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A Note on the Automorphic Langlands Group

To Robert Moody on his sixtieth birthday

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Abstract. Langlands has conjectured the existence of a universal group, an extension of the absolute Galois group, which would play a fundamental role in the classification of automorphic representations. We shall describe a possible candidate for this group. We shall also describe a possible candidate for the complexification of Grothendieck's motivic Galois group.

1

In 1977, Langlands postulated the existence of a universal group in the theory of automorphic forms [L5]. In Langlands's original formulation, the group would be an object in the category of complex, reductive, proalgebraic groups. It would be attached to a given number field F (or more generally, a global field), and would be an extension of the absolute Galois group

$$\Gamma_F = \mathrm{Gal}(\bar{F}/F)$$

($\bar{F}$ a fixed algebraic closure of F), by a connected, complex, reductive, proalgebraic group.

Kottwitz [K2] later pointed out that Langlands's group would be somewhat simpler if it were taken in the category of locally compact topological groups. In this formulation, the universal group would be an extension L_F of the absolute Weil group W_F [T] by a connected compact group. It would thus take its place in a sequence

$$L_F \to W_F \to \Gamma_F$$

of three locally compact groups, all having fundamental ties to the arithmetic of F. We shall work in this context. Our purpose is to describe a candidate for L_F.

With its ties to arithmetic, L_F could be expected to have many properties in common with W_F and Γ_F. In particular, it should have a local analogue L_{F_v}, for each

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valuation v of F, which fits into a commutative diagram

$$\begin{array}{ccccc} L_{F_v} & \longrightarrow & W_{F_v} & \longrightarrow & \Gamma_{F_v} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ L_F & \longrightarrow & W_F & \longrightarrow & \Gamma_F \end{array} \tag{1.1}$$

of continuous homomorphisms. The vertical embedding on the left would be determined only up to conjugacy, and would extend the well known conjugacy classes of embeddings $W_{F_v} \hookrightarrow W_F$ and $\Gamma_{F_v} \hookrightarrow \Gamma_F$ of local Weil and Galois groups. The local Langlands group L_{F_v} is known. It is given by

$$L_{F_v} = \begin{cases} W_{F_v}, & \text{if } v \text{ is archimedean,} \\ W_{F_v} \times \mathrm{SU}(2, \mathbb{R}), & \text{if } v \text{ is } p\text{-adic.} \end{cases}$$

(See [K2, Section 12)]). Observe that L_{F_v} is a (split) extension of W_{F_v} by a compact, simply connected Lie group (either $\{1\}$ or $\mathrm{SU}(2, \mathbb{R})$). Our candidate for the global Langlands group L_F will be a (non-split) extension of W_F by a product of compact, simply connected Lie groups.

The construction we shall give appears to be the most optimistic possible guess on the ultimate form of L_F. It is highly speculative, depending on hypotheses for which there is little evidence, in addition to well known conjectures. On the other hand, it seems to conform with what is generally believed about automorphic representations. Aside from any mistakes on my part, the remarks that follow undoubtedly reflect common views among mathematicians who have thought about the question. I am particularly indebted to Deligne and Langlands for enlightening conversations on the subject.

2

We first recall some conjectural properties of automorphic representations, especially Langlands's principle of functoriality. Suppose that G is a connected, reductive algebraic group over F, which we assume for simplicity is quasisplit. The main structural ingredient of functoriality is the L-group of G [L1], [K2, Section 1]. This is defined as a semidirect product

$${}^L G = \widehat{G} \rtimes W_F$$

of W_F with the dual group $\widehat{G}$, a complex reductive group that is in duality with G. (Recall that the Weil group acts on $\widehat{G}$ through a finite quotient of Γ_F, which can sometimes be used in place of W_F in the definition of ${}^L G$.) For each valuation v of F, the local L-group

$${}^L G_v = \widehat{G} \rtimes W_{F_v}$$

is defined the same way. The embedding of W_{F_v} into W_F gives a conjugacy class of embeddings of ${}^L G_v$ into ${}^L G$.

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An automorphic representation of G is an irreducible representation π of the adelic group $G(\mathbb{A})$ that is a constituent (in the precise sense of [L4]) of the regular representation of $G(\mathbb{A})$ on $L^2\big(G(F)\setminus G(\mathbb{A})\big)$. Automorphic representations decompose into local components. In fact, under a mild continuity condition, *any* irreducible representation π of $G(\mathbb{A})$ can be written as a (restricted) tensor product $\pi = \bigotimes_v \pi_v$ of irreducible representations of the local groups $G(F_v)$, almost all of which are unramified. We recall that if π_v is unramified, the group G is unramified at v. This means that v is p-adic, and the action of W_{F_v} on $\widehat{G}$ factors through the quotient of W_{F_v} by the inertia subgroup I_{F_v}. Moreover, π_v is obtained in a canonical fashion from an induced representation of $G(F_v)$, for which the inducing data can be represented by a semisimple conjugacy class $c(\pi_v)$ in ${}^LG_v$ that projects to the Frobenius element Frob_v in W_{F_v}/I_{F_v}. An irreducible representation π of $G(\mathbb{A})$ thus gives rise to a family of conjugacy classes

$$c^S(\pi) = \{c_v(\pi) = c(\pi_v) : v \notin S\}$$

in ${}^LG$. As usual, S denotes a finite set of valuations of F, outside of which G is unramified. The condition that π be automorphic is very rigid. It imposes deep and interesting relationships among the different conjugacy classes $c_v(\pi)$.

