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# Discrete automorphism groups of convex cones of finite type

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## ABSTRACT

We investigate subgroups of  $SL(n, \mathbb{Z})$  which preserve an open nondegenerate convex cone in  $\mathbb{R}^n$  and admit in that cone as fundamental domain a polyhedral cone of which some faces are allowed to lie on the boundary. Examples are arithmetic groups acting on self-dual cones, Weyl groups of certain Kac–Moody algebras, and they do occur in algebraic geometry as the automorphism groups of projective manifolds acting on their ample cones.

## Introduction

This story begins with the seemingly innocuous Theorem 2.2, which might have a place in the theory of linear programming. It is easily stated, even in an introduction: if  $V$  is a finite-dimensional real vector space,  $L \subset V$  a lattice and  $C$  an open nondegenerate convex cone in  $V$ , then the convex hull of  $C \cap L$  is locally polyhedral in the sense that its intersection with any bounded polyhedron is a polyhedron. This is our basic tool for our investigation of the linear automorphism groups  $\Gamma$  of  $C$  which preserve  $L$  and possess a natural finiteness property. That property has a number of equivalent formulations, one of which is that there exists a convex cone spanned by a finite subset of  $L \cap \bar{C}$  whose  $\Gamma$ -orbit contains  $C$ . This turns out to be of a self-dual nature in the sense that the same is then true for the contragredient action of  $\Gamma$  on  $V^*$  relative to the duals of  $C$  and  $L$ . Since the invariant lattice  $L$  is secondary to the resulting  $\mathbb{Q}$ -structure  $V(\mathbb{Q})$  on  $V$ , we call  $(V(\mathbb{Q}), C, \Gamma)$  a *polyhedral triple*.

Examples of polyhedral triples abound and provide sufficient justification for investigating this situation in its own right, even if the motivation lies elsewhere (more on this at the end of this introduction). First of all there is the case when  $C$  is a self-dual homogeneous cone and  $\Gamma$  is an arithmetic subgroup of the automorphism group of  $C$  (the most classical instances of which are perhaps the Lobatchevski cones and the cone of positive definite quadratic forms in a fixed number of variables). This is the case studied in detail by Ash in [AMRT10] and, indeed, much of the present paper generalizes his work to our setting.

Another class of examples constitute the cones attached to irreducible Coxeter groups that are neither finite nor of affine type (these are indeed nondegenerate convex). Related to this class are the hyperbolic reflection groups studied by Vinberg and Nikulin. A number of examples occur in algebraic geometry by taking for  $V$  the Néron–Severi space of a projective manifold, for  $C$  its ample cone (or dually, the cone spanned by the homology classes of curves) and for  $\Gamma$  the image of the representation of the automorphism group of the manifold.

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The first section recalls (or discusses) some of the basics of the theory of convex sets. In §2 we prove the fundamental theorem stated above. Its first applications are in §3. The following section introduces the central notion of this paper (that of a polyhedral triple) and we derive a number of properties of the corresponding group. In §5 we determine the structure of a stabilizer of a face, with most of our results being summed up by Theorem 5.1.

This paper is intended to be the first instalment of a series on semi-toric compactification. It is a ‘spin-off’ of my ancient unpublished preprint [Loo85] to the extent that it is only about the geometry of discrete groups acting on convex cones, the complex-analytic story being relegated to sequels. Yet I believe that this material forms a natural whole and that the results have an interest of their own, independent of the motivating application originally envisaged (namely to develop a common generalization of the compactifications of Baily and Borel [BB66] and Mumford *et al.* [AMRT10] of locally symmetric varieties).

*A notational convention.* If a group  $\Gamma$  acts on a set  $X$  and  $Y \subset X$ , then we denote by  $N_\Gamma(Y)$ , respectively  $Z_\Gamma(Y)$ , the subgroup of  $\gamma \in \Gamma$  which leave  $Y$  invariant, respectively pointwise fixed, so that  $\Gamma(Y) := N_\Gamma(Y)/Z_\Gamma(Y)$  can be understood as a group of permutations of  $Y$ .

### 1. Convex cones and kernels

We begin with recalling some definitions that are standard in the theory of convex sets. A subset  $C$  of a real finite-dimensional vector space  $V$  is called a *cone* if it is nonempty and invariant under scalar multiplication with positive numbers (so  $C$  need not contain the origin). The set of linear forms on  $V$  that are  $\geq 0$  on  $C$  is a closed convex cone in the dual vector space  $V^*$ , called the *cone dual to  $C$*  and denoted  $C^*$ . The interior of this dual, which we shall refer to as the *open dual* of  $C$ , will be denoted by  $C^\circ$ . It is set of linear forms whose zero set meets the closure of  $C$  in the origin only. For a closed convex cone  $C$  in  $V$ , we have  $C^{**} = C$ : according to [Roc70, Corollary 11.7.1]  $C$  is the intersection of the half spaces which contain it (if this collection of half spaces is empty, then this intersection is  $V$  by definition).

Let  $A$  be a finite-dimensional real affine space. Given a convex subset  $X \subset A$ , then the *relative interior* of  $X$  is the interior of  $X$  in its affine span; we denote it by  $\overset{\circ}{X}$  (other authors use  $\text{ri}(X)$ ). If this happens to be  $X$ , then we say that  $X$  is *relatively open*. A *face* of  $X$  is a nonempty subset  $F$  of  $X$  with the property that every segment in  $X$  which meets  $F$  is either contained in  $F$  or meets  $F$  in an end point. A face is closed in  $X$ . Any intersection of faces of  $X$  is a face of  $X$  and the relative interiors of the faces decompose  $X$ . If  $f : X \rightarrow \mathbb{R}$  is the restriction of an affine-linear function, then the set of points of  $X$  where  $f$  assumes its infimum is a face of  $X$ , but not every face of  $X$  is necessarily of that form. A face that can be so obtained is called *exposed*. A point of  $X$  that makes up a face, respectively an exposed face, by itself is also called an *extreme*, respectively *exposed, point* of  $X$ . The *star* of a face  $F$  of  $X$ ,  $\text{Star}_X(F)$ , is the collection of the faces of  $X$  that contain  $F$ . The union of the relative interiors of these faces will be denoted  $|\text{Star}_X(F)|$ . In case  $\text{Star}_X(F)$  is finite,  $|\text{Star}_X(F)|$  will be open in  $X$ , but this need not be so in general. We say that  $X$  is *nondegenerate* if it does not contain an affine line.

We denote by  $T(A)$  the vector space of translations of the affine space  $A$  and extend the range of that notation as follows. If  $X$  and  $Y$  are subsets  $A$ , then we denote by  $T(X, Y) \subset T(A)$  the set of translations that take  $X$  to  $Y$ . In the special case when  $X = Y$  is convex, then  $T(X, Y)$  is a convex cone (if  $t \in T(A)$  is such that  $t + X \subset X$  and  $\lambda > 0$ , then choose an integer  $n \geq \lambda$ ; we have  $nt + X \subset X$  and then also  $\lambda t + X \subset X$  by convexity), called the *recession cone* of  $X$ , and denoted  $T(X)$  instead (Rockafellar [Roc70] writes  $0^+X$ ). The recession cone of a *bounded* convex set is clearly reduced to the origin. According to [Roc70, Theorem 8.4], the converse holds for any

convex subset that is closed (and hence also when it is relatively open). In that case  $T(X)$  has a simple geometric interpretation: if we compactify  $A$  to a topological ball by adding as boundary the topological sphere of directions in  $T(A)$ , then the closure of a closed convex  $X \subset A$  intersects that boundary sphere in the subset defined by the directions in  $T(X)$ . We can also express this as follows: if  $\tilde{T}(A)$  denotes the dual of the space of affine-linear functions on  $A$ , so that  $A$  is naturally embedded in  $\tilde{T}(A)$  as an affine hyperplane and  $T(A)$  is embedded in  $\tilde{T}(A)$  as the linear hyperplane parallel to  $A$ , then the closure of the cone spanned by  $X$  meets  $T(A)$  in  $T(X)$ .

The following related notion is also useful.

**DEFINITION 1.1.** Let  $X$  be a subset of a finite-dimensional real affine space  $A$ . Then the *asymptotic space* of  $X$  is the intersection of all the linear subspaces  $W \subset T(A)$  for which the image of  $X$  in the affine quotient  $A/W$  is bounded. We denote this subspace  $\text{As}(X)$ .

In other words,  $\text{As}(X)$  is the smallest linear subspace  $W \subset T(A)$  with the property that the image of  $X$  in  $A/W$  is bounded. Notice that we can also characterize  $\text{As}(A)$  as the common zero set of the linear parts of the affine-linear  $\mathbb{R}$ -valued functions on  $A$  whose restriction to  $X$  is bounded.

It is clear that  $\text{As}(X)$  contains  $T(X)$ . But the latter need not span the former. For instance, if  $X$  is the solid parabola in  $\mathbb{R}^2$  defined by  $y > x^2$ , then  $T(X)$  is the nonnegative  $y$ -axis, whereas  $\text{As}(X) = \mathbb{R}^2$ .

We denote the convex hull of a subset  $Z$  of an affine space by  $[Z]$ .

In the remainder of this section,  $V$  is a finite-dimensional vector space and  $C$  an open nondegenerate convex cone in  $V$ . Then  $C^\circ$  is nondegenerate as well and we have  $(C^\circ)^\circ = C$ .

We first note that if  $F$  is a face of  $\bar{C}$ , then the elements of  $C^*$  that vanish on  $F$  make up a face  $F^\dagger$  of  $C^*$ . In fact, this is an exposed face of  $C^*$ : if  $x \in \hat{F}$ , then if  $\xi \in C^*$  (in other words,  $\xi|_C \geq 0$ ) is such that  $\xi(x) = 0$ , then we must have  $\xi|_F = 0$  and so  $\xi \in F^\dagger$ . This shows that  $F^\dagger$  is exposed by  $x$ .

Let us denote the annihilator of  $F^\dagger$  in  $V$  by  $V^F$ . So this is the common zero set of the  $\phi \in C^*$  which vanish on  $F$ . Clearly,  $F \subset V^F$ , and the projection  $\pi : V \rightarrow V/V^F$  is dual to the inclusion of the linear span of  $F^\dagger$  in  $V^*$ . Under this duality,  $\pi C$  can be identified with the open dual of  $F^\dagger$ . In particular,  $\pi C$  is an open nondegenerate convex cone in  $V/V^F$ . It may be characterized as the biggest projection of  $C$  that is a nondegenerate convex cone and has  $F$  in the preimage of its vertex.

In §2 we shall need the following lemma.

**LEMMA 1.2.** Let  $W \subset V$  be the asymptotic space of some subset of  $\bar{C}$ . Then for every nonempty compact  $B \subset C$ ,  $W$  is the asymptotic space of  $T(B, C) \cap W$ ; in other words, any  $\phi \in W^*$  which is bounded on  $T(B, C) \cap W$  must be zero.

The proof needs the following lemma.

**LEMMA 1.3.** Let  $F$  be a face of  $\bar{C}$  and  $q \in \hat{F}$ . Then for every compact  $B \subset C + V^F$  there exists a  $\lambda > 0$  such that  $B + \lambda q \in C$ .

*Proof.* We show that for every  $p \in C + V^F$  there exists a  $\lambda_p > 0$  such that  $p + \lambda_p q \in C$ . This suffices, for then  $p$  has a neighborhood  $B_p$  in  $V$  with the property that  $B_p + \lambda_p q \subset C$  and the lemma follows from the compactness of  $B$ . So suppose that for a given  $p \in C + V^F$  no such  $\lambda$  exists. Then  $(p + \mathbb{R}_{\geq 0}q) \cap C = \emptyset$ . Now notice that whenever  $p' \in (p + \mathbb{R}q) \cap \bar{C}$ , then  $p' + \mathbb{R}_{\geq 0}q \subset \bar{C} + \bar{C} = \bar{C}$ . So either  $(p + \mathbb{R}_{\geq 0}q) \cap \bar{C}$  is empty or is a ray contained in  $p + \mathbb{R}q$ . In both cases there exists by [Roc70, Theorem 11.6] a linear form  $\xi$  on  $V$  which is positive on  $C$  and

nonpositive on a ray in  $p + \mathbb{R}q$ . This implies that  $\xi(q) = 0$  (and hence  $\xi(p) \leq 0$ ). Since  $\xi|_C > 0$ , we have  $\xi|_{V^F} = 0$ , and hence  $\xi|_{C + V^F} > 0$ . But this contradicts the fact that  $p \in C + V^F$  and  $\xi(p) \leq 0$ .  $\square$

*Proof of Lemma 1.2.* We prove this with induction on  $\dim V$ . Choose a convex subset  $P \subset \bar{C}$  such that  $W = \text{As}(P)$ . We have  $T(P) \subset \bar{C}$ . Let  $F$  be the (unique) face of  $\bar{C}$  whose relative interior contains the relative interior of  $T(P)$ . If  $F = \{0\}$ , then  $T(P) = \{0\}$ . This implies that  $P$  is bounded and so  $W = V$  and the lemma clearly holds in that case. We therefore assume  $F \neq \{0\}$ . Denote the projection by  $\pi : V \rightarrow V/V^F$  and observe that  $\dim(V/V^F) < \dim V$ . Suppose  $\phi \in W^*$  is bounded on  $T(B, C) \cap W$ . We must show that  $\phi$  is zero.

First we prove that  $\phi$  factors through a linear form  $\phi' : \pi W \rightarrow \mathbb{R}$ . Choose  $x$  in the relative interior of  $T(P)$ . Then  $x \in \bar{C} \cap W$  and so  $\mathbb{R}_{\geq 0}x \subset T(B, C) \cap W$ . Since  $\phi$  is bounded on  $T(B, C) \cap W$ , it follows that  $\phi(x) = 0$ . Following Lemma 1.3 there exists for every  $u \in V^F \cap W$  a  $\lambda > 0$  such that  $\lambda x + u + B \subset C$ , i.e.,  $\lambda x + u \in T(B, C) \cap W$ . Since  $\phi(u) = \phi(\lambda x + u)$  it follows that  $\phi$  is bounded on the linear subspace  $V^F \cap W$ . So  $\phi|_{V^F \cap W} = 0$  and  $\phi$  factors through  $\pi W$ .

Observe that  $\pi W$  is the asymptotic subspace of a subset of  $\pi \bar{C}$ : a linear form is bounded on  $\pi P$  if and only if its pull-back to  $V$  is bounded on  $P$ . We claim that

$$\pi(T(B, C) \cap W) = T(\pi B, \pi C) \cap \pi W.$$

The inclusion  $\subset$  is clear, so let us prove  $\supset$ . Suppose  $w \in W$  is such that  $\pi(w) \in T(\pi B, \pi C)$ . This means that  $w + B \subset C + V^F$ . According to Lemma 1.3 there exists a  $u \in W \cap V^F$  such that  $u + w + B \subset C$  and so  $u + w \in T(B, C) \cap W$ . Since  $\pi(w) = \pi(u + w)$ , the claim follows.

