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# CHROMATIC SOLUTIONS

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**1. The problem of chromatic sums.** Early in the Seventies I sought the number of rooted $\lambda$-coloured triangulations of the sphere with $2p$ faces. In these triangulations double joins, but not loops, were permitted. The investigation soon took the form of a discussion of a certain formal power series $l(y, z, \lambda)$ in two independent variables $y$ and $z$.

The basic theory of $l$ is set out in [**1**]. There $l$ is defined as the coefficient of $x^2$ in a more complicated power series $g(x, y, z, \lambda)$. But the definition is equivalent to the following formula.

$$(1) \qquad l(y, z, \lambda) = \lambda(\lambda - 1)y + y \sum_T \{y^{n(T)} z^{2p(T)} P(T, \lambda)\}.$$

Here $T$ denotes a general rooted triangulation. $n(T)$ is the valency of its root-vertex, and $2p(T)$ is the number of its faces. $P(T, \lambda)$ is the chromatic polynomial of the graph of $T$.

By substituting 1 for $y$ in (1) we obtain the power series

$$(2) \qquad h(z, \lambda) = l(1, z, \lambda),$$

$$(3) \qquad h(z, \lambda) = \lambda(\lambda - 1) + \sum_T \{z^{2p(T)} P(T, \lambda)\}.$$

The enumerative problem mentioned above is the problem of determining $h$ for the value of $\lambda$ under consideration.

The colour-number $\lambda$ in (1), (2), and (3) can be generalized to values other than positive integers. For any fixed $\lambda$ we say that $l(y, z, \lambda)$ and $h(z, \lambda)$ are corresponding *chromatic sums*. I think of the "problem of chromatic sums" as that of determining $h(z, \lambda)$ for as many real or complex values of $\lambda$ as possible.

We note here some points of interest about $l$. First it is a power series in $z^2$ rather than $z$. However odd powers of $z$ do occur in the more general power series $g(x, y, z, \lambda)$. Next, $l$ is *z-restricted*, that is the coefficient of each power of $z^2$ is a polynomial in the other variable $y$. To prove this we observe that a triangulation $T$ with $2p$ faces has $3p$ edges and $p + 2$ vertices. The least possible value of $n(T)$ is 2. The maximum value of $n(T)$, for $2p$ faces, is attained when the deletion of the root-vertex and its incident edges reduces the graph of $T$ to a tree. This tree has $p + 1$ vertices and $p$ edges, whence $n(T) = 2p$. So the coefficient of $z^{2p}$ in $l$ is

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a polynomial in $y$ of degree at most $2p + 1$, and moreover this polynomial divides by $y^3$.

In [**4**] functional equations for $l$ are found for special values $B_n$ of $\lambda$, where

$$(4) \qquad B_n = 2 + 2\cos(2\pi/n),$$

and $n$ is an integer not less than 2. $B_n$ is often called "the $n$th Beraha number" by combinatorialists. The theory of [**4**] is simplified and extended in [**5**].

For small values of $n$ the equation for $l$ can be got by more direct graph-theoretical methods. This is done for $n = 5$ and $n = 6$ in [**2**] and [**3**] respectively. In each of these papers a parametric solution is found; $z^2$ and $h$ are each expressed as simple functions of a parameter $\xi$. In principle $h$ can be determined as a power series in $z^2$ by eliminating $\xi$. In these two papers the elimination is achieved, by applications of Lagrange's Theorem.

In the present paper similar parametric solutions are found for the cases $n = 7$ and $n = 8$, though the author has not yet attempted to extract numerical results from them. For larger values of $n$ parametric solutions can still be claimed, in terms of the parameter $\beta$ of Theorem 5.2. But the equation (64) relating $\beta$ to $z^2$ may be thought rather complicated.

**2. The basic equation.** The simplification in [**5**] involves some changes of notation. For a given $B_n$ it is found convenient to write

$$(5) \qquad \mu = \lambda - 2 = 2\cos(2\pi/n),$$

$$(6) \qquad \nu = (2 - \mu)^{-1} = 4\{\sin^2(\pi/n)\}^{-1}.$$

Next $y$ is replaced by a variable $V$ related to it as follows.

$$(7) \qquad y(1 - \nu + V) = 1.$$

Finally $l$ is replaced by the following expression $K$, regarded as a function of $V$ and $z^2$.

$$(8) \qquad K = \lambda^{-1}z^2 l(y, z, \lambda) + z^2(\nu - V)^{-1} + (\nu - V)^2 + V.$$

In [**4**] and [**5**] care is taken to explain that the work is being done in the field of quotients of formal power series in the basic variables. In this further study we can, because $l$ is $z$-restricted, regard $K$ as an element of the ring $R$ of formal power series in $z^2$ in which the coefficient of each power of $z^2$ is a rational function of $V$. In general we work in the quotient field $F$ of $R$. A member $X/Y$ of $F$ can be seen to be a member of $R$ unless the initial term, the coefficient of $z^0$, in $Y$ is zero.

The form given in [**5**] for the basic equation in the case $\lambda = B_n$ depends on whether $n$ is odd or even. In the first case we write $n = 2M - 1$, and in the second $n = 2M$. Let us also write $\theta = 0$ or 1 according as $n$ is even or odd, so that $M = (n + \theta)/2$.