The principle of functoriality is a fundamental conjecture of Langlands [L1]. To state it succinctly, let us write $\mathcal{C}^S_{\mathrm{aut}}(G)$ for the set of families

$$c^S = \{c_v : v \notin S\}$$

of semisimple conjugacy classes in ${}^LG$ such that $c^S = c^S(\pi)$, for some automorphic representation π of G. We then write $\mathcal{C}_{\mathrm{aut}}(G)$ for the set of equivalence classes of such families, c^S and $(c')^S$ being equivalent if the images of c_v and c'_v in ${}^LG_v/I_{F_v}$ are equal for almost all v. The principle of functoriality applies to a pair of (quasisplit) groups G' and G over F, and an L-homomorphism

$$\rho\colon {}^LG' \to {}^LG. \tag{2.1}$$

(In this context, an L-homomorphism is a continuous homomorphism that is analytic on $\widehat{G}'$ and semisimple on W_F, in the sense that the projection of $\rho(w)$ onto $\widehat{G}$ is semisimple for any $w \in W_F$, and that commutes with the two projections onto W_F.) In its basic form, functoriality asserts that the corresponding mapping

$$c' = \{c'_v\} \to \rho(c') = \{\rho(c'_v)\}, \quad c' \in \mathcal{C}_{\mathrm{aut}}(G'),$$

between families of conjugacy classes, sends $\mathcal{C}_{\mathrm{aut}}(G')$ to $\mathcal{C}_{\mathrm{aut}}(G)$. In other words, for any automorphic representation π' of G', there is an automorphic representation π of G such that

$$c_v(\pi) = \rho\big(c_v(\pi')\big) \pmod{I_{F_v}},$$

for almost all v.

We recall that functoriality has a critical application to the theory of L-functions. Suppose that

$$r\colon {}^LG \to \mathrm{GL}(n,\mathbb{C})$$

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is a finite dimensional representation of ${}^LG$. (As a special case of (2.1), r is assumed to be analytic on $\widehat{G}$, and continuous and semisimple on W_F.) Suppose also that $c^S = c^S(\pi)$ belongs to $\mathcal{C}^S_{\mathrm{aut}}(G)$, for some finite set S outside of which r and G are unramified. The (unramified) global L-function

$$L(s, c^S, r) = L^S(s, \pi, r) = \prod_{v \notin S} \det\bigl(1 - r(c_v)|\varpi_v|^{-s}\bigr)^{-1}$$

is then defined for s in some right half plane. It is conjectured that this function has meromorphic continuation to the complex plane, and satisfies an appropriate functional equation. In the special case that G equals $\mathrm{GL}(n)$, and r is the standard n-dimensional representation, the conjecture was proved by a matrix analogue [J] of the method of Tate. In the general case, one could apply the assertion of functoriality, with $\bigl(G, \mathrm{GL}(n), r\bigr)$ in place of (G', G, ρ). An affirmative answer to the functoriality conjecture would thus resolve the general problem of analytic continuation in terms of the special case that is known for $\mathrm{GL}(n)$. (See [L1] and [Bo]. For an elementary introduction to functoriality, see [A].)

3

The role of the group L_F would be to represent the functor

$$G \to \mathcal{C}_{\mathrm{aut}}(G).$$

More precisely, for any (quasisplit) G, there should be a surjective mapping

$$\phi \to c(\phi)$$

from the set of L-homomorphisms

$$\phi \colon L_F \to {}^LG,$$

taken up to conjugacy by $\widehat{G}$, onto the set $\mathcal{C}_{\mathrm{aut}}(G)$. (An L-homomorphism here means a continuous homomorphism that is semisimple, in the earlier sense, and that commutes with the projections onto W_F.) For a given ϕ, let

$$\phi_v \colon L_{F_v} \to {}^LG_v$$

be the restriction of ϕ to the subgroup L_{F_v} of L_F. Then ϕ should be unramified for almost all v, in the sense that ϕ_v factors through the projection of L_{F_v} onto W_{F_v}/I_{F_v}. The family $c(\phi)$ attached to ϕ would be defined by setting

$$c_v(\phi) = \phi_v(\mathrm{Frob}_v),$$

for each unramified v.

Recall that the local Langlands conjecture gives a classification of the (equivalence classes of) irreducible representations of $G(F_v)$. It asserts that the set of such representations is a disjoint union of finite sets Π_{ϕ_v}, parametrized by L-homomorphisms

$$\phi_v \colon L_{F_v} \to {}^L G_v,$$

taken again up to conjugacy by $\widehat{G}$. Further conjectures describe the individual L-packets Π_{ϕ_v} in terms of the finite groups

$$\pi_0(S_{\phi_v}) = S_{\phi_v}/S^0_{\phi_v},$$

where

$$S_{\phi_v} = \mathrm{Cent}_{\widehat{G}}\left(\mathrm{Image}(\phi_v)\right).$$

The latter are a part of the theory of endoscopy, which need not concern us here. We do, however, recall that the original local Langlands conjecture includes a characterization of tempered representations. The assertion is that the representations in Π_{ϕ_v} are tempered (that is, their characters are tempered distributions on $G(F_v)$) if and only if the image of ϕ_v projects to a relatively compact subset of $\widehat{G}$. This condition is equivalent to the requirement that ϕ_v map L_{F_v} to a subgroup $\widehat{K} \rtimes W_{F_v}$ of ${}^L G_v$, where $\widehat{K}$ is a compact real form of the complex group $\widehat{G}$. The local Langlands conjecture was proved for archimedean v by Langlands in [L3]. For p-adic v, it has recently been established in the case $G = \mathrm{GL}(n)$ by Harris and Taylor [HT] and Henniart [H].

