

Then from (4.3) and the second integral of (A7), we obtain

$$\begin{aligned} V(t_n) &= \lambda^2 \int_{-\infty}^{t_n} C_s(t_n,s) \Big(\int_s^{t_n} g(v,x(v),x(v-h))\,dv \Big)^2 ds \\ &\le \lambda^2 \Big(\frac{L}{\beta}\Big)^2 K^* \le \Big(\frac{L}{\beta}\Big)^2 K^* := \bar{K}. \end{aligned} \tag{4.5}$$

This proves that $V(t)$ is bounded by $\bar{K}$ for all t. Now from (4.2) we get

$$x(t) - \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big) = -\lambda \int_{-\infty}^t C(t,s)g(s,x(s),x(s-h))\,ds. \tag{4.6}$$

Integrating the right-hand side of the above equation by parts,

$$\lambda\Big[C(t,s) \int_s^t g(u,x(u),x(u-h))\,du\Big|_{s=-\infty}^t - \int_{-\infty}^t C_s(t,s) \int_s^t g(u,x(u),x(u-h))\,duds \Big].$$

Again, the first term vanishes at both limits by (A8). So squaring both sides of (4.6), we get

$$\begin{aligned} &\Big(x(t) - \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big)\Big)^2 \\ &\qquad = \lambda^2 \Big(\int_{-\infty}^t C_s(t,s) \int_s^t g(u,x(u),x(u-h))\,duds \Big)^2. \end{aligned} \tag{4.7}$$

Applying the Schwartz inequality on the right side of (4.7) and then using equation (4.3), the first integral of (A7), and the fact that $V(t)$ is bounded by $\bar{K}$ for all t as shown in (4.5), we can write

$$\begin{aligned}
&\Big(x(t) - \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big)\Big)^2 \\
&\qquad \le \lambda^2 \int_{-\infty}^{t} C_s(t,s)\,ds \int_{-\infty}^{t} C_s(t,s)\Big(\int_s^t g(v, x(v), x(v-h))\,dv\Big)^2 ds \\
&\qquad \le M^* V(t) \le M^* \bar{K}.
\end{aligned}$$

Taking the square root on both sides of the above inequality, one obtains,

$$\Big|x(t) - \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big)\Big| \le \sqrt{M^*\bar{K}}.$$

Then it follows from the above relation and from (4.4) that

$$\begin{aligned}
|x(t)| &\le \Big|x(t) - \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big)\Big| + \Big|\lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big)\Big| \\
&\le \sqrt{M^*\bar{K}} + (a+b)\||x\| + \bar{m},
\end{aligned}$$

from which we obtain,

$$\|x\| \le \frac{\bar{m} + \sqrt{M^*\bar{K}}}{1-(a+b)} := K. \qquad \blacksquare$$

Therefore, we have the following theorem.

Theorem 4.3 *Suppose assumptions* (A1)–(A4), *and* (A6)–(A8) *hold. Then there exists a continuous T-periodic solution of* (1.2).

Proof Proof of this theorem follows from Theorem 1.3. All required work has already been shown. ■

Remark We observed in this section that no size restrictions are placed on the functions C and g, although we had to use some sign conditions as shown in (A6) and (A7). However, we used these sign conditions only to obtain an *a priori* bound employing Liapunov's method, which requires construction of a suitable Liapunov functional. We used one such functional for equation (1.2). It is possible that one can employ an alternative method to Liapunov's, or perhaps construct an entirely different Liapunov functional that may not need these sign conditions. Therefore, our analysis indicates that the use of Theorem 1.3 to study periodic solutions of equations like (1.2) has potential for yielding better results than the use of Theorem 1.1 alone

Example Let

$$f(t,x,y) = a \sin t \Big(x + \frac{y}{1+y^2}\Big), \ 0 < a < \frac{1}{2},$$

$$g(t,x,y) = k \sin^2 t \Big(\frac{x}{1+x^2} + \frac{x}{1+x^2+y^2}\Big), \ k > 0.$$

One can easily varify that f and g satisfy (A1) and (A2), $|g(t,x,y)| \leq k$ and $xg(t,x,y) \geq 0$. Also,

$$\Big|\lambda f\Big(t, \frac{x}{\lambda}, \frac{y}{\lambda}\Big)\Big| \leq a|x| + \frac{a}{2}.$$

In this case, the condition of (A6) becomes

$$\Big[-(1-a)xg(t,x,y) + \frac{a}{2}|g(t,x,y)|\Big] \leq L - \beta|g(t,x,y)|,$$

which holds for a bounded g with appropriate L and β.

Remark Burton introduced a type of contraction called "large contraction" and proved that Krasnosel'skii's theorem holds if the contraction property of the mapping B is replaced by a large contraction [2]. Later Liu and Li [22] introduced a more general concept of contraction called a "separate contraction." In that article, the authors showed that every large contraction is a separate contraction, and that Krasnosel'skii's theorem, as well as Theorem 1.3, hold if the mapping B is a separate contraction. Therefore, the results of Sections 3 and 4 of our present paper hold if the function f of equation (1.2) defines a separate contraction or a large contraction. For definitions of separate contractions and large contractions, and for examples of functions f that define these types of contractions, we refer the reader to [22].

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