Our induction hypothesis can now be applied to  $\pi C$ ,  $\pi W$  and  $\pi B$  and this enables us to conclude that  $\phi' = 0$ .  $\square$

**DEFINITION 1.4.** A *kernel* for  $C$  is a nonempty convex subset  $K$  of  $\bar{C}$  with  $0 \notin \bar{K}$  and  $K + C \subset K$ . (Note that our definition differs slightly from that of Ash [AMRT10], in that he does not insist that  $K$  be convex.) Two kernels  $K_1$  and  $K_2$  for  $C$  are said to be *comparable* if  $\lambda K_1 \subset K_2 \subset \lambda^{-1} K_1$  for some  $\lambda > 0$ .

It is clear that comparability is an equivalence relation.

For any subset  $K$  of  $V$  we let  $K^\vee$  denote the set of  $\xi \in V^*$  with  $\xi|_K \geq 1$ . If  $K$  is a kernel for  $C$ , then it is easy to see that  $K^\vee$  is a closed kernel for  $C^\circ$ . Furthermore, if  $K_1$  and  $K_2$  are comparable kernels, then so are  $K_1^\vee$  and  $K_2^\vee$ .

**LEMMA 1.5.** *If  $K$  is a kernel for  $C$ , then  $K^{\vee\vee} = \bar{K}$ .*

*Proof.* Although this is essentially [Roc70, II.5.2, Proposition 1], we give a proof for completeness. Let us prove the nontrivial inclusion  $K^{\vee\vee} \subset \bar{K}$ . If  $x \notin \bar{K}$ , then there exists by the separating hyperplane theorem [Roc70, Theorem 11.5] a  $\xi \in V^*$  with  $\xi(x) < \inf_K \xi$ . It is clear that then  $\inf \xi|_{\bar{C}} \geq 0$  (and hence  $\inf \xi|_K \geq 0$ ). If  $\inf \xi|_K > 0$ , then normalize  $\xi$  such that  $\inf \xi|_K = 1$ . Then  $\xi \in K^\vee$ , and since  $\xi(x) < 1$ , we have  $x \notin K^{\vee\vee}$ . Suppose now  $\inf \xi|_K = 0$  (so that  $\xi(x) < 0$ ). The preceding argument applied to  $x = 0$  yields a  $\xi_0 \in V^*$  with  $\inf \xi_0|_K = 1$ . Choose  $t > -\xi_0(x)/\xi(x)$ . Then  $(\xi_0 + t\xi)(x) < 0$  and  $\xi_0 + t\xi \in K^\vee$ . So in this case  $x \notin K^{\vee\vee}$  also.  $\square$

Closed kernels have certain technical advantages over arbitrary ones and for that reason the following lemma is quite useful.

**LEMMA 1.6.** *Let  $\Lambda$  be a discrete subset of  $\bar{C}$ . Then the convex set  $[\Lambda] + \bar{C}$  is closed in  $V$  and every extreme point of  $[\Lambda] + \bar{C}$  is exposed and belongs to  $\Lambda$ . Moreover, every face of  $[\Lambda] + \bar{C}$  is of the form  $[M] + F$ , where  $M \subset \Lambda$  and  $F$  is a face of  $\bar{C}$ .*

*Proof.* Let  $K$  denote the closure of  $[\Lambda] + \bar{C}$ . We first show that every exposed point  $p$  of  $K$  is in fact in  $\Lambda$ . By definition there exists a  $\xi \in V^*$  with  $\xi(p) < \xi|_{K - \{p\}}$ . Since  $p + \bar{C} \subset K$ , it follows that  $\xi$  is positive on  $\bar{C} - \{0\}$ . This means that  $\xi|_{\bar{C}}$  is proper. Since  $\Lambda$  is closed in  $\bar{C}$ , it also follows that  $\xi|_{\Lambda}$  has a minimum. This is then also the minimum of  $\xi|_K$ , and so we must have  $p \in \Lambda$ .

Following Straszewicz’s theorem [Roc70, Theorem 18.6], the exposed points of  $K$  are dense in the set of extreme points of  $K$ . But  $\Lambda$  is discrete and so every extreme point of  $K$  must be an exposed point of  $K$ . It now follows from [Roc70, Theorem 18.6] that  $K \subset [\Lambda] + \bar{C}$ . The last assertion is a consequence of [Roc70, Theorem 18.5].  $\square$

## 2. Convex cones and lattices

The main result of this section is Theorem 2.2 below, which may not strike the reader as surprising. Nevertheless we shall see that it has interesting consequences, such as the *Siegel property*, Theorem 3.8.

Before we can state the result alluded to above, we recall, or introduce, some terminology pertaining to polyhedra.

DEFINITION 2.1. A *polyhedron* in a real finite-dimensional affine space is a subset of that space, which can be defined by finitely many affine-linear nonstrict inequalities (so of the form  $f \leq 0$ ). If the affine space has a  $\mathbb{Q}$ -structure and these affine-linear forms are definable over  $\mathbb{Q}$ , then we call this a *rational polyhedron*. A subset of the affine space is said to be (*rationally*) *locally polyhedral* if its intersection with every bounded (rational) polyhedron is a (rational) polyhedron.

We note here that every *bounded* polyhedron is the convex hull of a finite set (in some affine space), and is rational if we can take that subset to consist of rational points.

THEOREM 2.2. *Let  $V$  be a real finite-dimensional vector space,  $C \subset V$  an open nondegenerate convex cone and  $L \subset V$  a lattice. Then  $[C \cap L]$  is locally rationally polyhedral in  $V$  (relative to the  $\mathbb{Q}$ -structure on  $V$  defined by  $L$ ). In particular,  $[C \cap L]$  is closed in  $V$ .*

It will be convenient to prove a few preparatory results first.

LEMMA 2.3. *Let  $p_0 \in L$ ,  $B$  be a neighborhood of  $p_0$  in  $V$  and  $R$  a relatively open half line in  $V$  which is not contained in a proper linear subspace of  $V$  defined over  $\mathbb{Q}$ . Then  $[(B + R) \cap L]$  is a neighborhood of  $p_0 + R$  in  $V$ .*

*Proof.* Without loss of generality we may and will assume that  $p_0 = 0$ . It is a well-known fact that the image of a line in  $V$  not contained in a proper linear  $\mathbb{Q}$ -subspace of  $V$  has dense image in  $V/L$ . The same is true for a ray in such a line, such as  $R$ .

Let  $x \in R$ . We show that  $[(B + R) \cap L]$  is a neighborhood of  $x$  in  $V$ , more precisely, we will find  $p_1, \dots, p_N \in (B + R) \cap L$  such that  $x$  is in the interior of the convex hull of  $p_0 = 0, p_1, \dots, p_N$ . For this we choose nonempty open subsets  $U_1, \dots, U_N$  of  $B + R$  such that for every  $(u_1, \dots, u_N) \in U_1 \times \dots \times U_N$ ,  $x$  is in the interior of the convex hull of  $u_1, \dots, u_N$ . Since  $\rho(U_i + R) = V/L$ , there exist  $u_i \in U_i$  and  $t_i > 0$  with  $p_i := u_i + t_i x \in (U_i + R) \cap L$ ,  $i = 1, \dots, N$ . We can write  $x$  as a strictly convex linear combination of  $u_1, \dots, u_N$ :  $x = \sum_i \lambda_i u_i$  with all  $\lambda_i > 0$ ,  $\sum_i \lambda_i = 1$ . Then  $\sum_i \lambda_i p_i = (1 + \sum_i \lambda_i t_i)x$ , and so we have  $x = \sum_{j=0}^N \mu_j p_j$  with  $\mu_j = \lambda_j (1 + \sum_i \lambda_i t_i)^{-1}$  when  $j > 0$  and  $\mu_0 = (\sum_i \lambda_i t_i) (1 + \sum_i \lambda_i t_i)^{-1}$ . Hence  $p_1, \dots, p_N$  are as desired.  $\square$

In what follows we denote by  $\rho : V \rightarrow V/L$  the obvious map.



LEMMA 2.4. *Let  $W$  be a subspace of  $V$  defined over  $\mathbb{Q}$  and  $Y \subset W$  a convex subset such that  $W$  is the smallest subspace of  $V$  defined over  $\mathbb{Q}$  that contains  $\text{As}(Y)$ . Then the closure of  $\rho(Y)$  equals the subtorus  $\rho(W) = W/L \cap W$  of  $V/L$ .*

*Proof.* If  $Y$  is bounded, then  $\text{As}(Y) = 0$  and there is nothing to show. We therefore assume  $Y$  unbounded. Without loss of generality we may assume that  $Y$  is relatively open so that  $T(Y) \neq 0$ . Choose  $v_1$  in the relative interior of  $T(Y)$ . Then for every  $y \in Y$ ,  $\rho(y + \mathbb{R}_{\geq 0}v)$  is dense in  $\rho(y + \mathbb{R}v)$  and so  $\rho(Y)$  is dense in the image of  $Y_1 := Y + \mathbb{R}v_1$ . If  $\text{As}(Y) \neq \mathbb{R}v_1$ , we proceed in this manner and we eventually find that  $\rho(Y)$  is dense in  $\rho(Y + \text{As}(Y))$ . Now  $\rho(\text{As}(Y))$  is a connected subgroup of  $V/L$  and it is well known that the closure of such a group must be a subtorus. In the present case, this must be  $W/L \cap W$ . Since  $Y \subset W$ , it then also follows that  $\rho(Y + \text{As}(Y))$  is dense in  $\rho(W)$ .  $\square$

*Proof of Theorem 2.2.* We prove the theorem with induction on  $\dim V$ . Since there is nothing to show when  $V = \{0\}$ , we assume that  $\dim V > 0$ . The convex set  $[C \cap L] + \bar{C}$  is closed in  $V$  by Lemma 1.6 and contained in the open cone  $C$  (since  $C + \bar{C} \subset C$ ). Let  $P$  be a face of  $[C \cap L] + \bar{C}$ .

Step 1 is to prove that  $P$  is also a face of  $[C \cap L]$ . In particular, the affine span of  $P$  is defined over  $\mathbb{Q}$ . The proof proceeds as follows.

According to Lemma 1.6 (applied to  $\Lambda := L \cap C$ ), we have  $P = [P \cap L] + F$ , with  $F$  a face of  $\bar{C}$  and  $P \cap L \subset C$ . So it is enough to prove that for every  $p \in P \cap L$  and every ray  $R$  in  $F$ ,  $p + R \subset [C \cap L]$ . Denote by  $V_R$  the smallest subspace of  $V$  defined over  $\mathbb{Q}$  which contains  $R$  and let  $B$  be a convex neighborhood of 0 in  $V_R$  such that  $p + B \subset C$ . According to Lemma 2.3,  $[(p + B + R) \cap L]$  contains a neighborhood of  $p + R$  in  $V_R$ . Since  $p + B + R \subset C$  it follows that  $p + R \subset [C \cap L]$ .

Note that if we apply Step 1 to  $P = [C \cap L] + \bar{C}$ , we find that  $[C \cap L] = [C \cap L] + \bar{C}$  is closed in  $V$ .

We write  $W$  for the smallest subspace of  $V$  defined over  $\mathbb{Q}$  that contains  $\text{As}(P)$  and  $\pi : V \rightarrow V/W$  for the projection. So  $\pi P$  is bounded and  $\pi L$  is a lattice in  $V/W$ .

Step 2 is to show that if  $Q$  is face of  $[C \cap L]$  which contains  $P$ , then  $Q = \pi^{-1}\pi Q \cap [C \cap L]$  and hence  $\pi Q$  is a face of  $\pi[C \cap L]$ .

Clearly, the translation space of the affine span  $\text{Aff}(Q)$  of  $Q$  contains  $\text{As}(P)$ . Since the former is defined over  $\mathbb{Q}$ , it contains  $W$  as well. This implies that  $\pi^{-1}\pi Q \cap [C \cap L]$  equals  $\text{Aff}(Q) \cap [C \cap L]$ . Since  $Q$  is closed, the latter is just  $Q$ . This implies that  $\pi Q$  is a face of  $\pi[C \cap L]$  indeed: any segment in  $\pi[C \cap L]$  is of the form  $\pi\sigma$  for some segment  $\sigma$  in  $[C \cap L]$ , and if  $\pi\sigma$  meets  $\pi Q$ , then  $\sigma$  meets  $\pi^{-1}\pi Q \cap [C \cap L] = Q$  and we have  $\pi\sigma \cap \pi Q = \pi(\sigma \cap Q)$ . So the latter is either  $\pi\sigma$  or an end point of  $\pi\sigma$ .

Step 3 is to prove that  $[\pi C \cap \pi L] = \pi[C \cap L]$ .

We prove the nontrivial inclusion  $\subset$ , i.e., we show that if  $p \in C$  is such that  $(p + W) \cap L$  is nonempty, then  $(p + W) \cap C \cap L$  is nonempty.

Choose a compact convex neighborhood  $B$  of  $p$  in  $(p + W) \cap C$ . By Lemma 1.2,  $T(B, C) \cap \text{As}(P)$  and  $P$  have the same asymptotic space. Since  $T(B, C) \cap \text{As}(P)$  is convex, Lemma 2.4 applies here and tells us that the closure of  $\rho(T(B, C) \cap \text{As}(P))$  is  $\rho W$ .

Now  $B$  contains a nonempty open subset of  $p + W$  and hence of  $p + W + L = W + L$ . The same is true of  $-B$  and so  $\rho(-B)$  meets  $\rho(T(B, C) \cap \text{As}(P))$ . In other words, there exist  $b \in B$  and  $v' \in T(B, C) \cap \text{As}(P)$  such that  $b + v' \in L$ . It is clear that we also have  $b + v' \in C \cap (p + W)$ .

Step 4 is to show that  $\pi C$  is nondegenerate.

Suppose that it is not; then  $T(\pi C)$  contains a line  $l \subset V/W$ . Choose  $p \in P \cap L$ . It follows from Lemma 2.3 (applied to the two rays in  $\pi(p) + l$  emanating from  $\pi(p)$ ) that  $\pi(p) + l$  is in

the convex hull of  $\pi C \cap \pi L$ . By step 3, this convex hull is just  $\pi[C \cap L]$ . Let  $y \in l - \{0\}$ . Since  $\pi P$  is bounded, there exists a  $\mu > 0$  such that  $\pi(p) \pm \mu y \notin \pi P$ . Let  $p_{\pm} \in [C \cap L]$  be such that  $\pi(p_{\pm}) = \pi(p) \pm \mu y$ . Then  $\frac{1}{2}p_- + \frac{1}{2}p_+ \in \pi^{-1}\pi(p) \cap [C \cap L] = P$  by step 2, although  $p_{\pm} \notin P$ . This contradicts the fact that  $P$  is a face of  $[C \cap L]$ .