Returning to the hypothetical group L_F, we consider global L-homomorphisms ϕ as above. Any such ϕ would give rise to a localization ϕ_v at each v, and by the local Langlands conjecture, a corresponding local L-packet Π_{ϕ_v}. The representations π in the associated global L-packet

$$\Pi_\phi = \Big\{ \pi = \bigotimes_v \pi_v : \pi_v \in \Pi_{\phi_v}, \pi_v \text{ unramified for almost all } v \Big\}$$

would have the property that $c(\pi) = c(\phi)$. Some of these representations should be automorphic. The most important case occurs when the image of ϕ does not factor through any parabolic subgroup of ${}^L G$, or equivalently, when the group

$$S_\phi = \mathrm{Cent}_{\widehat{G}}(\mathrm{Image}\ \phi)$$

is finite modulo the center of $\widehat{G}$. In this case, the global theory of endoscopy gives a conjectural formula, in terms of the finite group

$$\pi_0(S_\phi) = S_\phi/S^0_\phi,$$

for the multiplicity of π in the space of cusp forms for G. There is no need to describe the formula. Our point is simply that the group L_F would be an extremely useful object to have. There are basic questions that would be difficult even to formulate

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without it. The question we were just considering is a case in point. It concerns a problem of obvious importance, that of determining the fibres of the mapping

$$\pi \to c(\pi)$$

from automorphic representations to automorphic families of conjugacy classes.

The tentative construction of L_F we are going to describe has two ingredients. The first is an indexing set $\mathcal{C}_F$. The second is an assignment of a locally compact group L_c to every element c in $\mathcal{C}_F$. This group will be given as an extension

$$1 \to K_c \to L_c \to W_F \to 1 \tag{3.1}$$

of W_F by a compact, semisimple, simply connected Lie group K_c. It will also be equipped with a localization

$$\begin{array}{ccc} L_{F_v} & \longrightarrow & W_{F_v} \\ \downarrow & & \downarrow \\ L_c & \longrightarrow & W_F, \end{array} \tag{3.2}$$

for each valuation v. With these ingredients, we will then be able to define L_F as a fibre product over W_F of the groups L_c. This will yield the required group as an extension

$$1 \to K_F \to L_F \to W_F \to 1 \tag{3.3}$$

of W_F by the compact group

$$K_F = \prod_{c \in \mathcal{C}_F} K_c.$$

Before we describe L_F in detail, it might be helpful to recall what happens in case that $G = T$ is a torus. Taking for granted that L_F has the general structure (3.3) above, we see that any L-homomorphism $\phi\colon L_F \to {}^L T$ is trivial on K_F. The set of $\widehat{T}$-orbits of such mappings can therefore be identified with the (continuous) cohomology group $H^1(W_F, \widehat{T})$. One of the earlier theorems in the subject was Langlands's classification [L2] of the automorphic representations of T as the quotient of $H^1(W_F, \widehat{T})$ by the subgroup

$$H^1_{\ell t}(F, \widehat{T}) = H^1_{\ell t}(W_F, \widehat{T}) = \ker\Big(H^1(W_F, \widehat{T}) \to \bigoplus_v H^1(W_{F_v}, \widehat{T})\Big).$$

From the case of a torus, we see that the mapping $\phi \to c(\phi)$ need not be bijective. For a given G, two L-homomorphisms ϕ and ϕ' from L_F to ${}^L G$ will be in the same fibre if and only if they are locally almost everywhere equivalent, which is to say that ϕ_v equals ϕ'_v for almost all v. I do not know whether one could hope for any sort of reasonable classification of the fibres. The question is important, since for non-abelian G, it is closely related to the failure of multiplicity one for representations in the space of cusp forms for G. Examples of nontrivial fibres have been systematically constructed for what would amount to homomorphisms that factor through a finite quotient of L_F ([B1], [Lars1], [Lars2]). Recently, S. Wang has found some simple examples of pairs of nonconjugate, pointwise conjugate homomorphisms from a compact, connected, simple group to a complex group [W].

4

We now give our tentative construction of L_F. It relies on both the local Langlands conjecture and the global functoriality conjecture. We assume both in what follows.

Suppose that G is a group over F, which we take to be quasisplit, semisimple, and simply connected. Since we are assuming functoriality, the L-functions

$$L(s, c^S, r), \quad c^S \in \mathcal{C}^S_{\mathrm{aut}}(G),$$

have meromorphic continuation, for any representation r of ${}^LG$. It follows from properties of standard L-functions of $\mathrm{GL}(n)$ that

$$\mathrm{ord}_{s=1}\big(L(s, c^S, r)\big) \geq [r : 1_{{}^LG}],$$

where the right hand side denotes the multiplicity of the trivial 1-dimensional representation of ${}^LG$ in r. Motivated by [L7], we shall say that a class $c \in \mathcal{C}_{\mathrm{aut}}(G)$ is *primitive* if for any r, we can choose some representative $c^S \in \mathcal{C}^S_{\mathrm{aut}}(G)$ of c such that

$$\mathrm{ord}_{s=1}\big(L(s, c^S, r)\big) = [r : 1_{{}^LG}].$$

The terminology is at least partially justified by the circumstance that if

$$c = \rho(c'), \quad c' \in \mathcal{C}_{\mathrm{aut}}(G), \tag{4.1}$$

for some L-homomorphism (2.1) whose image in ${}^LG$ is proper, there is a representation r with

$$[\rho \circ r : 1_{{}^LG'}] > [r : 1_{{}^LG}].$$

We write $\mathcal{C}_{\mathrm{prim}}(G)$ for the subset of primitive families in $\mathcal{C}_{\mathrm{aut}}(G)$.

We can now define the indexing set $\mathcal{C}_F$. Consider the set of pairs (G, c), where G is a quasisplit, simple, simply connected algebraic group over F, and c is a family (or rather an equivalence class of families) in $\mathcal{C}_{\mathrm{prim}}(G)$. Two pairs (G, c) and (G', c') will be said to be isomorphic if there is an isomorphism $G \to G'$ over F, and a dual isomorphism ${}^LG' \to {}^LG$ that takes c' to c. We define $\mathcal{C}_F$ to be the set of isomorphism classes of such pairs. We shall often denote an element in $\mathcal{C}_F$ by c, even though c is really only the second component of a representative (G, c) of an isomorphism class.