In the theory of [**5**] $K$ is called an "invariant". For each $B_n$ a second invariant $I(n)$ is found. It is shown that $I(n)$ must satisfy the equation

$$(9) \qquad I(n) = \sum_{i=0}^{M-2} g_i K^i,$$

where the coefficients $g_i$ are power series in $z^2$ independent of $V$. In Theorems 8.2 and 8.3 of [**5**] $I(n)$ is given as a polynomial in $V$ and $k/4 = K/(4 - \mu^2)$. The first theorem refers to the even case and the second to the odd. Unfortunately there is an error of sign in the second theorem; the expression $(V^{2r+1} - \nu^{2r+1})$ should be replaced by $(\nu^{2r+1} - V^{2r+1})$. Using $\theta$ we can combine the equations of the two theorems into one as follows.

$$(10) \qquad I(n) = (-1)^\theta n \sum_{r=0}^{M-\theta} \left\{ \frac{(-1)^r (M + r - 1)!}{(2r + \theta)!(M - r - \theta)!} \times \left(\frac{K}{(4 - \mu^2)}\right)^{M-r-\theta} (V^{2r+\theta} - \nu^{2r+\theta}) \right\}.$$

The "basic equation" with which this Section is concerned equates the right sides of (9) and (10). It seems that this equation contains enough information to allow us to determine the coefficients in the initially unknown power series $g_i$. When these series are known the basic equation reduces to an ordinary equation of degree $M - 1$ for $K$.

In (10) we are dealing with Chebyshev polynomials and we can make use of the identity

$$(11) \qquad (-1)^M \{\cos \{n \cos^{-1} (x/2)\}\} = \tfrac{1}{2}(-1)^\theta n \sum_{r=0}^{M-\theta} \frac{(-1)^r (M + r - 1)!}{(2r + \theta)!(M - r - \theta)!} x^{2r+\theta}.$$

Accordingly we can rewrite the basic equation as

$$(12) \qquad (-1)^M t^{-\theta} \{\cos \{n \cos^{-1} (V/2t)\} - \cos \{n \cos^{-1} (\nu/2t)\}\} = \sum_{i=2}^{M} q_i t^{-2i},$$

where

$$(13) \qquad q_i = \tfrac{1}{2}(4 - \mu^2)^{M-i} g_{M-i} \quad \text{and}$$

$$(14) \qquad t^2 = K/(4 - \mu^2).$$

The definition of $t^2$ needs some comment. From (5), (6) and (8) we have

$$(15) \qquad (\text{Coeff. of } z^0 \text{ in } K) = V^2 - \mu\nu V + \nu^2.$$

Hence the initial term, the coefficient of $z^0$, in the square roof of $K/(4 - \mu^2)$ is not a rational function of $V$, and so the square root is not a member of $F$. We have to think of $t$ as an imaginary over the field $F$. But the dependence of Equation (12) on the imaginary $t$ is only apparent; the equation is in fact a relation between two polynomials in $t^{-2}$.

Equation (12) combines Theorem 6.4 and 6.5 of [**5**].

**3. A preliminary study of the basic equation.** We write

$$Q = (\nu - V)K. \tag{16}$$

$Q$ is an element of $R$ in which the coefficient of each power of $z^2$ is a polynomial in $V$ divided by a power of $(1 - \nu + V)$, by (7) and (8). The same statement can be made for each of the derivatives of $Q$ with respect to $V$, of all orders. Hence we can substitute $\nu$ for $V$ in $Q$, or in any $V$-derivative of $Q$, to obtain a definite power series in $z^2$. By (8) we have

$$Q_\nu = z^2, \tag{17}$$

$$(\partial Q/\partial V)_\nu = -(\lambda^{-1}z^2h + \nu), \tag{18}$$

where the suffix indicates substitution of $\nu$ for $V$. More generally we note that each of the substituted $V$-derivatives of $Q$ has coefficients expressible in terms of those in the power series $l(y, z, \lambda)$ in $y$ and $z^2$.

In (12) we have an equation between two polynomials in $t^{-2}$. We can write in it

$$t^{-2} = (4 - \mu^2)(\nu - V)Q^{-1}. \tag{19}$$

Let us now multiply (12), expressed in polynomial form, by $Q^j$, where the integer $j$ satisfies $2 \leqq j \leqq M$. Let us then differentiate $j$ times with respect to $V$ and substitute $\nu$ for $V$. We find that

$$\begin{aligned}\tfrac{1}{2}(-1)^{\theta+1}n \sum_{r=0}^{j-\theta-1} \Bigg\{ &\frac{(-1)^r(M+r-1)!}{(2r+\theta)!(M-r-\theta)!}(4-\mu^2)^{r+\theta} \\ &\times \left[(\partial/\partial V)^j\left\{Q^{j-r-\theta}(\nu - V)^{r+\theta+1}\sum_{u=0}^{2r+\theta-1}(V^u\nu^{2r+\theta-u-1})\right\}\right]_\nu\Bigg\} \\ &= \sum_{i=2}^{j} q_i(4-\mu^2)^i[(\partial/\partial V)^j\{(\nu - V)^iQ^{j-i}\}]_\nu,\end{aligned}$$

that is,

$$\begin{aligned}\tfrac{1}{2}n \sum_{r=0}^{j-\theta-1} \Bigg\{ &\frac{(-1)^r(M+r-1)!}{(2r+\theta)!(M-r-\theta)!}(4-\mu^2)^{r+\theta}\frac{(-1)^r\cdot j!}{(j-r-\theta-1)!} \\ &\times \left[(\partial/\partial V)^{j-r-\theta-1}\left\{Q^{j-r-\theta}\sum_{u=0}^{2r+\theta-1}V^u\nu^{2r+\theta-u-1}\right\}\right]_\nu\Bigg\} \\ &= \sum_{i=2}^{j-1} q_i(-1)^i(4-\mu^2)^i\frac{j!}{(j-i)!}[(\partial/\partial V)^{j-i}Q^{j-i}]_\nu \\ &\qquad + q_j(-1)^j(4-\mu^2)^j\cdot j!.\end{aligned} \tag{20}$$

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We note that the left side of (20), expressed as a polynomial in $Q_\nu$ and the substituted $V$-derivatives of $Q$, divides by $Q_\nu$. It thus reduces to a power series in $z^2$ with a zero initial term. So by applying (20) for successive values $2, 3, 4, \ldots, M$ of $j$ we can obtain the following theorem.