Step 5 is to prove that if  $P$  is unbounded, then  $\text{Star}_{[C \cap L]}(P)$  has only finitely many members and  $|\text{Star}(P)_{[C \cap L]}|$  is a neighborhood of  $\dot{P}$  in  $[C \cap L]$ .

Since  $P$  is unbounded,  $W \neq \{0\}$ , and so  $\dim V/W < \dim V$ . Step 4 enables us to apply our induction hypothesis to  $\pi C$  and  $\pi L$ . Since  $[\pi C \cap \pi L] = \pi[C \cap L]$  (step 3), we find that  $\pi[C \cap L]$  is a locally rationally polyhedral subset of  $V/W$ . So  $\text{Star}_{\pi[C \cap L]}(\pi P)$  is finite. This implies that  $|\text{Star}_{\pi[C \cap L]}(\pi P)|$  is a neighborhood of the relative interior of  $\pi(P)$  in  $\pi([C \cap L])$  and that each member of  $\text{Star}_{\pi[C \cap L]}(\pi P)$  is obtained as in step 2, i.e., is the image of the face of  $[C \cap L]$  which contains  $P$  and which is also a face of  $[C \cap L] + \bar{C}$ . This implies the corresponding property for  $P$  with respect to  $[C \cap L]$ .

Step 6 is the conclusion.

We show that any bounded polyhedron  $\Pi$  in  $V$  meets only finitely many faces of  $[C \cap L]$ . Suppose that, on the contrary, there exists a sequence  $P_1, P_2, \dots$  of pairwise distinct faces of  $[C \cap L]$  with  $P_i \cap \Pi \neq \emptyset$ . Clearly,  $\bigcup_{i=1}^{\infty} P_i$  cannot be bounded: otherwise  $(\bigcup_{i=1}^{\infty} P_i) \cap L$  would be finite, and as each  $P_i$  is the convex hull of its intersection with  $L$ , only a finite number of the  $P_i$  could be distinct. This property is of course also true for the union over any subsequence of  $(P_i)_i$ . So, perhaps after passing to a subsequence, we can find sequences  $\{p_i \in P_i \cap \Pi\}_{i=1}^{\infty}$  converging to some  $p_{\infty} \in \Pi$  and  $\{q_i \in \dot{P}_i\}_{i=1}^{\infty}$  such that the intervals  $[p_i, q_i]$  converge to a closed half line emanating from  $p_{\infty}$ . Let  $P_{\infty}$  denote the face of  $[C \cap L]$  whose relative interior contains the relative interior of this half line. As  $P_{\infty}$  is unbounded,  $|\text{Star}_{[C \cap L]}(P_{\infty})|$  is by step 5 a neighborhood of  $\dot{P}_{\infty}$ , and so for  $i$  sufficiently large,  $q_i \in |\text{Star}(P_{\infty})|$  and hence  $P_i \supset P_{\infty}$ . According to step 5 only finitely many faces of  $[C \cap L]$  have that property, and so we get a contradiction.  $\square$

*Remark 2.5.* If it were true that a nondegenerate convex cone which is the linear projection of a closed nondegenerate cone is also closed, then the above proof could be simplified. That is however not so, as the following example due to Burt Totaro shows. The function  $x \in (-1, 1) \mapsto (1 - x^2)^{-1}$  is convex and so the open subset  $D \subset \mathbb{R}^2$  defined by  $y > 0$  and  $y(1 - x^2) > 1$  is convex as well. Note that if we put  $D$  in real projective space, then  $\partial D$  is smooth except for its unique point at infinity: there it has two distinct tangent lines (given by  $x = \pm 1$ ) which do not meet  $\bar{D}$  outside that point. We now take for  $C \subset \mathbb{R}^3$  the cone over  $\bar{D}$  and project along the axis defined by its point at infinity. So  $C$  is given by the inequalities  $|x| \leq z, y \geq 0, y(z^2 - x^2) \geq z^3$ , and we project  $C$  under  $(x, y, z) \mapsto (x, z)$ . Then the image of  $C$  is the union of the origin  $(0, 0)$  and the open cone defined by  $|x| < z$ . This is a nondegenerate convex cone, but it is not closed.

### 3. The Siegel property

In this section  $V$  is a real finite-dimensional vector space with a  $\mathbb{Q}$ -structure  $V(\mathbb{Q})$ . A lattice in  $V$  is always understood to be compatible with this  $\mathbb{Q}$ -structure, in other words, must be a subgroup of  $V(\mathbb{Q})$  of rank equal to  $\dim V$ .

We also fix an open convex nondegenerate cone  $C \subset V$ . The convex hull of  $\bar{C} \cap V(\mathbb{Q})$  clearly contains  $C$ ; we denote it by  $C_+$ . Similarly we have  $C_+^{\circ} \supset C^{\circ}$  in  $V^*$ .

We say that a subset  $K \subset V$  is *locally rationally polyhedral in  $C_+$*  if for every rationally polyhedral cone  $\Pi$  in  $C_+$ ,  $\Pi \cap K$  is a rational polyhedron (note that we do not require that  $K \subset C_+$ ).



PROPOSITION 3.1. For every lattice  $L$ ,  $[C \cap L]$  is a kernel for  $C$  and all such kernels belong to the same comparability class.

*Proof.* It is easy to see that  $[C \cap L]$  is a kernel. If  $L'$  is another lattice, then there exists a positive integer  $k$  such that  $kL \subset L' \subset (1/k)L$ . So  $k[C \cap L] \subset [C \cap L'] \subset (1/k)[C \cap L]$ .  $\square$

DEFINITION 3.2. A kernel for  $C$  is called a *core* if it is comparable with the convex hull of the intersection of  $C$  with some lattice in  $V$ . It is called a *cocore* if its dual is a core for  $C^\circ$ . Notice that given a lattice  $L$  in  $V$ , then  $[C^\circ \cap L^*]^\vee$  is a cocore which contains the core  $[C \cap L]$ .

DEFINITION 3.3. A collection  $\Sigma$  of convex cones in  $C_+$  is said to be a *locally rationally polyhedral decomposition* of  $C_+$  if the following conditions are fulfilled:

- (i) the relative interiors of the members of  $\Sigma$  are pairwise disjoint and their union is  $C_+$ ;
- (ii)  $\Sigma$  is closed under intersections and taking faces;
- (iii) if  $\Pi$  is a rationally polyhedral cone in  $C_+$ , then  $\Sigma|_\Pi := \{\sigma \cap \Pi\}_{\sigma \in \Sigma}$  is a finite collection of rationally polyhedral cones.

If moreover every  $\sigma \in \Sigma$  is a rationally polyhedral cone, then we omit ‘locally’, and call  $\Sigma$  a *rationally polyhedral decomposition* of  $C_+$ .

Remark 3.4. The collection of faces of  $C_+$  is obviously a locally rationally polyhedral decomposition of  $C_+$ . It is in fact the coarsest as it is refined by any other locally rationally polyhedral decomposition of  $C_+$ .

PROPOSITION 3.5. Let  $L \subset V(\mathbb{Q})$  be a lattice. For any face  $P$  of  $[C^\circ \cap L^*]$ , let  $\sigma(P)$  be the set of  $x \in V$  such that  $\xi \in [C^\circ \cap L^*] \mapsto \xi(x)$  assumes its infimum on all of  $P$ . Then  $\sigma(P)$  is a rationally polyhedral cone of dimension equal to  $\text{codim } P$ ,  $P \mapsto \sigma(P)$  is an injection which reverses inclusions, and  $\Sigma(C, L) := \{\sigma(P)\}_P$  is a rationally polyhedral decomposition of  $C_+$ .

Before we begin the proof we show the following proposition.

PROPOSITION 3.6. If  $L$  is a lattice in  $V(\mathbb{Q})$ , then  $[C^\circ \cap L^*]^\vee$  is a locally rationally polyhedral cocore for  $C$ .

This will be a consequence of the following result (which we shall later need in this general form for the proof of Proposition 4.11).

LEMMA 3.7. Let  $A$  be a real finite-dimensional affine space,  $P$  a polyhedron in  $A$ , and  $\Phi$  a collection of affine-linear functions on  $A$  that are  $\geq 0$  on  $P$  and such that for every  $p \in P$  and every  $t \in T(P)$ , the sets  $\{\phi(p)\}_{\phi \in \Phi}$  and  $\{d\phi(t)\}_{\phi \in \Phi}$  are discrete. For any finite subset  $S \subset \Phi$ , denote by  $P_S$  the set of  $p \in P$  such that all  $\phi \in S$  assume in  $p$  the same value and no member of  $\Phi$  takes in  $p$  a smaller value. Then  $\{P_S\}_{S \subset \Phi \text{ finite}}$  is a finite polyhedral decomposition of  $P$  and  $P_{\Phi \geq 1} := \{p \in P \mid \phi(p) \geq 1 \text{ for all } \phi \in \Phi\}$  is a polyhedron.

If  $A$  is endowed with a  $\mathbb{Q}$ -structure and, relative to this structure,  $P$  is a rational polyhedron and the members of  $\Phi$  are defined over  $\mathbb{Q}$ , then the  $P_S$  and  $P_{\Phi \geq 1}$  are rational polyhedra.

*Proof.* Let  $p_1, \dots, p_k$  enumerate the (finite) set of extreme points of  $P$ , and let  $t_{k+1}, \dots, t_l \in T(P)$  generate  $T(P)$  as a cone. Consider the subset of  $\mathbb{R}^l$  defined by

$$\Xi := \{(\phi(p_1), \dots, \phi(p_k), d\phi(t_{k+1}), \dots, d\phi(t_l)) \mid \phi \in \Phi\}.$$

By assumption the projection of  $\Xi$  on every coordinate is discrete and contained in  $\mathbb{R}_{\geq 0}$ . An inductive argument shows that there exists a finite subset  $\Xi_0$  of  $\Xi$  such that  $\Xi \subseteq \Xi_0 + \mathbb{R}_{\geq 0}^l$ . So if

$\Phi_0$  is a finite subset of  $\Phi$  which maps onto  $\Xi_0$ , then, for every  $\phi \in \Phi$  there exists a  $\phi_0 \in \Phi_0$  such that for all  $i, j$ ,  $\phi(p_i) \geq \phi_0(p_i)$  and  $d\phi(t_j) \geq d\phi_0(t_j)$ . In other words  $\phi \geq \phi_0$  on  $P$ . Hence every nonempty  $P_S$  is obtained by taking  $S \subset \Phi_0$ , and all such cover  $P$ . The set of  $p \in P$  with  $\phi(p) \geq 1$  for all  $\phi \in \Phi$  is already defined by restricting the index set to  $\Phi_0$  and is therefore a polyhedron.

The  $\mathbb{Q}$ -version is then straightforward. □

*Proof of Proposition 3.6.* Let  $\Pi \subset C_+$  be a rationally polyhedral cone in  $C_+$ . We must show that  $\Pi \cap [C^\circ \cap L^*]^\vee$ , i.e., the locus of  $p \in \Pi$  with  $\xi(p) \geq 1$  for all  $\xi \in C^\circ \cap L^*$ , is rationally polyhedral. It is enough to show that (the  $\mathbb{Q}$ -version of) Lemma 3.7 applies here with  $P = \Pi$  and  $\Phi = C^\circ \cap L^*$ . If  $p \in C \cap L$ , then clearly the set of  $\{\xi(q)\}_{\xi \in C^\circ \cap L}$  is a set of positive integers. So if  $p$  is a convex linear combination of such  $q$ , then  $\{\xi(p)\}_{\xi \in C^\circ \cap L}$  is in a semigroup of  $\mathbb{R}_{\geq 0}$  which is finitely generated as such. Hence it is discrete as a subset and bounded from below. □

*Proof of Proposition 3.5.* We only prove the last part of the statement, for everything else follows in a straightforward manner from Theorem 2.2. For  $p \in \bar{C} \cap L$ , the set  $\{\xi(p)\}_{\xi \in C^\circ \cap L^*}$  obviously consists of positive integers. So for  $x \in [\bar{C} \cap L] = C_+$ ,  $\{\xi(x)\}_{\xi \in C^\circ \cap L^*}$  is still a discrete subset of  $\mathbb{R}_{\geq 0}$ . This implies that the function  $\xi \in [C^\circ \cap L^*] \mapsto \xi(x)$  has a minimum, so that  $x \in \sigma(P)$  for some face  $P$  of  $[C^\circ \cap L^*]$ . This proves property (i) of Definition 3.3. Property (ii) is easy. As for (iii), let  $\Pi$  be a rationally polyhedral cone in  $C_+$ . Then  $\Pi \cap [C^\circ \cap L^*]^\vee$  is a rational polyhedron by Proposition 3.6. Since  $\sigma(P) \cap \Pi$  is the cone over a face of  $\Pi \cap [C^\circ \cap L^*]^\vee$  or reduced to the origin, the collection  $\{\sigma(P) \cap \Pi\}_P$  is finite and consists of rationally polyhedral cones. □

Here is an interesting application.

**THEOREM 3.8 (Siegel property).** *Let  $\Gamma$  be a subgroup of  $GL(V)$  which leaves  $C$  and a lattice in  $V(\mathbb{Q})$  invariant. Then  $\Gamma$  has the Siegel property in  $C_+$ : if  $\Pi_1$  and  $\Pi_2$  are polyhedral cones in  $C_+$ , then the collection  $\{\gamma\Pi_1 \cap \Pi_2\}_{\gamma \in \Gamma}$  is finite. Moreover, if  $F_i$  denotes the face of  $C_+$  whose relative interior contains the relative interior of  $\Pi_i$ , then the set of  $\gamma \in \Gamma$  with  $\gamma\overset{\circ}{\Pi}_1 \cap \overset{\circ}{\Pi}_2 \neq \emptyset$  is a finite union of right  $Z_\Gamma(F_1)$ -cosets (hence also a finite union of left  $Z_\Gamma(F_2)$ -cosets). In particular,  $\Gamma$  acts properly discontinuously on  $C$ .*

*Proof.* The first assertion follows from the second if we apply it to the relative interiors of the (finitely many) faces of  $\Pi_1$  and  $\Pi_2$ . In order to prove the second assertion, we first observe that the set of  $\gamma \in \Gamma$  with  $\gamma\overset{\circ}{\Pi}_1 \cap \overset{\circ}{\Pi}_2 \neq \emptyset$  is indeed a union of right  $Z_\Gamma(F_1)$ -cosets (and also of left  $Z_\Gamma(F_2)$ -cosets). Let  $L \subset V(\mathbb{Q})$  be a  $\Gamma$ -invariant lattice. Since  $\overset{\circ}{\Pi}_i$  is covered by finitely many members of  $\Sigma(C, L)$  whose relative interiors are contained in  $F_i$ , we may assume that  $\Pi_i \in \Sigma(C, L)$ ,  $i = 1, 2$ . If now  $\gamma\overset{\circ}{\Pi}_1 \cap \overset{\circ}{\Pi}_2 \neq \emptyset$ , then we must have  $\gamma\Pi_1 = \Pi_2$ . So we only need to show that  $G_i := N_\Gamma(\Pi_i)/Z_\Gamma(F_i)$  is finite. Since  $\Pi_i$  meets the relative interior of  $F_i$ , the collection  $\text{Star}_{\Sigma(C, L)}(\Pi)|_{F_i}$  of members of  $\Sigma(C, L)|_{F_i}$  containing  $\Pi_i$  must be finite. This is clearly acted on by  $G_i$ . It then suffices to see that some member  $\Pi'_i$  of  $\text{Star}_{\Sigma(C, L)}(\Pi)|_{F_i}$  has finite  $G_i$ -stabilizer. This is clear if we take  $\Pi'_i$  to be maximal, i.e., with the property that its relative interior is open in  $F_i$ , for then this stabilizer will act faithfully on  $\Pi'_i$  and hence (since  $\Pi'_i$  is a rationally polyhedral cone) must be finite. □

### 4. Pairs of polyhedral type

In this section  $V$  continues to denote a real finite-dimensional vector space equipped with a rational structure  $V(\mathbb{Q}) \subset V$  and  $C$  is an open nondegenerate convex cone in  $V$ .