Suppose that c belongs to $\mathcal{C}_F$. Since the associated group G is simply connected, the complex dual group $\widehat{G}$ is of adjoint type. We write K_c for a compact real form of its simply connected cover $\widehat{G}_{\mathrm{sc}}$. The Weil group W_F operates on $\widehat{G}$, and hence also on K_c, by an action that factors through a finite quotient of the Galois group Γ_F. Since G need not be absolutely simple, K_c is generally only semi-simple. However, the action of W_F on K_c does factor to a transitive permutation representation of W_F on the set of simple factors of this group. The center $Z(\widehat{G}_{\mathrm{sc}})$ of $\widehat{G}_{\mathrm{sc}}$ is of course finite, and coincides with the center of K_c. In order to define the extension L_c, we need to construct an element in $H^2\big(W_F, Z(\widehat{G}_{\mathrm{sc}})\big)$.

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The group $Z(\widehat{G}_{\mathrm{sc}})$ is dual to the center $Z(G)$ of G. We choose an embedding of $Z(G)$ into a torus Z over F. We can assume that Z is induced, in the sense that it is isomorphic to a finite product

$$\prod_i \mathrm{Res}_{E_i/F}\big(\mathrm{GL}(1)\big),$$

for finite extensions E_i of F. In particular, we assume that $H^1(F,Z)$ is trivial. For example, we could take Z to be a maximal torus T in G over F that is contained in an F-rational Borel subgroup. Having chosen Z, we obtain a short exact sequence

$$1 \to Z(G) \to Z \to Z^\vee \to 1$$

of diagonalizable groups over F, with $Z^\vee$ being another torus. We also have a dual exact sequence

$$1 \to Z(\widehat{G}_{\mathrm{sc}}) \to \widehat{Z^\vee} \to \widehat{Z} \to 1$$

of diagonalizable groups over $\mathbb{C}$, equipped with actions of Γ_F.

The compact (totally disconnected) group

$$Z(G,\mathbb{A}) = \prod_v Z(G,F_v)$$

maps to a closed subgroup $\big(Z(F)\setminus Z(\mathbb{A})\big)_G$ of $Z(F)\setminus Z(\mathbb{A})$. The first exact sequence above, combined with the vanishing of the group

$$H^1_{\ell t}(F,Z) = \ker\Big(H^1(F,Z) \to \bigoplus_v H^1(F_v,Z)\Big),$$

gives rise to an exact sequence

$$1 \to \big(Z(F)\setminus Z(\mathbb{A})\big)_G \to Z(F)\setminus Z(\mathbb{A}) \to Z^\vee(F)\setminus Z^\vee(\mathbb{A}).$$

Let us write $\Pi(H)$ for the group of continuous (quasi-) characters on any locally compact abelian group H. Then

$$\Pi\Big(\big(Z(F)\setminus Z(\mathbb{A})\big)_G\Big) \cong \mathrm{coker}\Big(\Pi\big(Z^\vee(F)\setminus Z^\vee(\mathbb{A})\big) \to \Pi\big(Z(F)\setminus Z(\mathbb{A})\big)\Big).$$

The group $H^1_{\ell t}(F,\widehat{Z})$ is also trivial, since it is dual to $H^1_{\ell t}(F,Z)$. The Langlands correspondence for tori then gives an isomorphism between the groups $\Pi\big(Z(F)\setminus Z(\mathbb{A})\big)$ and $H^1(W_F,\widehat{Z})$. It follows easily that

$$\Pi\Big(\big(Z(F)\setminus Z(\mathbb{A})\big)_G\Big) \cong \mathrm{coker}\big(H^1(W_F,\widehat{Z^\vee}) \to H^1(W_F,\widehat{Z})\big). \tag{4.2}$$

Finally, the second exact sequence above yields an exact sequence of cohomology

$$\cdots \to H^1(W_F,\widehat{Z^\vee}) \to H^1(W_F,\widehat{Z}) \to H^2\big(W_F, Z(\widehat{G}_{\mathrm{sc}})\big) \to \cdots$$

It follows that there is a canonical injection

$$\Pi\Big(\big(Z(F)\setminus Z(\mathbb{A})\big)_G\Big) \to H^2\big(W_F, Z(\widehat{G}_{\mathrm{sc}})\big). \tag{4.3}$$

Since c belongs to $\mathcal{C}_{\mathrm{aut}}(G)$, there is an automorphic representation π of G such that $c = c(\pi)$. In fact, because c belongs to the subset $\mathcal{C}_{\mathrm{prim}}(G)$ of $\mathcal{C}_{\mathrm{aut}}(G)$, we would expect π to be cuspidal. As an irreducible representation of $G(\mathbb{A})$, π has a central character on the group $Z(G,\mathbb{A})$. The central character is in turn the pullback of a character χ_c on $\big(Z(F)\setminus Z(\mathbb{A})\big)_G$, as one deduces without difficulty by considering the case $Z = T$ mentioned above. Having obtained an element χ_c in $\Pi\Big(\big(Z(F)\setminus Z(\mathbb{A})\big)_G\Big)$, we need only apply the injection (4.3). The image of χ_c is a class in $H^2\big(W_F, Z(\widehat{G}_{\mathrm{sc}})\big)$, which determines an extension L_c of W_F by K_c. (For a discussion of local central characters, see [L3, p. 119–122] and [Bo, p. 43].)