3.1. *In both the even and the odd cases each $q_i$ is a power series in $z^2$ with a zero initial term. Moreover its coefficients are expressible in terms of those of $l$.*

The cases $j = 2$ and $j = 3$ of (20) are of special interest. We consider first the even case, noting that the second sum on the left of (20) becomes empty when $r = 0$. It is found that

$$(21) \qquad q_2 = \frac{M^2\nu z^2}{(4-\mu^2)} \quad (\theta = 0).$$

$$(22) \qquad q_3 = \frac{M^2\nu z^2\{\lambda^{-1}z^2h + \nu\}}{(4-\mu^2)^2} - \frac{\frac{1}{2}M^2z^4}{(4-\mu^2)^2} - \frac{(M+1)M^2(M-1)\nu^3z^2}{6(4-\mu^2)} \quad (\theta = 0).$$

Equations (17) and (18) are used.

In the odd case we get the following rather different equations.

$$(23) \qquad q_2 = \frac{(2M-1)z^2}{2(4-\mu^2)} \quad (\theta = 1),$$

$$(24) \qquad q_3 = \frac{(2M-1)z^2\{\lambda^{-1}z^2h + \nu\}}{2(4-\mu^2)^2} - \frac{(2M-1)M(M+1)\nu^2z^2}{4(4-\mu^2)} \quad (\theta = 1).$$

Since we have stated our objective as the determination of $h$ in terms of $z^2$ we can now redefine our enumerative problem as that of relating $q_3$ to $q_2$. But perhaps we should resign ourselves to the necessity of determining all the $q_j$ as functions of $z^2$.

## 4. Reducers and conducers.

Let

$$(25) \qquad f = \sum_j f_j z^{2j}$$

be any power series in $z^2$ with real or complex numbers as coefficients. We consider the operation of substituting $f$ for $V$ in an element

$$(26) \qquad A = \sum_{j=0}^{\infty} A_j z^{2j}$$

of $R$. Here $A_j$ is a quotient $X_j/Y_j$ of two polynomials $X_j$ and $Y_j$ in $V$. We can substitute $f$ for $V$ in $X_j$ and $Y_j$, getting power series $[X_j]_f$ and $[Y_j]_f$ respectively in $z^2$. If the series $[Y_j]_f$ has a non-zero initial term we

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can also evaluate the quotient

$$[A_j]_f = [X_j]_f/[Y_j]_f$$

as a power series in $z^2$. If this can be done for each $j$ we can obtain from $A$ the well-defined power series

$$[A]_f = \sum_{j=0}^{\infty} z^{2j}[A_j]_f,$$

which we call the result of substituting $f$ for $V$ in $A$.

Evidently the procedure fails only if $V - f_0$ is a factor of $Y_j$ for some $j$. In particular we can substitute $f$ for $V$ in $K$ whenever $f_0$ is not $\nu$ or $\nu - 1$.

It is convenient to use the symbol $U_s$, where $s$ is a positive integer, to denote any number that can be expressed in the following form. It is a polynomial in the quantities $f_j$ with $j < s$ and the coefficients $l_{ij}$ of the products $y^i z^{2j}$ in $l(y, z, \lambda)$, again with $j < s$. Moreover the coefficients in this polynomial are to have determinable real values. For example we can write

$$(27) \qquad [(\partial/\partial z^2)^s[K]_f]_0 = [(\partial/\partial z^2)^s(f^2 - \mu\nu f + \nu^2)]_0 + U_s,$$

where the suffix 0 indicates that we are to take the initial term. As a simplification of (27) we write also

$$(28) \qquad [(\partial/\partial z^2)^s[K]_f]_0 = s!(2f_0 - \mu\nu)f_s + U_s.$$

The symbol $U_s$ need not represent the same number in each occurrence.

We use (28) to prove the following existence theorem.

4.1. *Let $\alpha$ be a real number which is not $\nu$, $\nu - 1$ or $2\nu/\mu$. Then there exists a real power series $f$ in $z^2$ such that $f_0 = \alpha$ and such that*

$$(29) \qquad f^2 = \beta[K]_f$$

*for some real number $\beta$. Moreover we then have*

$$(30) \qquad \beta = \frac{f_0^{\,2}}{f_0^{\,2} - \mu\nu f_0 + \nu^2}.$$

*Proof.* If (29) is to hold we must have

$$f_0^2 = \beta(f_0^2 - \mu\nu f_0 + \nu^2)$$

by (15). But

$$(31) \qquad f_0^2 - \mu\nu f_0 + \nu^2 = (f_0 - \tfrac{1}{2}\mu\nu)^2 + \tfrac{1}{4}\nu^2(4 - \mu^2).$$

Hence the expression on the left of (31) is positive for each real $f_0$, by (5) and (6). We are therefore justified in writing (30).

By formal differentiation of (29) and the use of (28) we have

$$s!2f_0f_s = s!(2f_0 - \mu\nu)f_s\beta + U_s,$$

that is

$$(32) \qquad (f_0{}^2\mu\nu - 2\nu^2 f_0)f_s = U_s.$$

If $f_0 = 0$ the theorem holds. In that case $f$ is the zero power series in $z^2$ and $\beta = 0$. In the remaining case, by hypothesis,

$$f_0{}^2\mu\nu - 2\nu^2 f_0 \neq 0.$$

Hence, $f_0$ being known, we can find $f_1, f_2, f_3$ and so on in terms of the numbers $l_{ij}$ by (32), using the appropriate number $U_s$ in each case. The sequence of the $f_i$ defines a formal power series $f$ in $z^2$, as in (25), and this series satisfies (29).

This completes the proof of 4.1. We can add that (32) determines $f_s$ uniquely in terms of the preceding coefficients $f_i$. So we have the

COROLLARY. *Under the conditions of* 4.1 *the series $f$ is uniquely determined.*

If the initial term of $f$ is not $\nu$ or $\nu - 1$ we have the series $[K]_f$ in $z^2$. The initial term of this series is $f_0{}^2 - \mu\nu f_0 + \nu^2$, which is positive by (31). Accordingly there is a uniquely determined power series $\phi$ in $z^2$ which is a square root of

$$[K]_f/(4 - \mu^2)$$

and has a positive initial term $\phi_0$. It is convenient to regard $\phi$ as the result of substituting $f$ for $V$ in $t$.