PROPOSITION–DEFINITION 4.1. *Let  $\Gamma$  be a subgroup of  $GL(V)$  which stabilizes  $C$  and some lattice in  $V(\mathbb{Q})$ . Then the following conditions are equivalent:*

- (i) *there exists a polyhedral cone  $\Pi$  in  $C_+$  with  $\Gamma \cdot \Pi = C_+$ ;*
- (ii) *there exists a polyhedral cone  $\Pi$  in  $C_+$  with  $\Gamma \cdot \Pi \supset C$ ;*
- (iii) *for every  $\Gamma$ -invariant lattice  $L \subset V(\mathbb{Q})$ ,  $\Gamma$  has finitely many orbits in the set of extreme points of  $[C \cap L]$ ;*
- (iv) *for some  $\Gamma$ -invariant lattice  $L \subset V(\mathbb{Q})$ ,  $\Gamma$  has finitely many orbits in the set of extreme points of  $[C \cap L]$ ;*
- (i)<sup>\*</sup>–(iv)<sup>\*</sup> *the corresponding property for the contragradient action of  $\Gamma$  on  $C^\circ$ .*

Moreover, in case (ii) we necessarily have  $\Gamma \cdot \Pi = C_+$ . If one of these equivalent conditions is fulfilled, we say that  $(V(\mathbb{Q}), C, \Gamma)$  is a polyhedral triple, or simply that  $(C_+, \Gamma)$  is of polyhedral type.

*Proof.* The implications (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (iv) are obvious.

We prove (ii)  $\Rightarrow$  (iii). Let  $\Pi$  be as in (ii). Without loss of generality we may assume that  $\Pi$  is rationally polyhedral. Let  $S$  denote the set of extreme points of  $[C \cap L]$ . Then we must show that  $S \cap \Pi$  is finite. Every extreme point of  $[C \cap L]$  is in  $C \cap L$  and hence  $S \cap \Pi \subset C \cap L \cap \Pi$ . Let  $v_1, \dots, v_r$  denote the set of primitive integral generators of the extremal rays of  $\Pi$ . Any  $e \in S \cap \Pi$  has the property that  $e - v_i \notin C \cap \Pi$  for all  $i$ . This implies that if we write  $e = \sum_{i=1}^r \lambda_i v_i$  with  $\lambda_i \geq 0$ , then  $\lambda_i \leq 1$  for all  $i$ . So  $S \cap \Pi$  is contained in a compact set (a continuous image of  $[0, 1]^r$ ) and hence finite.

Proof of (iv)  $\Rightarrow$  (i)<sup>\*</sup>. In Proposition 3.5 we have defined a rationally polyhedral decomposition  $\Sigma := \Sigma(C^\circ, L^*)$  of  $C_+^\circ$ . This decomposition is  $\Gamma$ -invariant, and the correspondence  $P \mapsto \sigma(P)$  between faces of  $[C \cap L]$  and members of  $\Sigma$  is equivariant. Now extreme points of  $[C \cap L]$  correspond to maximal members of  $\Sigma$ . So if  $S$  is a system of  $\Gamma$ -representatives in the collection of extreme points of  $[C \cap L]$ , then  $\sum_{e \in S} \sigma(\{e\})$  is a rationally polyhedral cone in  $C_+^\circ$  whose  $\Gamma$ -orbit equals  $C_+^\circ$ .

These implications, together with their dual forms, prove the equivalence of (i) through (iv)<sup>\*</sup>. As for the last assertion, we choose a rationally polyhedral cone  $\Pi_1 \subset C_+$  such that  $\Gamma \cdot \Pi_1 = C_+$  (which exists in view of (ii)  $\Rightarrow$  (i)) and prove that  $\Gamma \cdot \Pi \supset \Pi_1$ . By the Siegel property, Theorem 3.8, the collection  $\{\gamma(\Pi) \cap \Pi_1 \mid \gamma \in \Gamma\}$  has only finitely many distinct members, so  $(\Gamma \cdot \Pi) \cap \Pi_1$  is closed. Since  $\Gamma \cdot \Pi \supset C$  and the latter contains the interior of  $\Pi_1$ , it follows that  $\Gamma \cdot \Pi \supset \Pi_1$ . □

An important class of examples is singled out by the proposition below, which is essentially due to Ash [AMRT10].

PROPOSITION 4.2. *Let  $\mathcal{G}$  be a reductive  $\mathbb{Q}$ -algebraic subgroup of the general linear group of  $V$ . Assume that  $C$  is an orbit of the identity component  $G$  of  $\mathcal{G}(\mathbb{R})$ . Then  $\Gamma := G \cap GL(L)$  is an arithmetic group in  $\mathcal{G}$  and  $(C_+, \Gamma)$  is of polyhedral type.*

*Sketch of proof.* The  $G$ -stabilizer of a point of  $C$  is a maximal compact subgroup of  $G$  so that  $C$  is in fact the symmetric space of  $G$ . The reduction theory for arithmetic groups shows that a fundamental domain for the arithmetic group  $\Gamma$  acting in  $C$  is contained in a finite union of so-called Siegel sets in  $C$  (see for instance [AMRT10] for the definition). Hence it suffices to show that a Siegel set is contained in a rational polyhedral cone in  $C$ . This is however the easy part of the proof of [AMRT10, ch. II, Theorem 4.1]. □

*Example 4.3.* Another interesting class of examples not contained in the one above arises in the theory of Coxeter groups. Let  $(n_{ij})$  be a nonsingular, integral  $l \times l$  generalized Cartan matrix [Loo80] without components of finite type. Let  $W \subset GL_l(\mathbb{Z})$  be the (Weyl) group generated by the reflections  $s_i : x \mapsto x - \sum_j n_{ij}x_j$  and let  $I$  denote the  $W$ -orbit of the fundamental chamber  $\Pi \subset \mathbb{R}^l$  defined by  $x_i \geq 0, i = 1, \dots, l$ . It is known that  $I$  is nondegenerate convex [Loo80], and so  $(I, W)$  is of polyhedral type. The dual construction (for the contragradient action of  $W$  on  $(\mathbb{R}^n)^*$ ) yields a nondegenerate convex cone  $\check{I} \subset (\mathbb{R}^n)^*$  which with  $W$  also forms a pair of polyhedral type. But its closure is in general *not* the dual of  $I$ .

Somewhat more general situations, which have been investigated by Vinberg [Vin72a], also give examples of polyhedral triples.

*Example 4.4.* Algebraic geometry can provide interesting and highly nontrivial examples of polyhedral triples. If  $X$  is a complex compact manifold, then take for  $V$  the Néron–Severi group of  $X$  tensored with  $\mathbb{R}$ , for  $C$  the cone in  $V$  spanned by the ample classes (we assume this set to be nonempty) and for  $\Gamma$  the image  $\text{Aut}(X)$  in  $GL(V)$ . It is known that  $(C, V, \Gamma)$  is a polyhedral triple for many surfaces, among them K3 surfaces (Sterk [Ste85]) and Enriques surfaces (Namikawa [Nam85]). David Morrison’s cone conjecture [Mor93] asserts that this should also hold for the Kähler cone of a Calabi–Yau manifold  $X$  with  $h^{2,0}(X) = 0$ . This too has been verified in some cases. See also [Tat10].

*Question 4.5.* Given a pair of polyhedral type  $(C_+, \Gamma)$ , do  $\Gamma$  and the cone generated by the  $\Gamma$ -orbit of a rational point of  $C_+ - \{0\}$  form a pair of polyhedral type?

There is in general no subgroup  $\Gamma$  of  $GL(V(\mathbb{Q}))$  which forms with  $C_+$  a pair of polyhedral type. But if there is one, then the next result says that all such subgroups belong to a single commensurability class. (Recall that two subgroups of some group are said to be *commensurable* if their intersection is of finite index in each of them, and that this is an equivalence relation.)

**PROPOSITION 4.6.** *Let  $(C_+, \Gamma)$  be a pair of polyhedral type, and let  $\Gamma'$  be a subgroup of  $GL(V(\mathbb{Q}))$  which stabilizes  $C$ . Then  $(C_+, \Gamma')$  is of polyhedral type if and only if  $\Gamma'$  is commensurable with  $\Gamma$ .*

*Proof. ‘If’:* Let  $L \subset V(\mathbb{Q})$  be a lattice stabilized by  $\Gamma$ . If  $\Gamma'$  contains  $\Gamma$  as a subgroup of finite index, then  $\Gamma' \cdot L$  is contained in a finite union of lattices, and hence generates a lattice  $L' \subset V(\mathbb{Q})$ . Clearly,  $\Gamma'$  stabilizes  $L'$ . It then follows from Definition 4.1(i), that  $(C_+, \Gamma')$  is of polyhedral type. If on the other hand  $\Gamma'$  is a subgroup of finite index of  $\Gamma$ , then choose a finite system  $S \subset \Gamma$  of representatives of left cosets of  $\Gamma'$  in  $\Gamma$ . If  $\Pi$  is a rationally polyhedral cone in  $C_+$  such that  $\Gamma \cdot \Pi = C_+$ , then  $\Pi' = \sum_{s \in S} s(\Pi)$  is a rationally polyhedral cone satisfying  $\Gamma' \cdot \Pi' = C_+$ , and so  $(C_+, \Gamma)$  is in this case of polyhedral type, too.

*‘Only if’:* Assume that  $(C_+, \Gamma)$  is of polyhedral type. If  $L' \subset V(\mathbb{Q})$  is a lattice stabilized by  $\Gamma'$ , then  $L' \supset kL$ , for some  $k \in \mathbb{N}$ , and so  $L \supset L' \cap L \supset kL$ . Since  $L/kL$  is finite, the group of  $\gamma \in \Gamma$  stabilizing  $L' \cap L$  is of finite index in  $\Gamma$ , and hence its action on  $C_+$  is of polyhedral type. A similar assertion holds for the group of  $\gamma' \in \Gamma'$  which stabilize  $L \cap L'$ . So without loss of generality we can assume that  $\Gamma$  and  $\Gamma'$  both stabilize a lattice  $L \subset V(\mathbb{Q})$ . It is enough to prove that the group  $\Gamma''$  of  $\gamma \in GL(V)$  which leave both  $L$  and  $C$  invariant contains  $\Gamma$  and  $\Gamma'$  as subgroups of finite index. Let  $\Pi$  be a rationally polyhedral cone in  $C_+$  such that  $\Gamma \cdot \Pi \supset C$ , and let  $S$  denote the set of  $\gamma'' \in \Gamma''$  with  $\gamma''(\Pi) \cap \Pi \cap C \neq \emptyset$ . By the Siegel property, Theorem 3.8,  $S$  is finite. For every  $\gamma'' \in \Gamma''$  there exists a  $\gamma \in \Gamma$  such that  $\gamma(\Pi)$  meets  $\gamma''(\Pi) \cap C$ , so that  $\gamma^{-1}\gamma'' \in S$ . This proves that  $\Gamma'' = S \cdot \Gamma$  and hence that  $\Gamma$  is of finite index in  $\Gamma''$ . For the same reason,  $\Gamma'$  is of finite index in  $\Gamma''$ . □

In the remainder of this section,  $(C_+, \Gamma)$  is of polyhedral type and  $L \subset V(\mathbb{Q})$  is a  $\Gamma$ -invariant lattice.

**PROPOSITION 4.7.** *Let  $\Sigma$  be a  $\Gamma$ -invariant locally rationally polyhedral decomposition of  $C_+$ , and let  $\sigma \in \Sigma$ . Then  $(\sigma, \Gamma(\sigma))$  is of polyhedral type, and if  $F$  denotes the smallest face of  $C_+$  which contains  $\sigma$ , then  $\text{Star}_\Sigma(\sigma)$  decomposes into a finite number of  $Z_\Gamma(F)$ -equivalence classes.*

*Proof.* Let  $\Pi$  be a rationally polyhedral cone in  $C_+$  with  $\Gamma \cdot \Pi = C_+$ . Then  $\Pi$  meets the relative interiors of only finitely many members of  $\Sigma$ . Let  $\gamma_1, \dots, \gamma_N \in \Gamma$  be such that  $\gamma_1(\mathring{\sigma}), \dots, \gamma_N(\mathring{\sigma})$  are the  $\Gamma$ -translates of  $\mathring{\sigma}$  which meet  $\Pi$ . Then  $\Pi_1 := (\gamma_1^{-1}(\Pi) + \dots + \gamma_N^{-1}(\Pi)) \cap \sigma$  is a rationally polyhedral cone. For every  $x \in \mathring{\sigma}$ , there exists a  $\gamma \in \Gamma$  such that  $\gamma(x) \in \Pi$ . Then  $\gamma(\mathring{\sigma}) \cap \Pi \neq \emptyset$  and so  $\gamma(\mathring{\sigma}) = \gamma_\nu(\mathring{\sigma})$  for some  $\nu \in \{1, \dots, N\}$ . This implies that  $\gamma_\nu^{-1}\gamma$  leaves  $\sigma$  invariant and maps  $x$  into  $\Pi_1$ . So  $\Gamma(\sigma) \cdot \Pi_1 \supset \mathring{\sigma}$ . As every rationally polyhedral cone in  $C_+$  intersects  $\sigma$  in a rationally polyhedral cone, we have  $(\mathring{\sigma})_+ = \sigma$ . This proves the first assertion.