It is instructive to describe the group L_c a little more concretely. Given the induced torus Z above, we set

$$\tilde{G} = (G\times Z)/Z(G),$$

for the diagonal embedding of $Z(G)$ into $G\times Z$. The exact sequence

$$1 \to Z \xrightarrow{\varepsilon} \tilde{G} \to G_{\mathrm{ad}} \to 1$$

is then a z-extension [K1] of the adjoint group G_{ad}. In particular, the derived group $\tilde{G}_{\mathrm{der}}$ of $\tilde{G}$ equals the original simply connected group G. There is a second exact sequence

$$1 \to G \to \tilde{G} \xrightarrow{\varepsilon^\vee} Z^\vee \to 1,$$

as well as an associated pair of dual exact sequences

$$1 \to \widehat{G}_{\mathrm{sc}} \to \widehat{\tilde{G}} \xrightarrow{\widehat{\varepsilon}} \widehat{Z} \to 1$$

and

$$1 \to \widehat{Z^\vee} \xrightarrow{\widehat{\varepsilon^\vee}} \widehat{\tilde{G}} \to \widehat{G} \to 1.$$

Let $\tilde{K}_c$ be the normalizer of K_c in $\widehat{\tilde{G}}$. Then $\tilde{K}_c$ is the product of a compact real form of $\widehat{\tilde{G}}$ with the center $\widehat{Z^\vee}$. We choose a 1-cocycle z_c from W_F to $\widehat{Z}$ that represents χ_c under the isomorphism (4.2). We can then take the subgroup

$$L_c = \{g\times w \in \tilde{K}_c \rtimes W_F : \widehat{\varepsilon}(g) = z_c(w)\} \tag{4.4}$$

of ${}^L\tilde{G}$ for a realization of the extension (3.1). Its isomorphism class is of course independent of the choice of z_c.

We are expecting that there exists an "extension" of the family c to $\tilde{G}$. More precisely, there should be a cuspidal automorphic representation $\tilde{\pi}$ of $\tilde{G}$ such that c is the image of the family $\tilde{c} = c(\tilde{\pi})$ under the projection from ${}^L\tilde{G}$ onto ${}^LG$. We assume that $\tilde{\pi}$ exists, and that its central character on $Z(F)\setminus Z(\mathbb{A})$ extends χ_c. The central character of $\tilde{\pi}$ is then dual (under the Langlands correspondence for tori) to a 1-cocycle

z_c that represents χ_c, as above, and from which we can form the group (4.4). We are also assuming the local Langlands classification for the groups $\tilde{G}(F_v)$. The local components $\tilde{\pi}_v$ of $\tilde{\pi}$ then determine L-homomorphisms

$$L_{F_v} \to {}^L\tilde{G}_v.$$

Now the original condition that c is primitive, together with our assumption that functoriality holds, implies that $\tilde{\pi}$ satisfies the general analogue of Ramanujan's conjecture. To be precise, the argument of [L1, p. 56–59] (in combination with the classification [L3] of arbitrary representations in terms of tempered representations) can be used to prove that the local components $\tilde{\pi}_v$ of $\tilde{\pi}$ are all tempered (up to a possible twist of $\tilde{\pi}$ by a real valued automorphic character of $Z^\vee$). For each v, the image of L_{F_v} is therefore contained in the subgroup $\tilde{K}_c \rtimes W_{F_v}$ of ${}^L\tilde{G}_v$. Our condition on the central character of $\tilde{\pi}$ implies that the image of L_v in $\tilde{K}_c \rtimes W_F$ is contained in the subgroup (4.4). We therefore obtain a localization (3.2) for each v.

There is a rather serious additional hypothesis that has been implicit in the constructions above. We assume that the mappings (3.2) we have just defined are independent of the choice of $\tilde{\pi}$. This hypothesis applies also to the isomorphism class of the extension L_c, defined by (4.4) in terms of the central character of $\tilde{\pi}$. That is to say, we assume that the original central character χ_c on $\big(Z(F)\backslash Z(\mathbb{A})\big)_G$ is independent of the earlier choice of automorphic representation π with $c(\pi) = c$. Such a hypothesis is needed in order that the basic objects (3.1) and (3.2), taken up to isomorphism, depend only on c. We shall discuss it briefly in the next section.

We have now assembled, under various hypotheses, the necessary ingredients for L_F. They are the indexing set $\mathcal{C}_F$, the extension (3.1) of W_F attached to each c in $\mathcal{C}_F$, and the local mapping (3.2) attached to each c and v. We can now define L_F as the fibre product

$$L_F = \prod_{c \in \mathcal{C}_F} (L_c \to W_F)$$

of the extensions (3.1). This yields the required extension (3.3) of W_F, together with the local embeddings (1.1).

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The construction we have given for L_F represents the simplest form the group could possibly take. It may well turn out to be overly optimistic. However, as far as I can see, the construction is not in conflict with any properties of automorphic representations, either proved or conjectured. In any case, it raises some interesting questions.

The most obvious question is perhaps that of why the kernels K_c in (3.1) should be simply connected. This condition on L_F implies that for G and $\tilde{G}$ as in the last section, any "primitive" L-homomorphism

$$L_F \to {}^L G = \widehat{G} \rtimes W_F$$

extends to an L-homomorphism from L_F to ${}^L\tilde{G}$. It is more or less equivalent to our assumption in Section 4 that for any $c \in \mathcal{C}_{\mathrm{prim}}(G)$, there is a cuspidal automorphic representation $\tilde{\pi}$ of $\tilde{G}$ such that the automorphic family $\tilde{c} = c(\tilde{\pi})$ projects to c. Why should such a $\tilde{\pi}$ exist?