Since $f$ can be substituted for $V$ in $K$ it can be substituted also in the basic equation. The result, expressed in trigonometrical form, is as follows

$$(33) \qquad (-1)^M\{\cos\{n\cos^{-1}(f/2\phi)\} - \cos\{n\cos^{-1}(\nu/2\phi)\}\} = \sum_{i=2}^{M} q_i{}^{\theta-2i}.$$

We can get another equation involving $f$ and $\phi$ by first differentiating the basic equation by $V$ and then substituting. We take note of the following identities.

$$(34) \qquad (d/dx)\{\cos\{n\cos^{-1}(x/2)\}\} = \tfrac{1}{2}n\frac{\sin\{n\cos^{-1}(x/2)\}}{\{1-(x^2/4)\}^{1/2}}$$
$$= \tfrac{1}{2}(-1)^\theta n\sum_{r=0}^{M-\theta}\frac{(-1)^r(M+r-1)!x^{2r+\theta-1}}{(2r+\theta-1)!(M-r-\theta)!}.$$

We can now write our new equation as

$$(35) \qquad \frac{1}{2\phi}\left[\frac{d(t^2)}{dV}\right]_f\Bigg[-\frac{nf\sin\{n\cos^{-1}(f/2\phi)\}}{2\phi^2\{1-(f^2/4\phi^2)\}^{1/2}} + \frac{n\nu\sin\{n\cos^{-1}(\nu/2\phi)\}}{2\phi^2\{1-(\nu^2/4\phi^2)\}^{1/2}}$$
$$- (-1)^M\sum_{i=2}^{M}(2i-\theta)q_i\phi^{\theta-2i-1}\Bigg] + \frac{n\sin\{n\cos^{-1}(f/2\phi)\}}{2\phi\{1-(f^2/4\phi^2)\}^{1/2}} = 0.$$

Let us say that the power series $f$ is a *reducer*, with respect to the given $n$, if it satisfies the two following conditions.

(i) $f_0$ is not $\nu$, $\nu - 1$, $2\nu/\mu$ or $\mu\nu/2$.
(ii) $f = 2\phi \cos (p\pi/n)$

for some integer $p$.

The second condition is equivalent to the assertion that

$$(36) \qquad \sin \{n \cos^{-1} (f/2\phi)\} = 0.$$

If $f$ is a reducer we say that $\phi$ is the corresponding *conducer*.

If (35) is to be interpreted as a relation between power series in $z^2$ we must have

$$(37) \qquad 4\phi_0^2 \neq f_0^2,\ 4\phi_0^2 \neq \nu^2.$$

But if $4\phi_0^2 = f_0^2$ or $\nu^2$ we have

$$4f_0^2 - 4\mu\nu f_0 + 4\nu^2 = (4 - \mu^2)f_0^2 \quad \text{or} \quad (4 - \mu^2)\nu^2,$$

by (15), whence

$$(\mu f_0 - 2\nu)^2 = 0 \quad \text{or} \quad (2f_0 - \mu\nu)^2 = 0.$$

Hence (37) is always satisfied if $f$ is a reducer.

By (15) the initial term of the power series $[d(t^2)/dV]_f$ is that of $[d(V^2 - \mu\nu V + \nu^2)/dV]_f$, which is $(2f_0 - \mu\nu)$. Hence if $f$ is a reducer the power series $[d(t^2)/dV]_f$ is non-zero.

Using these results we can state Equations (33) and (35) in simpler forms valid for the case in which $f$ is a reducer.

4.2. *Let $f$ be a reducer, and $\phi$ the corresponding conducer. Then*

$$(38) \qquad (-1)^M\{(-1)^p - \cos \{n \cos^{-1} (\nu/2\phi)\}\} = \sum_{i=2}^{M} q_i \phi^{\theta-2i}$$

*and*

$$(39) \qquad (-1)^M(d/d\phi)\{\cos \{n \cos^{-1} (\nu/2\phi)\}\} = \sum_{i=2}^{M} (2i - \theta)q_i \phi^{\theta-2i-1}.$$

We can get some information about initial terms by applying 3.1. Using that theorem in conjunction with (34) and (39) we find that

$$\sin \{n \cos^{-1} (\nu/2\phi_0)\} = 0.$$

Hence

$$(40) \qquad \nu = 2\phi_0 \cos (m\pi/n)$$

for some integer $m$. Since $\phi_0$ is by definition positive we can restrict $m$ to satisfy $0 \leqq m \leqq \frac{1}{2}n$. If $n$ is even we exclude the case $m = \frac{1}{2}n$ since it makes $\phi_0$ infinite. We can now state the range of $m$ as from 0 to $M - 1$ in both the even and the odd cases. But we shall see that Condition (i) of the definition of a reducer imposes a further restriction.

Let us write

$$w = \cos(m\pi/n).$$

The values of $f_0$ corresponding to $m$ can be found by substitution in (15). We find that

$$(4 - \mu^2)\nu^2 w^{-2} = 4f_0^2 - 4\mu\nu f_0 + 4\nu^2,$$

$$f_0 = \tfrac{1}{2}\{\mu\nu \pm \nu(4 - \mu^2)^{1/2}(w^{-2} - 1)^{1/2}\},$$

$$f_0 = \tfrac{1}{2}\nu\{\mu \pm (4 - \mu^2)^{1/2} \tan(m\pi/n)\}. \tag{41}$$

By (5) and (6) this can be written also as

$$f_0 = \frac{\nu \cos\{(m \pm 2)\pi/n\}}{\cos(m\pi/n)}. \tag{42}$$

Comparing this equation with (40) and with Condition (ii) of the definition of a reducer we see that the integers $p$ and $m$ have the same parity.