Next we fix an  $x_0 \in \mathring{\sigma} \cap V(\mathbb{Q})$  which is not a fixed point of a nonidentity element of  $\Gamma(\sigma)$ . We prove that for every  $\tau \in \Sigma$  with  $\tau \supset \sigma$ , there exists a  $\gamma_\tau \in \Gamma$  such that  $\gamma_\tau(x_0) \in \Pi$  and  $\gamma_\tau(\mathring{\tau}) \cap \Pi \neq \emptyset$ . This will imply the last assertion, for  $\Gamma x_0 \cap \Pi$  and  $\Sigma|_\Pi$  are finite. To see that such a  $\gamma_\tau$  exists, choose a rationally polyhedral cone  $\Pi_\tau \subset \tau$  with  $\Pi_\tau \cap \mathring{\tau} \neq \emptyset$  and  $\Pi_\tau \cap \sigma = \mathbb{R}_{\geq 0}x_0$ . Since  $\{\Pi_\tau \cap \gamma^{-1}(\Pi) \mid \gamma \in \Gamma\}$  is a finite collection of rationally polyhedral cones which covers  $\Pi_\tau$ , there exists a  $\gamma_\tau \in \Gamma$  with  $x_0 \in \gamma_\tau^{-1}(\Pi)$  and  $\Pi_\tau \cap \gamma_\tau^{-1}(\Pi) \neq \emptyset$ . So  $\gamma_\tau(x_0) \in \Pi$  and  $\gamma_\tau(\mathring{\tau}) \cap \Pi \neq \emptyset$ , as required. □

*Example 4.8.* Here is an example of a nontrivial situation to which the previous proposition applies. Let  $\langle , \rangle$  be a symmetric bilinear form on  $V$  of signature  $(1, \dim V - 1)$  defined over  $\mathbb{Q}$ , and let  $C$  be a connected component of the set of  $x \in V$  with  $\langle x, x \rangle > 0$ . We choose a lattice  $L$  in  $V(\mathbb{Q})$ , and let  $\Gamma := O(L) \cap \text{Aut}(C)$ . It follows from Proposition 4.2 that  $(V(\mathbb{Q}), C, \Gamma)$  is of polyhedral type. Suppose now that we have a collection  $\mathcal{H}$  of hyperplanes of  $V$  defined over  $\mathbb{Q}$  meeting  $C$ , which is a finite union of  $\Gamma$ -orbits.

**CLAIM.** *The collection of hyperplanes  $\mathcal{H}$  induces a  $\Gamma$ -invariant locally rationally polyhedral decomposition  $\Sigma$  of  $C_+$ .*

*Proof.* We must show that for every rationally polyhedral cone  $\Pi$  in  $C_+$ , the collection  $\{H \cap \Pi \mid H \in \mathcal{H}\}$  has only finitely distinct members. Given  $H \in \mathcal{H}$ , then  $C_+ \cap H$  and the group of  $\gamma \in O(L \cap H)$  which preserve  $C \cap H$  make up a pair of polyhedral type (of one dimension lower, but otherwise of the same type as  $(C_+, \Gamma)$ ). It is not hard to show that  $\Gamma(H)$  is of finite index in the latter group, and so by Proposition 4.6,  $(C_+ \cap H, \Gamma(H))$  is also of polyhedral type. Hence there exists a rationally polyhedral cone  $\Pi_H \subset C_+ \cap H$  such that  $\Gamma(H) \cdot \Pi_H = C_+ \cap H$ . By the Siegel property, Theorem 3.8, the collection  $\{\gamma(\Pi_H) \cap \Pi \mid \gamma \in \Gamma\}$  has finitely many distinct members. If we let  $H$  run over a representative system of  $\Gamma$ -equivalence classes in  $\mathcal{H}$ , we find that the same is true for the collection  $\{H \cap \Pi \mid H \in \mathcal{H}\}$ . □

This construction often yields locally rationally polyhedral decompositions of  $C_+$  for which the adverb ‘locally’ cannot be dropped, and thus also produces in view of Proposition 4.7 interesting new examples of pairs of polyhedral type. For instance, given a union  $T$  of conjugacy classes of reflections in  $\Gamma$ , then because such conjugacy classes are finite in number, the collection  $\mathcal{H}$  of fixed point hyperplanes of the members of  $T$  breaks up in a finite number of  $\Gamma$ -equivalence classes. The resulting locally rationally polyhedral decomposition of  $C_+$  is rationally polyhedral if and only if the subgroup of  $\Gamma$  generated by  $T$  is of finite index in  $\Gamma$ . This follows from work



of Vinberg [Vin72a]. But according to this very author [Vin72b], there are only a few cases in which the subgroup of  $\Gamma$  generated by its reflections is of finite index in  $\Gamma$ .

PROPOSITION 4.9. *Every  $\Gamma$ -invariant kernel for  $C$  contains a core and is contained in a cocore for  $C$ . Moreover  $[(\bar{C} - \{0\}) \cap L] + \bar{C}$  is a cocore for  $C$ , and, dually, we have that  $((C^* - \{0\}) \cap L^*)^\vee$  is a core for  $C$ .*

*Proof.* Choose a rationally polyhedral cone  $\Pi \subset C_+$  such that  $\Gamma \cdot \Pi = C_+$ . Let  $K$  be a  $\Gamma$ -invariant kernel for  $C$ . Since  $\Pi \cap (C^\circ \cap L^*)^\vee$  contains the convex hull of  $(\Pi - \{0\}) \cap L$  and  $0 \notin \bar{K} \cap \Pi$ , there exists a  $\lambda > 0$  such that  $\bar{K} \cap \Pi \subset \lambda(C^\circ \cap L^*)^\vee$ . This implies that the last set also contains  $\bar{K} \cap C$ . As  $\bar{K} \cap C$  is dense in  $\bar{K}$ , it follows that it even contains  $\bar{K}$ . Applying this to  $K^\vee$ , we also find that  $\bar{K} = K^{\vee\vee}$  contains a set of the form  $\mu[C \cap L]$  for some  $\mu > 0$ . Hence  $K$  contains  $2\mu[C \cap L]$ .

Let  $\Lambda$  denote the set of lattice points in  $\bar{C} - \{0\}$ . Clearly,  $(C^\circ \cap L^*)^\vee$  contains  $\Lambda$  and hence also  $[\Lambda] + \bar{C}$ . If  $\nu > 0$  is such that  $\Pi \cap (C^\circ \cap L^*)^\vee \subset \nu[(\Pi - \{0\}) \cap L]$ , then  $C \cap (C^\circ \cap L^*)^\vee \subset \nu[\Lambda]$ . Following Lemma 1.6,  $[\Lambda] + \bar{C}$  is closed, and since  $C \cap (C^\circ \cap L^*)^\vee$  is dense in  $(C^\circ \cap L^*)^\vee$ , it follows that the last space is contained in  $\nu[\Lambda] + \bar{C}$ . This proves that  $[\Lambda] + \bar{C}$  is a cocore for  $C$ . If we apply this to the dual situation and dualize, we find that  $((C^* - \{0\}) \cap L^*)^\vee$  is a core for  $C$ .  $\square$

Remark 4.10. It is in general not true that a cocore is contained in  $C_+$ . To see this, suppose that there exist a face  $F \neq \{0\}$  of  $C_+$ , and a proper face  $G$  of  $\bar{C}$  which contains  $F$  and whose relative interior does not contain any rational point. Then no cocore is contained in  $C_+$ : if  $L \subset V(\mathbb{Q})$  is a lattice, choose  $p \in \overset{\circ}{F} \cap L$ , so that  $p$  belongs to the typical cocore  $K := (C^\circ \cap L^*)^\vee$ . Hence  $K \supset p + \bar{C} \supset p + \overset{\circ}{G}$ , and the last space is a nonempty open subset of  $\overset{\circ}{G}$  which by assumption does not meet  $C_+$ .

To be more concrete, let  $V$  be the space of symmetric bilinear forms on  $\mathbb{R}^n$ ,  $n \geq 3$ , with its standard rational structure, and let  $C$  be the cone of positive definite forms. (We are in a special case of Proposition 4.2 if we take  $\mathcal{G} := \text{PSL}_n$ .) Choose an irrational line  $l$  in  $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ , and let  $F$ , respectively  $G$ , be the cone of positive semi-definite forms on  $\mathbb{R}^n$  whose nilspace contains  $\mathbb{R}^{n-1}$ , respectively  $l$ . Then  $F$  is the half line spanned by  $x_n^2$  and is a face of  $C_+$ , whereas  $G$  is a face of  $\bar{C}$  which contains  $F$ , but has no rational points in its relative interior.

So while a  $\Gamma$ -invariant kernel need not be contained in  $C_+$ , the following proposition shows that its intersection with  $C_+$  is quite nice (compare [AMRT10, Proposition II 5.22]).

PROPOSITION 4.11. *Let  $K$  be a  $\Gamma$ -invariant kernel for  $C$ . Then the following are equivalent:*

- (i)  $K$  is locally rationally polyhedral in  $C_+$ ;
- (ii) there exists a finite union  $S$  of  $\Gamma$ -orbits in  $(\bar{C} - \{0\}) \cap V(\mathbb{Q})$  such that  $K \cap C_+ = [S] + C_+$ ;
- (i)\*  $K^\vee$  is locally rationally polyhedral in  $C_+^\circ$ ;
- (ii)\* there exists a finite union  $S^*$  of  $\Gamma$ -orbits in  $(C^* - \{0\}) \cap V(\mathbb{Q})$  such that  $K^\vee \cap C_+^\circ = [S^*] + C_+^\circ$ .

Moreover, if one of these conditions is fulfilled, then we can take for  $S$  the set  $E$  of extreme points of  $K$  and we have  $\bar{K} = [E] + \bar{C}$  and every bounded face of  $K$  is a rational polyhedron.

*Proof.* (i)  $\Rightarrow$  (ii) plus the last clause: Let  $\Pi \subset C_+$  be a rationally polyhedral cone such that  $\Gamma \cdot \Pi = C_+$ . Since  $K \cap \Pi$  is a rational polyhedron, the set  $E_0$  of its extreme points is a finite set of rational points with the property that  $K \cap \Pi = ([E_0] + C_+) \cap \Pi$ . So if we let  $E := \Gamma \cdot E_0$ , then  $K \cap C_+ = [E] + C_+$  and  $E$  is the set of extreme points of  $K \cap C_+$ .

Let  $k \in \mathbb{N}$  be such that  $E_0 \subset (1/k)L$ . Then  $E \subset (1/k)L$ , which shows that  $E$  is discrete in  $V$ . According to Lemma 1.6, this implies that  $[E] + \bar{C}$  is closed in  $V$  and that every bounded face of



$[E] + \bar{C}$  is spanned by a finite subset of  $S$  and is therefore a rational polyhedron. In particular, it is a face of  $K \cap C_+$ . Since  $K \cap C_+ = [S] + C_+$  is dense in  $K$ , we have  $\bar{K} = [S] + \bar{C}$  and so every bounded face of  $K$  is also one of  $\bar{K}$ . Hence it is of the stated form.

(ii)  $\Rightarrow$  (i)\*: Arguing as above we find that  $S$  is contained in some lattice in  $V(\mathbb{Q})$ . We then conclude from Lemma 3.7 that  $([S] + \bar{C})^\vee$  is locally rationally polyhedral in  $C_+$ .

The proposition now follows from the proven implications and their dual versions. □

DEFINITION 4.12. We call a function  $f : C_+ \rightarrow \mathbb{R}$  *admissible* if  $f$  is continuous and for every rationally polyhedral cone  $\Pi \subset C_+$ , the set of  $(x, t) \in \Pi \times \mathbb{R}$  with  $f(x) \geq t$  is a rationally polyhedral cone. So

$$C_f := \{(x, t) \in C \times \mathbb{R} \mid f(x) > t\}$$

is an open nondegenerate convex cone in  $V \times \mathbb{R}$  with  $C_{f,+} = \{(x, t) \in C_+ \times \mathbb{R} \mid f(x) \geq t\}$ . The interest of such a function lies in the fact that it determines a decomposition  $\Sigma(f)$  of  $C_+$ : the members of this decomposition are simply the projections of the faces of  $C_f$  which do not contain the negative  $t$ -axis. An alternative characterization of  $\Sigma(f)$  is that it is the coarsest locally rationally polyhedral decomposition of  $C_+$  with the property that  $f$  is linear on each member.

We return to Example 4.8 and prove that the decomposition described there comes from an admissible function. Fix a maximal member  $\sigma \in \Sigma$ , and let for every  $H \in \mathcal{H}$ ,  $\xi_H$  be the unique indivisible element of  $L^*$ , which defines  $H$  and is  $\geq 0$  on  $\sigma$ . For  $x \in C_+$ , we define

$$f(x) = \sum_{H \in \mathcal{H}} \min\{\xi_H(x), 0\}.$$

The sum involves at most a finite number of nonzero terms, since at most finitely many  $H \in \mathcal{H}$  will separate  $x$  from  $\sigma$ . It is easily verified that  $f$  is admissible and that  $\Sigma(f) = \Sigma$ . Notice that  $f$  transforms under  $\gamma \in \Gamma$  as follows:  $f\gamma^{-1} = f + \sum_{H \in \mathcal{H}(\gamma)} \xi_H$ , where  $\mathcal{H}(\gamma)$  denotes the collection of  $H \in \mathcal{H}$  which separate  $\gamma^{-1}(\sigma)$  from  $\sigma$ . So  $f$  is not  $\Gamma$ -invariant (unless  $\mathcal{H} = \emptyset$ ), but  $\gamma \mapsto f\gamma^{-1} - f$  is a 1-cocycle on  $\Gamma$  with values in the  $\Gamma$ -representation  $V(\mathbb{Q})^*$  (in fact, even in  $L^*$ ). But if this cocycle happens to be a coboundary, then by definition there exists a  $\rho \in V^*(\mathbb{Q})$  such that  $f - \rho$  is  $\Gamma$ -invariant. Observe that this function is also admissible and defines the same decomposition as  $f$ . This phenomenon occurs in the theory of generalized root systems, where  $\rho$  appears as what some authors call a ‘Weyl vector’.

Interesting examples of  $\Gamma$ -invariant admissible functions (and hence of  $\Gamma$ -invariant locally rationally polyhedral decompositions) are obtained from  $\Gamma$ -invariant locally rationally polyhedral kernels.

LEMMA 4.13. *Let  $K$  be a locally rationally polyhedral kernel for  $C^\circ$ , invariant under  $\Gamma$ . Then every  $x \in C_+$  has a minimum on  $K$ , and if we denote this minimum by  $f_K(x)$ , then  $f_K$  is a  $\Gamma$ -invariant admissible function on  $C_+$  and  $K^\vee \cap C_+ = \{x \in C_+ \mid f_K(x) \geq 1\}$ .*

*Proof.* Let  $E$  denote the set of extreme points of  $K$ . By Proposition 4.11,  $\bar{K} = [E] + \bar{C}$ , and so it follows that for  $x \in C_+$ ,  $\inf_K x = \inf_{\bar{K}} x = \inf_E x$ . Write  $x = \lambda_1 x_1 + \dots + \lambda_m x_m$  with  $x_\mu$  a rational point of  $C_+$  and  $\lambda_\mu \geq 0$ . Since  $E$  is contained in a lattice in  $V(\mathbb{Q})^*$ ,  $x_\mu(E)$  will be a discrete subset of  $\mathbb{R}_{\geq 0}$ ,  $\mu = 1, \dots, m$ . Hence the same is true for  $x(E)$ . In particular,  $x(E)$  has a minimum.