I do not have anything particularly useful to say about the question. One could always replace it with the broader question of whether elements in the larger set $\mathcal{C}_{\mathrm{aut}}(G)$ also extend to $\tilde{G}$. In this form, it can be posed for the Weil group W_F instead of L_F. Does any L-homomorphism ϕ from W_F to ${}^LG$ extend to an L-homomorphism from W_F to ${}^L\tilde{G}$? The answer is yes. For we can identify ϕ with an element in the relevant cohomology group $H^1(W_F, \widehat{G})$, which can in turn be placed in an exact sequence

$$\cdots \to H^1(W_F, \widehat{\tilde{G}}) \to H^1(W_F, \widehat{G}) \to H^2(W_F, \widehat{Z^\vee}) \to \cdots$$

Labesse has shown that ϕ maps to zero in $H^2(W_F, \widehat{Z^\vee})$, and is hence the image of an element in $H^1(W_F, \widehat{\tilde{G}})$ [Lab, Théorème 7.1]. (The theorem of Labesse holds if F is replaced by any local or global field. Earlier special cases were treated in [L3, Lemma 2.10] and [L6, Lemma 4]).

One could consider the question directly in terms of representations. An answer of sorts would presumably follow from some generalization of Lemma 6.2 of [LL]. This lemma was used in [LL] in support of a comparison of stable trace formulas, in the special case that $G = \mathrm{SL}(2)$ and $\tilde{G} = \mathrm{GL}(2)$. In general, any comparison of stable trace formulas would begin with an analysis of conjugacy classes. The analogue of a tempered, cuspidal automorphic representation of G would be a strongly regular conjugacy class in $G(\mathbb{A})$ that is the image of an elliptic conjugacy class in $G(F)$. A global L-packet of such representations would be analogous to a corresponding stable conjugacy class. There is of course an embedding of $G(\mathbb{A})$ into $\tilde{G}(\mathbb{A})$. One checks that the associated mapping of strongly regular, stable conjugacy classes is injective. This might be construed as heuristic evidence that the transpose mapping between class functions takes L-packets of tempered, cuspidal, automorphic representations of $\tilde{G}$ surjectively to the set of such packets for G.

Another question concerns the last hypothesis in Section 4. We have assumed that the building blocks (3.1) and (3.2) of L_F depend on c alone, and not the choice of automorphic representation $\tilde{\pi}$ of $\tilde{G}$. The failure of this hypothesis would seem to leave no alternative but to enlarge the indexing set $\mathcal{C}_F$, since a given pair (G, c) might require several indices to accommodate different families of localizations (3.2). With such a change, there could be two primitive, locally almost everywhere equivalent L-homomorphisms

$$\phi, \phi' \colon L_F \to {}^LG, \tag{5.1}$$

corresponding to two different primitive factors of L_F (over W_F). This would be contrary to conjecture. Indeed, one could compose both ϕ and ϕ' with an irreducible, faithful, finite dimensional representation of ${}^LG$. The resulting pair of irreducible representations

$$r, r' \colon L_F \to \mathrm{GL}(n, \mathbb{C})$$

would parametrize the same cuspidal automorphic representation of $\mathrm{GL}(n)$, by the theorem of strong multiplicity one, yet would be inequivalent, by construction. Such a property of L_F is not to be expected. For the irreducible, n-dimensional representation of L_F are supposed to be in *bijection* with the cuspidal automorphic representations of $\mathrm{GL}(n)$. Is there any way to study the question, albeit heuristically, directly in terms of primitive automorphic representations of the group G?

Let us now assume that the group L_F we have defined really is the automorphic Langlands group. Namely, we assume that for any G, the mapping $\phi \to c(\phi)$ takes the set of L-homomorphisms $\phi\colon L_F \to {}^LG$ surjectively to $\mathcal{C}_{\mathrm{aut}}(G)$. What does this general hypothesis have to say about automorphic representations?

It certainly implies functoriality. If ρ is an L-homomorphism from ${}^LG'$ to ${}^LG$ and $c' = c(\phi')$ belongs to $\mathcal{C}_{\mathrm{aut}}(G')$, then $c = \rho(c')$ equals $c(\phi)$, where ϕ is the composition

$$L_F \xrightarrow{\phi'} {}^LG' \xrightarrow{\rho} {}^LG.$$

Therefore, the general hypothesis on L_F does imply that c belongs to $\mathcal{C}_{\mathrm{aut}}(G)$.

However, the general hypothesis also implies the existence of some automorphic families that cannot be explained solely in terms of functoriality. Suppose that ϕ is an arbitrary L-homomorphism from L_F to ${}^LG$. The connected component of 1 in the image of ϕ projects to a subgroup of $\widehat{G}$. Let $\widehat{H}_\phi$ be the Zariski closure in $\widehat{G}$ of this subgroup. Then $\widehat{H}_\phi$ is a complex reductive subgroup of $\widehat{G}$, which is normalized by $\phi(L_F)$. Set

$$\mathcal{H}_\phi = \widehat{H}_\phi \phi(L_F).$$

The projection of ${}^LG$ onto W_F then yields a short exact sequence

$$1 \to \widehat{H}_\phi \to \mathcal{H}_\phi \to W_F \to 1.$$

The action of $\mathcal{H}_\phi$ on $\widehat{H}_\phi$ by conjugation factors to a homomorphism from Γ_F to the group of outer automorphisms of $\widehat{H}_\phi$. This action in turn determines a well defined quasisplit group H_ϕ over F, for which $\widehat{H}_\phi$ is a complex dual group. However, $\mathcal{H}_\phi$ does not have to be isomorphic (over W_F) to ${}^LH_\phi$. There need not be a section $W_F \to \mathcal{H}_\phi$ whose image gives an L-action [K2, Section 1] by conjugation on $\widehat{H}_\phi$. To attach an L-group to $\mathcal{H}_\phi$, we can choose a z-extension

$$1 \to Z_\phi \to \tilde{H}_\phi \to H_\phi \to 1$$

of H_ϕ over F. This gives rise to a dual L-embedding of $\mathcal{H}_\phi$ into ${}^L\tilde{H}_\phi$, and a diagram of L-homomorphisms

$$\begin{tikzcd} L_F \arrow[r] \arrow[dr, "\tilde{\phi}"] & \mathcal{H}_\phi \arrow[r] & {}^LG \\ & {}^L\tilde{H}_\phi. & \end{tikzcd} \tag{5.2}$$