Consider the case $m = 0$. There is only one corresponding value of $f_0$, namely $\frac{1}{2}\mu\nu$, by (41). But this value is excluded by Condition (i).

Now suppose $m = 1$. We deduce from (5) and (6) that

$$\cos(\pi/n) = \tfrac{1}{2}(2 + \mu)^{1/2}, \ \sin(\pi/n) = \tfrac{1}{2}(2 - \mu)^{1/2},$$

$$\tan(\pi/n) = \{(2 - \mu)/(2 + \mu)\}^{1/2}.$$

Substituting this in (41) we find that

$$f_0 = \tfrac{1}{2}\nu\{\mu \pm (2 - \mu)\} = \nu \quad \text{or} \quad \nu(\mu - 1).$$

But

$$\nu(\mu - 1) = \nu(-\nu^{-1} + 1) = \nu - 1,$$

by (6). Hence $f_0$ is $\nu$ or $\nu - 1$, and both these values are excluded.

Take next the case $m = 2$. By (5) we have

$$\cos(2\pi/n) = \tfrac{1}{2}\mu, \ \sin(2\pi/n) = \tfrac{1}{2}(4 - \mu^2),$$

$$\tan(2\pi/n) = (4 - \mu^2)^{1/2}/\mu.$$

Hence, by (41),

$$f_0 = (\nu/2\mu)\{\mu^2 \pm (4 - \mu^2)\},$$

$$f_0 = 2\nu/\mu \quad \text{or} \quad \nu(\mu^2 - 2)/\mu.$$

The first of these is an excluded value. We deduce that if $m = 2$ we must take the negative sign in (41) and the positive sign in (42).

Now by (15) $f_0$ determines $\phi_0$ uniquely. By (40) and the restriction on $m$ distinct values of $m$ determine distinct values of $\phi_0$. Hence no value of

$m$ exceeding 2 can correspond to an excluded value of $f_0$. If $m$ is 2 or more the two values of $f_0$ given by (41) are distinct. Hence if $m = 2$ the second value of $f_0$ is not an excluded value.

Using Theorem 4.1 we summarize our results as follows.

4.3. *There exist $M - 2$ distinct conducers, one satisfying*

$$2\phi_0 \cos (m\pi/n) = \nu$$

*for each integer $m$ such that $2 \leqq m \leqq M - 1$.*

When $m$ exceeds 2 there are two distinct corresponding reducers $f$, with $f_0$ given by (41). The corresponding conducers have the same value $\phi_0$ for their initial terms. Indeed we shall show in the next section that the two conducers are equal. Let us say that the two reducers corresponding to the same value of $m$ (exceeding 2) are *conjugate.*

**5. Parametric equations.** We can express (38) and (39) in simpler form by writing

$$(43) \qquad p_r = q_r + \tfrac{1}{2}n \frac{(-1)^r (M + r - \theta - 1)!\nu^{2r-\theta}}{(2r - \theta)!(M - r)!},$$

where $0 \leqq r \leqq M$ and $q_0 = q_1 = 0$. If we also write $\psi$ for the reciprocal of $\phi$ then we can, by (11) and (34), replace (38) and (39) by the following equations (44) and (45) respectively.

$$(44) \qquad (-1)^{M+m} = \sum_{r=\theta}^{M} p_r \psi^{2r-\theta},$$

$$(45) \qquad 0 = \sum_{r=\theta}^{M} (2r - \theta) p_r \psi^{2r}.$$

We refer to (44) and (45) as the *parametric equations* for the given value of $n$. There are at least $M - 2$ parameters $\psi$, by 4.3, and for each one we have the above two equations. Moreover $p_0$, $p_1$ and $p_2$ are known in terms of $z^2$, by (21), (23) and (43). This leaves $M - 2$ unknown functions $p_r$. We may hope to determine them, and $M - 2$ parameters $\psi$, from $2M - 4$ parametric equations.

In the even case $\theta = 0$ we have, by (43),

$$(46) \qquad p_0 = 1, \quad p_1 = -\tfrac{1}{2}M^2\nu^2.$$

In the odd case $\theta = 1$ we can take $p_0$ to be zero. By (43) we have

$$(47) \qquad p_1 = -\tfrac{1}{2}(2M - 1)\nu.$$

Let us now differentiate (44) by $z^2$. Applying (45) to the result we find that

$$(48) \qquad \sum_{r=2}^{M} (dp_r/d(z^2))\psi^{2r} = 0.$$

5.1. *Conjugate reducers have the same conducer.*

*Proof.* Suppose that some two conjugate reducers $f$ and $g$ have different conducers. Then we can assert (48) for $M - 1$ distinct parameters $\psi = \psi_1, \psi_2, \ldots, \psi_{M-1}$.

The derivatives in (48) are not all zero, by (21), (23) and (43). But in (48) we have $M - 1$ homogeneous linear equations for the $M - 1$ derivatives. Hence the determinant of the matrix $\{\psi_i^{2j}\}$ must vanish. The necessary and sufficient condition for this is that $\psi_i^2 = \psi_j^2$ for some two distinct suffixes $i$ and $j$. But then $\psi_i = \psi_j$ since each conducer has a positive initial term, and we have a contradiction.

We write

$$(49) \qquad p_r' = dp_r/d(z^2).$$

We put also $p_r = p_r' = 0$ when $r > M$.

5.2. *There exists a power series $\beta$ in $z^2$ such that*

$$(50) \qquad \lambda p_{r+1}' = \beta p_r' - (2r - \theta)p_r$$

*for each integer $r \geqq 2$.*

*Proof.* The squares of our $M - 2$ parameters are the roots of the equation

$$(51) \qquad \sum_{r=0}^{M-2} p_{r+2}'x^r = 0,$$

by (48). But they are also roots of

$$(52) \qquad \sum_{r=0}^{M} (2r - \theta)p_r x^r = 0,$$

by (45). In (52) the coefficient of $x^0$ is zero if $\theta = 0$. So in both the even and the odd case we can reduce the degree of (52) by dividing by $x$. Then

$$(53) \qquad \sum_{r=0}^{M-1} (2r + 2 - \theta)p_{r+1}x^r = 0.$$

The squares of our parameters are still roots of this equation. For the reciprocal of a conducer cannot be zero.