Now let  $\Pi$  be a rationally polyhedral cone in  $C_+$ . Then  $\Pi \cap K^\vee$  is a rational polyhedron (which may be empty). Let  $\phi : \Pi \rightarrow \mathbb{R}_{\geq 0}$  be the function characterized by  $\phi(\lambda x) = \lambda\phi(x)$ ,  $x \in \Pi, \lambda \in \mathbb{R}_{\geq 0}$ , and  $\{x \in \Pi \mid \phi(x) \geq 1\} = \Pi \cap K^\vee$ . Then the set of  $(x, t) \in \Pi \times \mathbb{R}$  with

$\phi(x) \geq t$  is a rationally polyhedral cone, and it is clear that  $\phi = f_K|_{\Pi}$ . So  $f_K$  is admissible and  $K^\vee \cap C_+ = \{x \in C_+ \mid f_K(x) \geq 1\}$ . □

Let us now consider the special case when  $K$  is a  $\Gamma$ -invariant locally rationally polyhedral core for  $C^\circ$ . Then  $K^\vee$  is a  $\Gamma$ -invariant locally rationally polyhedral cocore for  $C$ . For every rationally polyhedral cone  $\Pi \subset C_+$ ,  $K^\vee \subset \Pi$  is a rational polyhedron which meets each extremal ray of  $\Pi$  (for  $K^\vee$  is comparable with the standard cocore  $[C^\circ \cap L^*]^\vee$ , which has that property by Proposition 3.5). It is then easily seen that the bounded faces of  $K^\vee \subset \Pi$  lie on bounded faces of  $K^\vee$ . So the cone spanned by the union of the bounded faces of  $K^\vee$  coincides with  $C_+$ .

According to Proposition 4.11 every bounded face of  $K^\vee$  is a rational polyhedron, so that  $\Sigma(K) := \Sigma(f_{K^\vee})$  is in fact a rationally polyhedral decomposition of  $C_+$ . Moreover, the faces of  $K$  parameterize in a bijective manner the faces of  $\Sigma(f_{K^\vee})$  by assigning to a face  $P$  of  $K$  the cone  $\sigma(P)$  of  $x \in V$  with the property that  $x|_K$  assumes its infimum on all of  $P$ . In particular,  $C_+ = \cup \sigma(P)$  is the set of  $x \in V$  which have a minimum on  $K$ . Notice that  $P \mapsto \sigma(P)$  reverses inclusions and that  $\dim P + \dim \sigma(P) = \dim V$ . This generalizes the construction of  $\Sigma(C, L)$  of Proposition 3.5, for the latter is obtained if we take  $K = [C^\circ \cap L^*]$ .

*Application 4.14* (Construction of a polyhedral  $\Gamma$ -fundamental domain in  $C_+$ ). Choose  $\xi \in C^\circ \cap V^*(\mathbb{Q})$ . Then it follows from Proposition 4.11 that  $K = [\Gamma\xi] + C^*$  is a  $\Gamma$ -invariant locally rationally polyhedral kernel. If  $\lambda \in \mathbb{N}$  is such that  $\lambda\xi \in L^*$ , then  $\lambda K \subset [C^\circ \cap L^*]$  and so by Proposition 4.9  $K$  is a core for  $C^\circ$ . Every extreme point of  $K$  corresponds to a maximal element of  $\Sigma(K)$ . Since  $\Gamma\xi$  is the set of extreme points of  $K$ , it follows that  $\Gamma$  is transitive on the collection of maximal members of  $\Sigma(K)$ . So

$$\sigma := \sigma(\{\xi\}) = \{x \in C_+ \mid \xi(\gamma x) \geq \xi(x) \text{ for all } \gamma \in \Gamma\}$$

is a rationally polyhedral cone with the property that  $\Gamma \cdot \sigma = C_+$  and  $\gamma(\sigma) \cap \sigma = \emptyset$  if  $\gamma \in \Gamma$  does not fix  $\xi$ . In particular,  $\sigma$  is a fundamental domain in  $C_+$  if  $\Gamma_\xi = \{1\}$ . It also follows that  $C_+$  is just the set of  $x \in \bar{C}$  which have a minimum on  $\Gamma x$ .

If we take  $\xi \in \mathring{F} \cap V^*(\mathbb{Q})$ , where  $F$  is a proper face of  $C_+^\circ - \{0\}$ , then the corresponding decomposition  $\Sigma(K)$  is also of interest. Again,  $\Gamma$  is then transitive on the maximal members of  $\Sigma(K)$ , and the stabilizer of  $\sigma(\{\xi\})$  (which is one such member) is  $\Gamma_\xi$ . Notice that  $\Gamma_\xi$  contains  $Z_\Gamma(F)$  as a subgroup of finite index; in general this will be an infinite group.

We finally mention two consequences of having a polyhedral fundamental domain.

**COROLLARY 4.15.** *The group  $\Gamma$  is finitely presented.*

*Proof.* Let  $\Pi$  be a rationally polyhedral cone in  $C_+$  such that  $\Gamma \cdot \Pi = C_+$ . So  $\Gamma \cdot (\Pi \cap C) = C$ . As is well known (and easy to prove), the mere fact that  $C$  is connected now implies that  $\Gamma$  is generated by the  $\gamma \in \Gamma$  for which  $\gamma(\Pi) \cap \Pi$  is a codimension one face of  $\Pi$  which meets  $C$ . This is clearly a finite set. Similarly, the fact that  $C$  is simply connected implies that a complete set of relations among these generators is indexed by the codimension two faces of  $\Pi$  which meet  $C$ . □

The other consequence involves the property VFL (which stands for having *Virtuellement une résolution Libre de type Finie*) for groups. Recall that a (discrete) group  $G$  is said to be VFL of dimension  $\leq d$  if there exists a subgroup  $H \subset G$  of finite index such that the trivial  $H$ -module  $\mathbb{Z}$  admits a resolution of length  $\leq d$  by free finite rank  $\mathbb{Z}[H]$ -modules.

We show that  $\Gamma$  has this property. This is based on a standard construction, which we briefly recall. Consider the  $\Gamma$ -invariant decomposition  $\Sigma$  of  $C_+$  constructed from the  $\Gamma$ -stable lattice  $L \subset V(\mathbb{Q})$  in Application 4.14. It has a canonical ‘barycentric’ subdivision defined as follows: every member  $\sigma \in \Sigma$ , being a rational polyhedral cone, has finitely many extremal rays. The sum

of the integral generators of these rays spans a ray  $R_\sigma$  with  $\mathring{R}_\sigma \subset \mathring{\sigma}$ . Now  $\Sigma$  is naturally refined by a decomposition  $\Sigma'$  whose members  $\neq \{0\}$  are indexed by the strictly monotonous sequences  $\sigma_\bullet := (\sigma_0 \supseteq \sigma_1 \supseteq \dots \supseteq \sigma_k \neq 0)$  in  $\Sigma$ , the associated polyhedral cone being  $\langle \sigma_\bullet \rangle := R_{\sigma_0} + \dots + R_{\sigma_k}$ . Notice that if  $\sigma_k$  meets  $C$ , then  $\langle \sigma_\bullet \rangle \subset C \cup \{0\}$  so that its projectivization  $P(\langle \sigma \rangle)$  is a polyhedron entirely contained in the open contractible  $P(C) \subset P(V)$ . If we denote by  $\Sigma'_f$  the subcollection of  $\sigma_\bullet \in \Sigma'$  with that property, then the union  $P|\Sigma'_f| \subset P|\Sigma|$  of such polyhedra (often called the *spine* of  $\Sigma$ ) is a polyhedral complex of dimension  $\leq \dim V - 1$  that is invariant under  $\Gamma$  and whose polyhedral cells decompose into finitely many  $\Gamma$ -equivalence classes.

There is a natural  $\Gamma$ -equivariant deformation retraction of  $P(C)$  onto this spine as follows. If  $\sigma_\bullet = (\sigma_0 \supseteq \sigma_1 \supseteq \dots \supseteq \sigma_k \neq 0)$  is any strictly monotonous sequence in  $\Sigma$  with  $\sigma_0 \cap C \neq \emptyset$ , then let  $r \in \{0, \dots, k\}$  be the highest index for which  $\sigma_r$  still meets  $C$  and denote by  $\sigma_\bullet^C$  the truncation  $\sigma_0 \supseteq \dots \supseteq \sigma_r$ . There is a natural deformation retraction of the improper polyhedron  $P(\langle \sigma_\bullet \rangle) \cap P(C)$  onto spinal polyhedron  $P(\langle \sigma_\bullet^C \rangle)$ . It is compatible with inclusion and so this results in a  $\Gamma$ -equivariant deformation retraction of  $P(C)$  onto the spine  $P|\Sigma'_f|$ . In particular,  $P|\Sigma'_f|$  is contractible.

**COROLLARY 4.16.** *The group  $\Gamma$  is of type VFL of dimension  $\leq \dim V - 1$ .*

*Proof.* Denote by  $\Gamma'$  the kernel of the representation of  $\Gamma$  on  $L/3L$ . A well-known theorem of Serre asserts that  $\Gamma'$  is torsion free. So  $\Gamma'$  acts freely on  $P(C)$  and hence also on  $P|\Sigma'_f|$ . Upon replacing  $\Gamma'$  by a smaller subgroup (still of finite index in  $\Gamma$ ) we may assume that the  $\Gamma'$ -stabilizer of any  $\sigma \in \Sigma$  which meets  $C$  is trivial. The result is that the cells of  $P|\Sigma'_f|$  have the same property. The associated chain complex therefore provides a resolution of length  $\leq \dim V - 1$  of the trivial  $\Gamma'$ -module  $\mathbb{Z}$  by free finite rank  $\mathbb{Z}[\Gamma']$ -modules. □

*Question 4.17.* Corollary 4.16 implies among other things that the cohomology of  $\Gamma$  with values in a finite-dimensional  $\mathbb{Q}$ -vector space that is also a representation of  $\Gamma$  is finite dimensional. A case of particular interest is  $H^1(\Gamma, V)$  (which has a  $\mathbb{Q}$ -structure for which  $H^1(\Gamma, V)(\mathbb{Q}) = H^1(\Gamma, V(\mathbb{Q}))$ ). Any  $c \in H^1(\Gamma, V)$  is representable by a cocycle, i.e., a map  $\gamma \in \Gamma \mapsto c_\gamma \in V$  satisfying  $c_{\gamma_1\gamma_2} = c_{\gamma_1} + \gamma_1(c_{\gamma_2})$ . This defines an action of  $\Gamma$  on a copy  $V_c$  of  $V$  by affine-linear transformations defined by the rule  $\gamma_c(v) := c_\gamma + \gamma(v)$  (so its linear part is the given action). The  $\Gamma$ -action on  $V_c$  has fixed point if and only if the class  $c$  is zero. We can make it depend linearly on  $c$  by choosing representative cocycles for a basis of  $H^1(\Gamma, V)$  and then extending linearly the resulting actions. This yields an exact sequence of  $\Gamma$ -representations

$$0 \rightarrow V \rightarrow \tilde{V} \rightarrow H^1(\Gamma, V) \rightarrow 0,$$

where  $\Gamma$  of course acts trivially on  $H^1(\Gamma, V)$ . It is universal for that property in a sense we don't bother to make precise. If we choose the basis in  $H^1(\Gamma, V)(\mathbb{Q})$  and let the representative cocycles take their values in  $V(\mathbb{Q})$ , then  $\tilde{V}$  acquires a  $\mathbb{Q}$ -structure preserved by  $\Gamma$ . We can do even better and take the basis in the image of  $H^1(\Gamma, L) \rightarrow H^1(\Gamma, V)$ , let the representative cocycles take their values in  $L$  and get a lattice  $\tilde{L}$  in  $\tilde{V}(\mathbb{Q})$  preserved by  $\Gamma$ .

Does there exist an open nondegenerate convex cone  $\tilde{C}$  in  $\tilde{V}$  which forms with  $\Gamma$  a pair of polyhedral type and is such that  $\tilde{C}_+$  contains  $C_+$  as a face?

### 5. The stabilizer of a face

Throughout this section, we fix a polyhedral triple  $(V(\mathbb{Q}), C, \Gamma)$  in the sense of Proposition 4.1 and a face  $F$  of  $C_+$ . Our principal goal is to describe the structure of the  $\Gamma$ -stabilizer of  $F$ .

We begin with a bit of notation. We let  $F^\dagger$  stand for the set of  $\xi \in C_+^0$  which vanish on  $F$ . This is clearly an exposed face of  $C_+^0$  and its annihilator contains  $F$ . We shall find that  $F^{\dagger\dagger} = F$ , but at this point it is not even clear whether  $F \neq C_+$  implies  $F^\dagger \neq \{0\}$ . We denote the linear span of  $F$  in  $V$  by  $V_F$  and write  $V^F$  for the annihilator of  $F^\dagger$  (it will turn out that there is no conflict with that same notation used in §1). So we have a flag of  $\mathbb{Q}$  vector spaces defined over  $\mathbb{Q}$ :

$$0 \subset V_F \subset V^F \subset V.$$

We further put  $T_F := V^F/V_F$  and denote the projections

$$\pi_F : V \rightarrow V/V_F, \quad \pi^F : V \rightarrow V/V^F,$$

so that the latter is the composite of  $\pi_F$  and the projection

$$q_F : V/V_F \rightarrow V/V^F.$$

Observe that we have a perfect duality  $V/V^F \times V_{F^\dagger}^* \rightarrow \mathbb{R}$  and that under this duality  $\pi^F(C)$  is identified with the open dual of  $F^\dagger$ . It is in particular a nondegenerate convex cone. Let us begin with stating one of the main results of this section. Denote by  $N_\Gamma(F)$  the  $\Gamma$ -stabilizer of  $F$ . It acts on  $F$  and  $F^\dagger$  and so we have a group homomorphism  $N_\Gamma(F) \rightarrow \Gamma(F) \times \Gamma(F^\dagger)$ .

The following theorem sums up the main content of this section.