The earlier discussion implies that the family $c(\tilde{\phi})$ belongs to $\mathcal{C}_{\mathrm{aut}}(\tilde{H}_\phi)$. It is in fact the family $c(\tilde{\pi})$ we have assumed may be attached to an element in $\mathcal{C}_{\mathrm{prim}}(\tilde{H}_{\phi,\mathrm{der}})$. Of course, the existence of $c(\tilde{\phi})$ does not alter the fact that ${}^L H_\phi$ need not embed in ${}^L G$. The discussion shows that the automorphic family $c(\phi) \in \mathcal{C}_{\mathrm{aut}}(G)$ need not be a strictly functorial image of any primitive family c.

How general is this phenomenon? Suppose that we have a diagram of L-homomorphisms

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & {}^L G \\ \big\downarrow & & \\ {}^L\tilde{H}, & & \end{array}$$

for a quasisplit group H, an extension $\mathcal{H}$ of W_F by $\widehat{H}$, and a z-extension $\tilde{H}$ of H. When can we expect those families $\tilde{c} \in \mathcal{C}_{\mathrm{aut}}(\tilde{H})$ that factor through $\mathcal{H}$ to map to families c in $\mathcal{C}_{\mathrm{aut}}(G)$? It would be interesting to find an explanation that does not rely on the existence of L_F. The conjectural theory of endoscopy gives an answer in one case. It is the case that H is an endoscopic group, or more generally a twisted endoscopic group, for G. (See [LS, Section 4.4], [KS, Section 2.2].)

The automorphic Langlands group, in whatever form it might take, would be as significant for the limits it places on automorphic representations as for what it implies about their existence. The general hypothesis on L_F above implies that any automorphic family $c \in \mathcal{C}_{\mathrm{aut}}(G)$ is of the form $c(\phi)$, and hence gives rise to a diagram (5.2). In particular, it implies the existence of the group $\mathcal{H}_\phi$. This was one of the main reasons for Langlands's original introduction of a universal automorphic group. The existence of $\mathcal{H}_\phi$ may be regarded as a kind of converse to functoriality. It means that a general family $c \in \mathcal{C}_{\mathrm{aut}}(G)$ can be obtained from a primitive family $c_{\mathrm{prim}} \in \mathcal{C}_{\mathrm{prim}}(\tilde{H}_{\phi,\mathrm{der}})$ through a diagram (5.2). Even though (5.2) does not always represent an embedding of L-groups, it is really only a mild generalization of the data (2.1) of functoriality.

The existence of the group $\mathcal{H}_\phi$ is a fundamental premise of Langlands's recent paper [L7]. As an essential supplement to functoriality, the question is obviously very deep. One is tempted to believe that a successful attack on functoriality will also need to establish the existence of $\mathcal{H}_\phi$. This is part of the appeal of the methods proposed in [L7], provisional as they may be.

6

Langlands's second main reason for introducing a universal automorphic group was to make precise the conjectural relationship between automorphic representations and motives. The theory of motives is due to Grothendieck. It is based on the "standard conjectures" for algebraic cycles, and includes the existence of a universal group $\mathcal{G}_F$ whose representations classify motives over F. (See [Klei], [S2] and [DM].) We shall consider the motivic Grothendieck group only in its simplest form, which is as a group over $\mathbb{C}$. In other words, we identify $\mathcal{G}_F$ with its group of complex points (relative to the $\mathbb{Q}$-rational structure defined by a fixed embedding of F into $\mathbb{C}$). Then $\mathcal{G}_F$ is a proalgebraic extension of Γ_F by a connected, complex, reductive, proalgebraic group $\mathcal{G}_F^0$.

By the 1970's Deligne and Langlands were prepared to conjecture very general

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relations between motives and automorphic representations. Langlands expressed them in the form of a conjectural mapping from the universal automorphic group to the motivic Grothendieck group. In the present context this amounts to a continuous homomorphism from L_F to $\mathcal{G}_F$, determined up to conjugation by $\mathcal{G}_F^0$, such that the diagram

$$\begin{array}{ccc} L_F & \longrightarrow & \mathcal{G}_F \\ \downarrow & & \downarrow \\ \Gamma_F & \longrightarrow & \Gamma_F \end{array} \tag{6.1}$$

is commutative. (See [L5, Section 2]. For further discussion of the relations between automorphic representations and motives, see [C] and [R].)

The tentative construction of L_F in Section 4 can be adapted to the motivic Grothendieck group. In this last section, we shall describe a tentative construction of $\mathcal{G}_F$, together with the corresponding homomorphism from L_F to $\mathcal{G}_F$.

Not all automorphic representations can be attached to motives. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of a reductive (quasisplit) group G over F. Since the local Langlands conjecture has been established for archimedean fields, we can attach an L-homomorphism

$$\phi_v \colon L_{F_v} = W_{F_v} \to {}^L G_v$$

to each archimedean component π_v of π. Suppose that G_{der} is simply connected. With this condition, we shall say that π is of type A_0 if for every finite dimensional representation r of ${}^L G$ whose kernel contains a subgroup of finite index in W_F, the associated representations $r \circ \phi_v$ of archimedean Weil groups W_{F_v} are of Hodge type. In other words, the restriction of $r \circ \phi_v$ to the subgroup $\mathbb{C}^*$ of W_{F_v} is a direct sum of characters of the form

$$z \to z^{-p} \bar{z}^{-q}, \quad z \in \mathbb{C}^*, \ p, q \in \mathbb{Z}.$$

We shall say that an automorphic family $c \in \mathcal{C}_{\mathrm{aut}}(G)$ is of the type A_0 if π is of type A_0, for any automorphic representation π of G with $c(\pi) = c$. It is the subset $\mathcal{C}_{\mathrm{aut},0}(G)$ of families in $\mathcal{C}_{\mathrm{aut}}(G)$ of type A_0 that are thought to have motivic significance.