We deduce that the left of (51) is a factor of the left of (53). We can therefore write

$$(54) \qquad (\alpha + \beta x) \sum_{r=0}^{M-2} p_{r+2}'x^r = \sum_{r=0}^{M-1} (2r + 2 - \theta)p_{r+1}x^r.$$

Here each of $\alpha$ and $\beta$ is a constant, a power series in $z^2$ or perhaps a quotient of two such power series. Such a quotient could be written as a power series multiplied by a negative power of $z^2$.

From the coefficients of $x^0$ and $x^1$ in (54) we find

(55) $\alpha p_2' = (2 - \theta)p_1,$

(56) $\alpha p_3' + \beta p_2' = (4 - \theta)p_2.$

Taking all the coefficients into account we have the general rule

(57) $\alpha p_{r+1}' + \beta p_r' = (2r - \theta)p_r,$

valid for all positive integers $r$, and of which (55) and (56) are special cases.

In the even case we have

(58) $p_1 = -\frac{1}{2}M^2\nu^2, \quad p_2' = M^2\nu/(4 - \mu^2),$

by (46) and (21). In the odd case we have

(59) $p_1 = -\frac{1}{2}(2M - 1)\nu, \quad p_2' = \frac{1}{2}(2M - 1)/(4 - \mu^2),$

by (47) and (23). In both cases

(60) $\alpha = -\nu(4 - \mu^2) = -\lambda,$

by (55), (5) and (6). Since $p_2'$ is a non-zero constant we deduce from (56) that $\beta$ is indeed a power series in $z^2$. The theorem follows, by (57).

We conclude this section with a comment on Theorem 5.2. Let us define an operator $W_r$ as follows.

(61) $W_r = D^{-1}(\beta D - (2r - \theta)),$

where $D$ is the operation of differentiating by $z^2$. Then

(62) $p_{r+1} = \lambda^{-1}W_r p_r$

for $r \geqq 2$, by 5.2. The initial term in $p_{r+1}$ is known, by (43) and Theorem 3.1. Hence the constant of integration implied by the symbol $D^{-1}$ can be determined. By iteration of (62) we have

(63) $p_{r+1} = \lambda^{-r+1}W_r W_{r-1} \ldots W_3 W_2 p_2.$

In particular we can put $r = M$ in (63). Then $p_{r+1} = 0$ and

(64) $W_M W_{M-1} \ldots W_3 W_2 p_2 = 0.$

Equation (64) involves only the one unknown function $\beta$; it would seem to be of some theoretical interest as an equation for that power series. If $\beta$ were known we could use (62) to find $p_3$, $p_4$, $p_5$ and so on.

**6. The case $n = 5$.** In this case

(65) $\mu = \tau^{-1},$

where $\tau$ is the positive root of the equation

(66) $\quad x^2 = x + 1.$

It is convenient to have the following table of powers of $\tau$.

TABLE I.

| | | | |
|---|---|---|---|
| $2\tau =$ | $1 + \sqrt{5}$ | $2\tau^{-1} =$ | $-1 + \sqrt{5}$ |
| $2\tau^2 =$ | $3 + \sqrt{5}$ | $2\tau^{-2} =$ | $3 - \sqrt{5}$ |
| $2\tau^3 =$ | $4 + 2\sqrt{5}$ | $2\tau^{-3} =$ | $-4 + 2\sqrt{5}$ |
| $2\tau^4 =$ | $7 + 3\sqrt{5}$ | $2\tau^{-4} =$ | $7 + 3\sqrt{5}$ |
| $2\tau^5 =$ | $11 + 5\sqrt{5}$ | $2\tau^{-5} =$ | $-11 + 5\sqrt{5}$ |

Using this Table in conjunction with (5) and (6) we find that

(67) $\quad \lambda = \tau^2,\ \nu = \tau/\sqrt{5},\quad 4 - \mu^2 = \sqrt{5}\cdot\tau.$

From (23) we have

(68) $\quad q_2 = \sqrt{5}\cdot z^2/2\tau,$

and from (24)

$$q_3 = \tfrac{1}{2}\left\{\tau^{-4}z^4h + \frac{\tau^{-1}z^2}{\sqrt{5}}\right\} - \frac{3\tau z^2}{\sqrt{5}},$$

(69) $$q_3 = \tfrac{1}{2}\tau^{-4}z^4h - \frac{\tau^3 z^2}{2\sqrt{5}}.$$

Applying (43) we obtain the following formulae for the $p_r$.

(70) $\quad p_1 = -\tfrac{1}{2}\sqrt{5}\cdot\tau,$

(71) $$p_2 = \tfrac{1}{2}\sqrt{5}\cdot\tau^{-1}z^2 + \frac{\tau^3}{2\sqrt{5}},$$

(72) $$p_3 = \tfrac{1}{2}\tau^{-4}z^4h - \frac{\tau^3z^2}{2\sqrt{5}} - \frac{\tau^5}{50\sqrt{5}}.$$

In the present case $M = 3$ and there is only one conducer $\phi$, corresponding to the value 2 of $m$. The associated parameter $\psi = \phi^{-1}$ satisfies the following two equations, by (44) and (45)

(73) $\quad -1 = p_1\psi + p_2\psi^3 + p_3\psi^5,$

(74) $\quad 0 = p_1\psi + 3p_2\psi^3 + 5p_3\psi^5.$

From these we deduce

(75) $\quad 2p_2 = -5\phi^3 + 2\sqrt{5}\cdot\tau\phi^2,$

(76) $\quad 2p_3 = 3\phi^5 - \sqrt{5}\cdot\tau\phi^4.$

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Using (71) and (72) we can now express $z^2$ and $z^4h$ in terms of $\phi$, as follows.