**THEOREM 5.1.** *The image of the projection  $N_\Gamma(F) \rightarrow \Gamma(F) \times \Gamma(F^\dagger)$  is of finite index in the latter and the elements in its kernel that act trivially on  $T_F$  form a free abelian subgroup  $U_\Gamma(F)$  of finite index in that kernel. The action of  $U_\Gamma(F)$  on  $V$  is 2-step unipotent and is given by a unique homomorphism*

$$u \in U_\Gamma(F) \mapsto \sigma_u \in \text{Hom}(V/V_F, V^F)$$

with the following properties:

- (i)  $\sigma_u$  maps  $T_F$  to  $V_F$  and the induced maps

$$j_u : V/V^F \rightarrow T_F, \text{ respectively } k_u : T_F \rightarrow V_F,$$

are such that  $u(x) = x + \sigma_u(x') + \frac{1}{2}k_u j_u(x'')$ , where  $x'$ , respectively  $x''$ , denote the images of  $x$  in  $V/V_F$ , respectively  $V/V^F$ ;

- (ii) for  $u, v \in U_\Gamma(F)$ , we have  $k_u j_v = k_v j_u$ ;
- (iii)  $k_u j_u$  maps  $\pi^F(C)$  to  $F - \{0\}$ , unless  $u = 1$ .

*Example 5.2.* This theorem is well illustrated by the following basic example. Take for  $V$  the space  $\text{Sym}^2 W$  of symmetric tensors in  $W \otimes W$ , where  $W$  is a real finite-dimensional vector space with a  $\mathbb{Q}$ -structure and let  $C \subset V$  be the cone of positive ones. Then  $C_+$  is the cone spanned by the pure squares  $w \otimes w$  with  $w \in W(\mathbb{Q})$ . Alternatively,  $C_+$  consists of the semipositive symmetric tensors whose annihilators are defined over  $\mathbb{Q}$ . So a face  $F$  of  $C_+$  is given by a subspace  $W' \subset W$  defined over  $\mathbb{Q}$  and then consists of the semipositive elements in  $\text{Sym}^2 W'$ . We have  $V_F = \text{Sym}^2 W'$ ,  $V^F = W' \circ W$  (i.e., the span of the tensors  $w' \otimes w + w \otimes w'$ , with  $w' \in W'$ ), so that  $T_F = V^F/V_F$  may be identified with  $(W/W') \otimes W'$  and  $V/V^F$  with  $\text{Sym}^2(W/W')$ . The open dual  $C^\circ$  is the cone of positive definite quadratic forms on  $W$  and under this identification, the relative interior  $F^\dagger$  may be identified with the cone of positive definite quadratic forms on  $W/W'$ . The group  $\text{GL}(W)$  acts on  $(V, C)$  and the stabilizer of  $F$  is the stabilizer of  $W'$ . The latter maps onto  $\text{GL}(W') \times \text{GL}(W/W')$  (its Levi quotient) with kernel an abelian unipotent group  $U(F)$  that can be identified with the vector group  $\text{Hom}(W/W', W')$ . The map  $\sigma : U(F) \rightarrow \text{Hom}(V/V_F, V^F)$  is identified with the map

$$\text{Hom}(W/W', W') \rightarrow \text{Hom}(\text{Sym}^2 W / \text{Sym}^2 W', W' \circ W),$$

which assigns to  $u \in \text{Hom}(W/W', W')$  the map  $\text{Sym}^2(W/W') \rightarrow W' \circ W$  characterized by  $w \otimes w + \text{Sym}^2 W' \mapsto u(\bar{w}) \otimes w + w \otimes u(\bar{w})$  (here  $w \in W$  and  $\bar{w}$  is its image in  $W/W'$ ). Notice that this induces maps

$$\begin{aligned} j_u &: \text{Sym}^2(W/W') \rightarrow (W/W') \otimes W', & \bar{w} \otimes \bar{w} &\mapsto \bar{w} \otimes u(\bar{w}) \\ k_u &: (W/W') \otimes W' \rightarrow \text{Sym}^2 W', & \bar{w} \otimes k &\mapsto u(\bar{w}) \otimes k + k \otimes u(\bar{w}) \end{aligned}$$

so that  $k_u j_v = u \otimes v + v \otimes u : \text{Sym}^2(W/W') \rightarrow \text{Sym}^2 W'$ . We note that if  $(w_i)_i$  is a basis of  $W$ , then  $\frac{1}{2} k_u j_u = u \otimes u$  sends  $\sum_i w_i \otimes w_i$  to  $\sum_i u(w_i) \otimes u(w_i)$ , which is zero only when  $u = 0$ .

If  $\Gamma \subset \text{SL}(W)$  is arithmetic, then we have a similar description for a  $\Gamma$ -stabilizer of  $F$  (which is of course the  $\Gamma$ -stabilizer of  $W'$ ).

We shall denote the kernel of  $N_\Gamma(F) \rightarrow \Gamma(F) \times \Gamma(F^\dagger)$  by  $Z_\Gamma(F \times F^\dagger)$  (this is in agreement with our notational convention if we let  $\Gamma$  act on  $V \times V^*$  diagonally). Furthermore,  $L$  stands for some  $\Gamma$ -invariant lattice in  $V(\mathbb{Q})$ .

**LEMMA 5.3.** *Let  $\Sigma$  be a  $\Gamma$ -invariant locally rationally polyhedral decomposition of  $C_+$ , and let  $\sigma \in \Sigma$  be such that  $\dot{\sigma}$  is open in  $F$ . Then every point of  $\pi_F(C_+)$  is in the  $\pi_F$ -image of the relative interior of a unique member of  $\text{Star}_\Sigma(\sigma)$ , and the projection  $\pi_F$  maps the members of  $\text{Star}_\Sigma(\sigma)$  onto a locally rationally polyhedral decomposition  $\pi_{F*}\text{Star}_\Sigma(\sigma)$  of  $\pi_F(C_+)$ .*

*If  $\Sigma$  is in fact rationally polyhedral and  $P$  is a rationally polyhedral cone in  $\pi^F(C_+)$  whose preimage in  $\pi_F(C_+)$  is denoted  $\tilde{P}$ , then the restriction of  $\pi_{F*}^\dagger \text{Star}_\Sigma(\sigma)$  to  $\tilde{P}$  has only finitely many  $Z_\Gamma(F \times F^\dagger)$ -orbits.*

*Proof.* Given  $x \in C_+$ , choose a rationally polyhedral cone  $\Pi$  in  $C_+$  which intersects  $\dot{\sigma}$  and contains  $x$ . Since  $\Sigma \upharpoonright \Pi$  is a finite decomposition into rationally polyhedral cones, there is a  $y \in \Pi \cap \dot{\sigma}$  such that  $x + y$  is in the relative interior of a member of  $\text{Star}_\Sigma(\sigma)$ . This member is clearly independent of the choice of  $y$ . The first part of the lemma now follows easily.

For last clause of the lemma we may assume that  $\tilde{P} \subset \pi^F(C)$ . It suffices to show that the collection of  $\tau \in \text{Star}_\Sigma(\sigma)$  whose image in  $\pi^F(C_+)$  meets  $\tilde{P}$  is finite modulo  $Z_\Gamma(F \times F^\dagger)$ . But this follows from the fact that the collection  $\text{Star}_\Sigma(\sigma)$  is finite modulo  $Z_\Gamma(F)$  by Proposition 4.7 and the Siegel property of the image of the latter group in  $\pi^F(C)_+$ .  $\square$

**COROLLARY 5.4.** *We have  $\pi^F(C_+) = \pi^F(C)_+$  and the image of  $Z_\Gamma(F)$  in  $\Gamma(F^\dagger)$  is a subgroup of the latter of finite index (or equivalently, we can say that the image of  $N_\Gamma(F)$  in  $\Gamma(F) \times \Gamma(F^\dagger)$  is a subgroup of finite index).*

*Proof.* Choose a  $\Gamma$ -invariant rationally polyhedral decomposition  $\Sigma$ . It follows from Proposition 4.7 that there exists a rational polyhedral cone  $\Pi$  in  $C_+$  whose  $Z_\Gamma(F)$ -orbit contains  $|\text{Star}_\Sigma(\sigma)|$ . Then Lemma 5.3(i) implies that  $Z_\Gamma(F) \cdot \pi^F(\Pi) = \pi^F(C_+)$ . The corollary now follows from Proposition 4.1.  $\square$

**Remark 5.5.** In contrast to the first assertion of the above corollary, it may happen that  $\pi_F(C)_+$  is strictly greater than  $\pi_F(C_+)$ .

**PROPOSITION 5.6.** *Let  $G$  be a face of  $C_+$  which contains  $F$ .*

(i) *The common zero set of the set of rational linear forms on  $V_G$  which are  $\geq 0$  on  $G$  and vanish on  $F$  is  $V^F \cap V_G$ .*

(ii) *The assignment  $F \mapsto F^\dagger$  sets up a bijection between the faces of  $C_+$  and those of  $C_+^\circ$  which reverses the inclusion relation. In particular,  $F^{\dagger\dagger} = F$  and  $F$  is an exposed face of  $C_+$ .*



(iii) The projections  $\pi_F$ , respectively  $\pi^F$ , map  $G$  onto a face of  $\pi_F(C_+)$ , respectively  $\pi^F(C_+)$ ; this sets up a bijection between the collection of faces of  $C_+$  which contain  $F$ , the collection of faces of  $\pi_F(C_+)$ , and the collection of faces of  $\pi^F(C_+)$ .

(iv) The dual of  $\pi^F(\overset{\circ}{G})$  is naturally identified with the closure of  $\pi_{G^\dagger}^\dagger(F^\dagger)$ .

*Proof.* We first prove (i) under the additional hypothesis that  $G = G^{\dagger\dagger}$ . (This assumption becomes superfluous once we have proved (ii).) We have to show that every rational linear form  $\xi$  on  $V_G$  which is  $\geq 0$  on  $G$  extends to a rational linear form on  $V$  which is  $\geq 0$  on  $C$ . Since  $G = G^{\dagger\dagger}$ , this follows from Corollary 5.4 applied to  $G^\dagger$ : such a  $\xi$  lies in  $\pi^G C_+^\circ$ .

We next prove a special case of (ii): we claim that if  $F^\dagger = \{0\}$ , then  $F = C_+$ . Choose  $x \in \overset{\circ}{F} \cap L$ . Then for every nonzero integral  $\xi \in C^*$  we have  $\xi(x) \geq 1$  (otherwise  $F^\dagger \neq \{0\}$ ) and hence  $x \in [(C^* - \{0\}) \cap L^*]^\vee$ . According to Proposition 4.9 the last set is a core for  $C$  and hence contained in  $C$ . So  $x \in C$  and hence  $F = C_+$ .

Now we prove (ii) in general. Clearly,  $F^{\dagger\dagger} = V^F \cap C_+ \supset F$ . We may apply the above to  $G := F^{\dagger\dagger}$  and find that there is no rational linear form on  $V_G$  which is  $\geq 0$  on  $G$  and zero on  $F$ . Then  $F = G$  by the special case.

(iii) Consider the face of  $\pi^F(C_+)$  whose relative interior contains  $\pi^F(G)$ . Its preimage  $H$  in  $C_+$  is then a face of  $C_+$  whose relative interior intersects  $(V_G + V^F) \cap C_+$ , and therefore also  $V^G \cap C_+$  (for we have  $V^G \supset V^F$ ). This last set is equal to  $G$  (by (ii)). So  $H = G$  and hence  $\pi^F(G)$  is a face of  $\pi^F(C_+)$ . The assertion follows from this.

(iv) By (ii),  $\pi^{G^\dagger}(C^\circ)$  can be regarded as the open dual of  $\overset{\circ}{G}$ . Then applying (ii) once more to the face  $F$  of  $G$  shows that  $\pi^F(\overset{\circ}{G})$  can be identified with the open dual of  $\pi^{G^\dagger}(C_+^\circ) \cap \text{Ann}(F)$ . By (iii), this last set is just  $\pi^{G^\dagger}(F^\dagger)$ . □

**COROLLARY 5.7.** Every rational linear form on the linear span of  $F$  which is  $\geq 0$  on  $F$  extends to a rational linear form on  $V$  which is  $\geq 0$  on  $C$ .

*Proof.* This follows from the fact that  $\pi_{F^\dagger}^\dagger$  maps  $C_+^\circ$  onto  $(\pi^{F^\dagger} C^\circ)_+$  (by Corollary 5.4 and the fact that  $\pi^{F^\dagger} C^\circ$  can be identified with the open dual of  $\overset{\circ}{F}$  (by Proposition 5.6(iii)). □

We shall need the following proposition.

**PROPOSITION 5.8.** Let a real affine space  $A$  of finite dimension, an affine lattice  $A_{\mathbb{Z}} \subset A$ , an open convex subset  $D$  of  $A$  and a group  $\Delta$  of affine-linear transformations of  $A$  which leave both  $A_{\mathbb{Z}}$  and  $D$  invariant be given. Assume that  $\Delta$  has only finitely many orbits in  $A_{\mathbb{Z}} \cap D$ . Then the asymptotic space of  $D$  coincides with its recession cone:  $\text{As}(D) = \text{T}(D)$  (i.e.,  $D + \text{As}(D) = D$ ),  $\text{As}(D)$  is defined over  $\mathbb{Q}$ , and  $\Delta$  acts on the affine space  $A/\text{As}(D)$  via a finite quotient.

*Proof.* If  $\text{As}(D) = \{0\}$ , then  $D$  is bounded and there is nothing to show. We therefore assume that  $\text{As}(D) \neq \{0\}$ . Then  $\text{T}(D) \neq 0$  [Roc70, Theorem 8.4]. We first show that  $\text{T}(D)$  is a linear space. If that is not the case, then let  $R \subset \text{T}(D)$  be a ray such that a line spanned by it is not contained in  $\text{T}(D)$ . Choose an open ball  $B \subset D$ . Then for any integer  $n > 0$ , there exists in  $B + R$  an interval  $[x_n, y_n]$  whose end points lie in  $A_{\mathbb{Z}}$  and which contains at least  $n + 1$  lattice points. Denote by  $\Phi$  the collection of affine-linear maps  $A \rightarrow \mathbb{R}$  that are integral on  $A_{\mathbb{Z}}$ , whose linear part is positive on  $\text{T}(D) - \text{As}(D)$  and whose minimum on  $D \cap A_{\mathbb{Z}}$  is 0. This is a nonempty  $\Delta$ -invariant set and so if  $x \in D \cap A_{\mathbb{Z}}$ , then the nonnegative integer  $\min_{f \in \Phi} f(x)$  only depends on the orbit  $\Delta x$ . We write  $m(\Delta x)$  for this number. Now for every  $f \in \Phi$ ,  $f(y_n) \geq n + f(x_n) \geq n$  and so  $m(\Delta y_n) \geq n$ . This contradicts the fact that  $\Delta$  has finitely many orbits in  $D \cap A_{\mathbb{Z}}$ .