The motivic analogue of the Weil group is the Taniyama group ([L5, Section 5], [MS], [D]). In our context of complex motivic groups, the Taniyama group is an extension

$$1 \to \mathcal{S}_F \to \mathcal{T}_F \to \Gamma_F \to 1$$

of Γ_F by the complex Serre group $\mathcal{S}_F$ ([S1], [L5, Section 4], [S2, Section 7]), a complex proalgebraic torus. There is a natural homomorphism

$$w \to t(w), \quad w \in W_F,$$

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from W_F to $\mathcal{T}_F$, defined up to conjugation in $\mathcal{T}_F$ by its connected component $\mathcal{S}_F$, such that the diagram

$$\begin{tikzcd} W_F \arrow[rr] \arrow[dr] & & \mathcal{T}_F \arrow[dl] \\ & \Gamma_F & \end{tikzcd} \tag{6.2}$$

is commutative [L5, p. 226–227]. This is a reflection of the fact that there is a canonical bijection between (continuous, semisimple) representations of W_F of type A_0, and (proalgebraic) representations of $\mathcal{T}_F$.

We write $\mathcal{C}_{F,0}$ for the set of elements in $\mathcal{C}_F$ of type A_0. An element in $\mathcal{C}_{F,0}$ is thus an isomorphism class of pairs (G, c), where G is a quasisplit, simple, simply connected group over F, and c is a family in

$$\mathcal{C}_{\mathrm{prim},0}(G) = \mathcal{C}_{\mathrm{prim}}(G) \cap \mathcal{C}_{\mathrm{aut},0}(G).$$

Suppose that c belongs to $\mathcal{C}_{F,0}$. That is, c is the second component of a representative (G, c) of an isomorphism class in $\mathcal{C}_{F,0}$. We shall write $\mathcal{D}_c$ for the complex, simply connected group $\widehat{G}_{\mathrm{sc}}$. The Taniyama group $\mathcal{T}_F$ acts on $\mathcal{D}_c$ through its projection onto Γ_F. We need to define an extension

$$1 \to \mathcal{D}_c \to \mathcal{G}_c \to \mathcal{T}_F \to 1 \tag{6.3}$$

of $\mathcal{T}_F$ by $\mathcal{D}_c$.

For the given element $c \in \mathcal{C}_{F,0}$, we choose a z-extension

$$1 \to Z \xrightarrow{\varepsilon} \tilde{G} \to G_{\mathrm{ad}} \to 1$$

as in Section 4. We have already assumed the existence of an automorphic representation $\tilde{\pi}$ of $\tilde{G}$ such that c is the image of the family $\tilde{c} = c(\tilde{\pi})$. We assume here that $\tilde{\pi}$ may be chosen to be of type A_0. In particular, we assume that the central character of $\tilde{\pi}$ is of type A_0. It follows from this that the corresponding L-homomorphism

$$w \to z_c(w) \times w$$

from W_F to ${}^L Z$ factors through the Taniyama group $\mathcal{T}_F$. In other words, we can write

$$z_c(w) = \zeta_c\big(t(w)\big), \quad w \in W_F,$$

for a proalgebraic morphism ζ_c from $\mathcal{T}_F$ to $\widehat{Z}$ such that the mapping $t \to \zeta_c(t) \times t$ is a homomorphism form $\mathcal{T}_F$ to $\widehat{Z} \rtimes \mathcal{T}_F$. Let us write $\tilde{\mathcal{D}}_c$ for the complex dual group $\widehat{\tilde{G}}$. The required extension (6.3) can then be defined by

$$\mathcal{G}_c = \{g \times t \in \tilde{\mathcal{D}}_c \rtimes \mathcal{T}_F : \widehat{\varepsilon}(g) = \zeta_c(t)\}. \tag{6.4}$$

The isomorphism class of $\mathcal{G}_c$ over $\mathcal{T}_F$ is independent of the choice of extension $\tilde{G} \to G$ and cocycle z_c. Comparing (6.4) with the construction (4.4) of L_c, we see that the mapping

$$g \times w \to g \times t(w)$$

is a continuous homomorphism from L_c to $\mathcal{G}_c$ that commutes with the projections onto Γ_F.

We have assembled the necessary ingredients for a tentative definition of $\mathcal{G}_F$. They consist of the subset $\mathcal{C}_{F,0}$ of $\mathcal{C}_F$, and the extension (6.3) attached to each c in $\mathcal{C}_{F,0}$. It remains only to define $\mathcal{G}_F$ as the proalgebraic fibre product

$$\mathcal{G}_F = \prod_{c \in \mathcal{C}_{F,0}} (\mathcal{G}_c \to \mathcal{T}_F). \tag{6.5}$$

This yields an extension

$$1 \to \mathcal{D}_F \to \mathcal{G}_F \to \mathcal{T}_F \to 1$$

of $\mathcal{T}_F$ by the complex, connected, proalgebraic group

$$\mathcal{D}_F = \prod_{c \in \mathcal{C}_{F,0}} \mathcal{D}_c.$$

The required mapping (6.1) of L_F is defined in the obvious way as the fibre product over (6.2) of the homomorphisms $L_c \to \mathcal{G}_c$.

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