(77) $z^2 = -5\tau\phi^3 + 2\tau^2\phi^2 - (\tau^4/5),$

(78) $\tau^{-9}z^4h = 3\phi^5 - \sqrt{5}\cdot\tau\phi^4 - \tau^4\phi^3 + \dfrac{2\tau^5\phi^2}{\sqrt{5}} - \dfrac{\tau^5(5\tau^2 - 1)}{25\sqrt{5}}.$

If we put

(79) $x = \sqrt{5}\cdot\tau^{-1}$

Equations (77) and (78) can be written as follows.

(80) $-5\tau^{-4}z^2 = (x - 1)(x - \tau)(x + \tau^{-1}),$

(81) $25\sqrt{5}\cdot\tau^{-9}z^4h = (x - \tau)^2(x + \tau^{-1})(3x^2 + (3\tau - 2)x - (4\tau + 1)).$

These two formulae can be identified with the parametric equations numbered (35) and (36) in [**2**]. There the symbol $u$ is to be interpreted as

$$(x + \tau^{-1})/\sqrt{5}.$$

A solution of the parametric equations is given in [**2**]. Here it remains for us to check the application of Equations (48) and (50).

From (75) and (76) we have

(82) $dp_2/d\phi = -15\phi^2 + 4\sqrt{5}\cdot\tau\phi,$

(83) $dp_3/d\phi = 15\phi^4 - 4\sqrt{5}\cdot\tau\phi^3.$

Using (71) we deduce that

(84) $2p_2' = \sqrt{5}\cdot\tau^{-1},$

(85) $2p_3' = -\sqrt{5}\cdot\tau^{-1}\phi^2.$

These results verify (48) which for $M = 3$ takes the form

(86) $p_2' + \psi^2p_3' = 0.$

From Equation (50) we have

(87) $\lambda p_3' = \beta p_2' - 3p_2,$

(88) $0 = \beta p_3' - 5p_3.$

From either of the last two equations we can deduce

(89) $\beta = -3\sqrt{5}\cdot\phi^3 + 5\tau^2\phi.$

**7. The case** $n = 6$**.** In this case we have

(90) $\lambda = 3, \mu = 1, \nu = 1, 4 - \mu^2 = 3.$

From (21) and (22) we find

(91) $q_2 = 3z^2$,

(92) $q_3 = (z^4h/3) - 3z^2 - (z^4/2)$.

Hence by (43)

(93) $p_0 = 1, p_1 = -9/2, p_2 = q_2 + 3, p_3 = q_3 - \frac{1}{2}$.

Again there is only one conducer $\phi$, corresponding to $m = 2$. Equations (44) and (45) can be written thus.

(94) $-2 = p_1\psi^2 + p_2\psi^4 + p_3\psi^6$,

(95) $0 = p_1 + 2p_2\psi^2 + 3p_3\psi^4$.

From these we find

(96) $p_2 = -6\phi^4 + 9\phi^2$,

(97) $2p_3 = 8\phi^6 - 9\phi^4$.

We can use the preceding equations to express $z^2$ and $z^4h$ in terms of $\phi$. We find

(98) $z^2 = -(\phi^2 - 1)(2\phi^2 - 1)$,

(99) $z^4h = 6(\phi^2 - 1)^2(\phi^4 + \phi^2 - 1)$.

These can be identified with the parametric equations numbered (24) and (27) in [**3**]. We are to put $\xi = \psi$.

From (96) and (97) we have

(100) $dp_2/d(\phi^2) = -12\phi^2 + 9$,

(101) $dp_3/d(\phi^2) = 12\phi^4 - 9\phi^2$.

Using (91) we deduce that

(102) $p_2' = 3, p_3' = -3\phi^2$.

This verifies (48), which still takes the form (86). From Equation (50) we now have

(103) $3p_3' = \beta p_2' - 4p_2, \quad 0 = \beta p_3' - 6p_3$.

From either of these equations we can deduce

(104) $\beta = 9\phi^2 - 8\phi^4$.

**8. The case $n = 7$.** In this case $M = 4$ and $\mu$ satisfies the equation

(105) $\mu^3 + \mu^2 - 2\mu - 1 = 0$.

In works on chromatic polynomials the values of $\lambda$ for $n = 5$ and $n = 7$ are sometimes called the gold and silver roots respectively.

We now have two distinct conducers, corresponding to $m = 2$ and $m = 3$. We denote them by $\phi_2$ and $\phi_3$ respectively. In (44) and (45) we now have four linear equations which we can solve for $p_1$, $p_2$, $p_3$ and $p_4$ in terms of $\phi_2$ and $\phi_3$.

The author found it convenient to write

$$(106)\quad v = \phi_3 - \phi_2, \quad u = v^{-2}\phi_2\phi_3.$$

Solving the four linear equations he then found

$$(107)\quad 2p_1 = -v(3 + 3u - u^2),$$

$$(108)\quad 2p_2 = v^3(1 + 3u + 4u^2 - 3u^3),$$

$$(109)\quad 2p_3 = -v^5(2u^2 + 2u^3 - 3u^4),$$

$$(110)\quad 2p_4 = v^7(u^4 - u^5).$$

But $p_1 = -7\nu/2$ by (47), so Equation (107) gives $v$ as a function of $u$. Accordingly we can write the following equations for $p_2$, $p_3$ and $p_4$ as functions of a single parameter $u$.

$$(111)\quad 2p_2 = \frac{(7\nu)^3(1 + 3u + 4u^2 - 3u^3)}{(3 + 3u - u^2)^3},$$

$$(112)\quad 2p_3 = -\frac{(7\nu)^5(2u^2 + 2u^3 - 3u^4)}{(3 + 3u - u^2)^5},$$

$$(113)\quad 2p_4 = \frac{(7\nu)^7(u^4 - u^5)}{(3 + 3u - u^2)^7}.$$

Since $p_2$ and $p_3$ have simple expressions in terms of $z^2$ and $z^4h$, by (23), (24) and (43), the above equations can in principle be used to determine $h$ as a power series in $z^2$. Leaving this for possible future investigation we go on to study the present forms of (48) and (50). By differentiation we have

$$(114)\quad \frac{2dp_2}{du} = \frac{(7\nu)^3u(12 - 24u + 16u^2 - 9u^3)}{(3 + 3u - u^2)^4},$$

$$(115)\quad \frac{2dp_3}{du} = \frac{-(7\nu)^5u(1 + 2u)(12 - 24u + 16u^2 - 9u^3)}{(3 + 3u - u^2)^6}$$

$$(116)\quad \frac{2dp_4}{du} = \frac{(7\nu)^7u^3(12 - 24u + 16u^2 - 9u^3)}{(3 + 3u - u^2)^8}.$$

Accordingly

$$(117)\quad \frac{dp_3}{dp_2} = \frac{-(7\nu)^2(1 + 2u)}{(3 + 3u - u^2)^2}.$$

$$(118)\quad \frac{dp_4}{dp_2} = \frac{(7\nu)^4u^2}{(3 + 3u - u^2)^4}.$$

By (23) and (43) we have

(119) $2p_2' = 7/(4 - \mu^2)$.

To find $2p_3'$ and $2p_4'$ we multiply the right sides of (117) and (118) respectively by $2p_2'$.

Equations (48) can be written

$$\phi_2^4 dp_2 + \phi_2^2 dp_3 + dp_4 = 0,$$
$$\phi_3^4 dp_2 + \phi_3^2 dp_3 + dp_4 = 0.$$

Their solution is

(120) $$dp_2 = \frac{-dp_3}{\phi_2^2 + \phi_3^2} = \frac{dp_4}{\phi_2^2\phi_3^2}.$$

But

$$\phi_2^2\phi_3^2 = u^2v^4, \quad \phi_2^2 + \phi_3^2 = v^2(1 + 2u),$$

by (106). Hence (120) is consistent with (117) and (118).

From (50) we get the three following equations.

$$\lambda p_3' = \beta p_2' - 3p_2,$$
$$\lambda p_4' = \beta p_3' - 5p_3,$$
$$0 = \beta p_4' - 7p_4.$$

From any one of these we can deduce that

(121) $$\beta = \frac{(4 - \mu^2)(7\nu)^3(u^2 - u^3)}{(3 + 3u - u^2)^3}.$$

We have checked our parametric equations (111)-(113) against (50). Since the check gives a solution $\beta$ of (64) as a function of $u$ it amounts almost to a new proof of these equations.

**9. The case** $n = 8$**.** In this case $M = 4$ and $\mu = \sqrt{2}$. Again we have two conducers $\phi_2$ and $\phi_3$, and four linear equations to solve for $p_1$, $p_2$, $p_3$ and $p_4$ in terms of them. We also have

$$p_1 = -8\nu^2$$

by (46).

This time the author wrote

(122) $u\phi_3^2 = \phi_3^2 - \phi_2^2$

and obtained the following parametric equations.

(123) $$p_2 = \frac{-8\nu^4u^3(6 - 15u + 8u^2)}{(1 - 2u)^2}$$

(124) $$p_3 = \frac{-32\nu^6u^6(3 - 6u + 2u^2)(1 - u)}{(1 - 2u)^3}$$

(125) $$p_4 = \frac{-32\nu^8u^9(2 - 3u)(1 - u)^2}{(1 - 2u)^4}.$$

For derivatives we can deduce the following equations.

$$(126)\quad \frac{dp_2}{du} = \frac{-16\nu^4 u^2(9 - 36u + 50u^2 - 24u^3)}{(1-2u)^3},$$

$$(127)\quad \frac{dp_3}{du} = \frac{-32\nu^6 u^5(2-u)(9 - 36u + 50u^2 - 24u^3)}{(1-2u)^4}$$

$$(128)\quad \frac{dp_4}{du} = \frac{-64\nu^8 u^8(1-u)(9 - 36u + 50u^2 - 24u^3)}{(1-2u)^5}$$

$$(129)\quad \frac{dp_3}{dp_2} = \frac{2\nu^2 u^3(2-u)}{(1-2u)},$$

$$(130)\quad \frac{dp_4}{dp_2} = \frac{4\nu^4 u^6(1-u)}{(1-2u)^2}.$$

We can use these results to verify (48), much as in Section 8. From Equation (50) we get the following equations for the parameter $\beta$.

$$\lambda p_3' = \beta p_2' - 4p_2,$$
$$\lambda p_4' = \beta p_3' - 6p_3,$$
$$0 = \beta p_4' - 8p_4.$$

From (21) and (43) we have

$$p_2' = 16/(4-\mu^2).$$

From this, Equations (123)-(130) and any one of the three $\beta$-equations we can deduce that

$$(131)\quad \beta = \frac{-4(4-\mu^2)\nu^3 u^3(1-u)(2-3u)}{(1-2u)^2}.$$

## References

1. W. T. Tutte, *Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$*, Can. J. Math. *25* (1973), 426–447.
2. ——— *Chromatic sums for rooted planar triangulations II: the case $\lambda = \tau + 1$*, Can. J. Math. *25* (1973), 657–671.
3. ——— *Chromatic sums for rooted planar triangulations III: the case $\lambda = 3$*, Can. J. Math. *25* (1973), 780–790.
4. ——— *Chromatic sums for rooted planar triangulations V: special equations*, Can. J. Math. *26* (1974), 893–907.
5. ——— *On a pair of functional equations of combinatorial interest*, Aequationes Math. *17* (1978), 121–140.

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