The same argument shows that  $T(D)$  is defined over  $\mathbb{Q}$ . If  $\pi : A \rightarrow A/T(D)$  is the projection, then  $D = \pi^{-1}\pi D$ , and so we must have  $T(\pi D) = \{0\}$ . This implies that  $\pi(D)$  is bounded, in other words  $\text{As}(D) \subset T(D)$ . The opposite inclusion is clear.  $\square$

**COROLLARY 5.9.** *Suppose that in the situation of the previous proposition  $\Delta$  acts on  $D$  with compact fundamental domain. Then  $D = A$ .*

*Proof.* Following Proposition 5.8,  $\text{As}(D)$  is a subspace defined over  $\mathbb{Q}$ ,  $D + \text{As}(D) = D$  and  $\Delta$  acts on  $A/\text{As}(D)$  via a finite group. As it acts with compact fundamental domain on  $D/\text{As}(D)$ , it follows that  $D/\text{As}(D)$  is compact. But  $D/\text{As}(D)$  is open in the affine space  $A/\text{As}(D)$ , and so this can only happen if  $D/\text{As}(D)$  is a singleton, i.e., if  $D = A$ .  $\square$

**COROLLARY 5.10.** *The closure of  $F^\dagger$  in  $V^*$  is just the set of  $\xi \in C^*$  which vanish on  $F$  (and hence is an exposed face of  $C^*$ ), and  $\pi_F(C)$  is invariant under the translations in  $T_F$ .*

*Proof.* Let  $\tilde{A}$  be an affine subspace of  $V$  parallel to  $V^F$  which is defined over  $\mathbb{Q}$  and meets  $C$ . We denote the images of  $\tilde{A}$ ,  $L \cap \tilde{A}$  and  $C \cap \tilde{A}$  in  $V/V_F$  by  $A$ ,  $A_{\mathbb{Z}}$  and  $D$  respectively. It follows from Proposition 5.6 that  $D$  is also the image of  $C_+ \cap \tilde{A}$ . If  $\Sigma$  and  $\sigma$  are chosen as in Lemma 5.3, then according to that lemma the restriction of  $\pi_F(\Sigma_\sigma)$  to  $D$  is a decomposition into compact rational polyhedra which is finite modulo  $Z_\Gamma(F \times F^\dagger)$ . So Corollary 5.9 applies and we find that  $D = A$ . This proves that  $\pi_F(C)$  is invariant under the translations in  $T_F$ .

The set of  $\xi \in C^*$  which vanish on  $F$  is an exposed face of  $C^*$  which contains  $F^\dagger$ . Any such  $\xi$  can be regarded as a linear form on  $V/V_F$  which is nonnegative on  $\pi_F C$ . Since  $\pi_F C$  is invariant under translations in  $T_F$ , it follows that  $\xi$  vanishes on  $V^F$ . So  $\xi$  is in the linear span of  $F^\dagger$ . The latter intersects  $C^*$  in the closure of  $F^\dagger$ , and thus the corollary follows.  $\square$

**LEMMA 5.11.** *The unipotent elements in  $Z_\Gamma(F \times F^\dagger)$  form a normal subgroup  $U_\Gamma(F)$  of finite index.*

*Proof.* We first prove that the characteristic polynomial of any  $\gamma \in Z_\Gamma(F \times F^\dagger)$  is a product of cyclotomic polynomials. This suffices: since there are only finitely many such polynomials of given degree, it follows that the set of eigenvalues of elements of  $Z_\Gamma(F \times F^\dagger)$  is finite. Now choose a strictly increasing (Jordan–Hölder) filtration  $0 = W_0 \subset W_1 \subset \dots \subset V(\mathbb{C})$  invariant under  $Z_\Gamma(F \times F^\dagger)$  such that the image  $G_i$  of  $Z_\Gamma(F \times F^\dagger)$  in  $\text{GL}(W_i/W_{i-1})$  is irreducible. Clearly, the set of traces of elements of  $G_i$  is finite and a well-known fact of representation theory (see for instance [CR81, proof of Burnside’s theorem (36.1)]) then implies that  $G_i$  is finite. Hence the group of  $\gamma \in Z_\Gamma(F \times F^\dagger)$  that act trivially on the quotients  $G_i/G_{i-1}$  is of finite index in  $\Gamma$  and coincides with the set of its unipotent elements. (This argument was pointed out to me by O. Gabber.)

The characteristic polynomial of  $\gamma \in Z_\Gamma(F \times F^\dagger)$  has integral coefficients and so will be a product of cyclotomic polynomials once we show that every eigenvalue of  $\gamma$  has absolute value one. Suppose this is not so: let  $m > 1$  be the maximal absolute value that occurs and denote by  $W$  the corresponding eigenspace of  $\gamma$  in  $V$ . Since  $\gamma$  acts trivially on  $V/V^F$ , we have  $W \subset V^F$ . Now choose a half line in  $C$  which is not contained in a proper eigenspace of  $\gamma$ . Then the translates of this half line under the positive powers of  $\gamma$  have a limiting half line contained in  $W \cap \bar{C}$ , and hence contained in  $V^F \cap \bar{C}$ . According to Corollary 5.10, this last intersection equals the closure of  $F$  in  $V$ . So  $W \cap V^F \neq \{0\}$ . But this is impossible as  $\gamma$  leaves  $V_F$  pointwise fixed.  $\square$

It follows from Corollary 5.4 and Lemma 5.11 that ‘up to finite groups’  $N_\Gamma(F)$  is an extension of  $\Gamma(F) \times \Gamma(F^\dagger)$  by  $U_\Gamma(F)$ . We shall now concentrate on the action of the latter on  $V$ . We will find among other things that this group is abelian.

Most of our information is obtained via the following proposition. In this proposition we regard the space of rays in a vector space as the boundary (the sphere at infinity) of any affine space over that vector space.

PROPOSITION 5.12. *Let  $A$  be a real affine space of finite dimension (with translation space denoted  $T$ ),  $T_0 \subset T$  a linear subspace,  $C_0 \subset T_0$  a closed nondegenerate convex cone and  $U$  a unipotent group of affine-linear transformations of  $A$  which leaves  $T_0$  pointwise fixed. We assume (i) that  $A/T_0$  is spanned by some  $U$ -orbit and (ii) that there exists a nonempty open subset  $D \subset A$  with the property that for every  $a \in D$  and  $1 \neq u \in U$ , the rays  $\{\mathbb{R}_{\geq 0}(u^k(a) - a)\}_{u \in U}$  have a limiting ray in  $C_0$ . Put  $A' := A/T_0$  and  $T' := T/T_0$  and let  $a \in A \mapsto a' \in A'$ , respectively  $t \in T \mapsto t' \in T'$ , denote the obvious projections.*

*Then  $U$  acts faithfully on  $A'$  as a group of translations which spans  $T'$ ; in particular,  $U$  acts trivially on  $T'$ . Moreover, there exists a unique map*

$$\sigma : A' \times T' \rightarrow T$$

*with the following properties:*

- (a)  $\sigma$  is affine-linear in the first variable and for every  $a \in A$ ,  $\sigma_{a'} : T' \rightarrow T$  is a linear section of the projection  $T \rightarrow T'$  and so  $\sigma$  induces a bilinear map

$$d\sigma : T' \times T' \rightarrow T_0$$

*characterized by the property that for  $a' \in A'$  and  $t'_1, t'_2 \in T'$ ,  $\sigma(a' + t'_1, t'_2) = \sigma(a', t'_1) + d\sigma(t'_1, t'_2)$ ;*

- (b)  $d\sigma$  is a  $C_0$ -positive symmetric form in the sense that it is symmetric, and for every nonzero  $t' \in T'$ , we have  $d\sigma(t', t') \in C_0 - \{0\}$ ;
- (c) if  $u \in U$  is identified with  $[u] \in T'$ , then for all  $a \in A$ ,

$$u(a) = a + \sigma(\pi(a), [u]) + \frac{1}{2}d\sigma([u], [u]).$$

*Proof.* We use induction on  $\dim(T')$ . To start the induction, assume  $T' = \{0\}$ . Then  $U$  must act on  $A$  as a group of translations. As  $U$  preserves  $D$ , it follows that  $U = \{1\}$  and we are done.

From now on we assume  $T' \neq \{0\}$  and  $U \neq \{1\}$ . Then  $T_0 \neq \{0\}$ , for the orbit of a unipotent transformation is either a singleton or has a limiting point at infinity.

Since  $U$  is unipotent, we can find a  $U$ -invariant hyperplane  $T_1$  of  $T$  containing  $T_0$ . Then  $U$  acts trivially on  $T/T_1$  and hence acts on  $A/T_1$  as a group of translations. We denote the ensuing homomorphism  $U \rightarrow T/T_1$  by  $\alpha$  and write  $U_1$  for its kernel. Choose a  $T_1$ -orbit  $A_1$  in  $A$  which intersects  $D$ . Clearly  $U_1$  leaves  $A_1$  invariant and one verifies easily that the triple  $(A_1, D \cap A_1, U_1)$  fulfills the hypotheses of the proposition. So by induction  $U_1$  acts faithfully on  $A'_1 := A_1/T_0$  as its full group of translations.

We can now prove the first assertion. Choose  $a \in D$ , and put  $e_i := (u - 1)^i(a)$ . Then  $e_1 \in T$ ,  $e_2 \in T_1$ ,  $e_3 \in T_0$ , and  $e_i = 0$  for  $i \geq 4$ . So

$$u^k(a) = a + \binom{k}{1}e_1 + \binom{k}{2}e_2 + \binom{k}{3}e_3$$

for all  $k \in \mathbb{Z}$ . If  $e_3 \neq 0$ , then the rays  $\{\mathbb{R}_{\geq 0}(u^k(a) - a)\}_{k \geq 0}$ , respectively  $\{\mathbb{R}_{\geq 0}(u^{-k}(a) - a)\}_{k \geq 0}$ , converge to  $\mathbb{R}_{\geq 0}e_3$ , respectively  $\mathbb{R}_{\leq 0}e_3$ , and so  $C_0$  would contain  $\mathbb{R}e_3$ . This contradicts the nondegeneracy of  $C_0$ . So  $e_3 = 0$ . By a similar argument it follows that  $e_2 \in C_0$ . Since this is true for all  $a \in D$ , it follows that  $u$  induces a translation in  $A'$ . If this translation is trivial, then  $e_1 \in T_0$  and  $e_2 = 0$ . But then  $C_0$  contains both  $e_1$  and  $-e_1$ , and since  $C_0$  is nondegenerate, this can only happen when  $e_1 = 0$ , i.e., when  $u = 1$ . This proves that  $U$  acts faithfully on  $A'$  as a translation group. Since  $A'$  is spanned by some  $U$ -orbit, this translation group must span  $T'$ .

To prove the remaining assertions, fix  $a_0 \in A$ , and a linear section  $s : T' \rightarrow T$  of  $T \rightarrow T'$ . Then  $A$  is parameterized by

$$(t', t_0) \in T' \times T_0 \mapsto a_0 + s(t') + t_0 \in A.$$

In terms of this parameterization the action of  $U$  on  $A$  is then given by

$$u(a_0 + s(t') + t_0) = a_0 + s(t' + [u]) + t_0 + \phi_u(a_0 + s(t')),$$

where  $\phi_u$  is an affine-linear map from  $A$  to  $T_0$ . The map  $\phi_u$  factors over  $A \rightarrow A'$  and is independent of  $a_0$ . So we can write  $\phi_u(a) = \phi(a', u')$ . Then the fact that  $u, v \in U$  commute implies the symmetry of  $d\phi$ . In particular,  $\phi$  is linear in the second variable. Hence for any  $k \in \mathbb{Z}$ ,

$$u^k(a_0) = a_0 + ks(u') + k\phi_u(a_0) + \frac{1}{2}k(k-1)d\phi([u], [u]),$$

where  $u' \in T'$  denotes the image of  $u \in U$ .

Suppose  $u \in U - \{1\}$  is nonzero. If  $d\phi([u], [u]) = 0$ , then the displayed formula shows that the orbit  $\{u^k(a_0) \mid k \in \mathbb{Z}\}$  has two opposite limiting rays (spanned by  $\pm(s(u') + \phi_u(a_0))$ ), which evidently contradicts our assumption. So  $d\phi([u], [u]) = 0$  is nonzero, and the same formula above shows that it must belong to  $C_0$ . So if we define  $\sigma$  by  $\sigma(a', t') := s(t') + \phi(a', t')$ , then  $\sigma$  has the asserted properties (the uniqueness of  $\sigma$  is easy).  $\square$

We return to the face  $F$  and recall that  $T_F := V^F/V_F$ .

**COROLLARY 5.13.** *The group  $U_\Gamma(F)$  acts trivially on  $T_F$ , so that we can define homomorphisms of groups*

$$\begin{aligned} j : U_\Gamma(F) &\rightarrow \text{Hom}(V/V^F, T_F) \text{ such that } u(x') = x' + j_u q_F(x'), \\ k : U_\Gamma(F) &\rightarrow \text{Hom}(T_F, V_F) \text{ such that } u(y) = y + k_u \pi_F(y), \end{aligned}$$

with  $x \in V$  and  $y \in V^F$ . Moreover, if  $x \in C$ , then its image  $x''$  in  $V/V^F$  has the property that the map  $u \in U_\Gamma(F) \mapsto j_u(x'') \in T_F$  is an isomorphism of groups whose image spans  $T_F$ .

*Proof.* Let  $x'' \in C(F)$ , let  $A$  denote its preimage in  $V$ , and set  $D = A \cap C$ . Then  $D$  is a nonempty open convex subset of  $A$  and by Corollary 5.10 we have  $T(D) = \bar{C} \cap V^F = \bar{F}$ . It follows from Lemma 5.3(ii) that  $U_\Gamma(F)$  acts with compact fundamental set on the image of  $D$  in  $A/V_F$ . Hence Proposition 5.12 applies (with  $C = \bar{F}$ ) and we find that  $U_\Gamma(F)$  acts in  $A/V_F$  faithfully as a group  $T_F$  of translations and this group spans  $T_F$ .  $\square$

We can now complete the proof of Theorem 5.1.

*Proof of Theorem 5.1.* Let  $s : T_F \rightarrow V^F$  be a linear section of the projection. Since  $u \in U_\Gamma(F)$  acts trivially on  $V_F$ , and as  $x' \mapsto x' + j_u(x'')$  on  $V/V_F$ , it follows that there exists a  $\phi_u \in \text{Hom}(V/V_F, V_F)$  such that

$$u(x) = x + sj_u(x'') + \phi_u(x' + \frac{1}{2}j_u(x'')).$$

Since  $u$  acts on  $V^F$  as  $x \mapsto x + k_u(x')$ , the restriction of  $\phi_u$  to  $T_F$  must be  $k_u$ . So  $\phi_u j_u(x'') = \frac{1}{2}k_u j_u(x'')$ . If we set

$$\sigma_u(x') := s j_u(x'') + \phi_u(x'),$$

then it also follows that the restriction of  $\sigma_u$  to  $T_F$  is  $k_u$ . It is clear that the map  $V/V^F \rightarrow V/V_F$  induced by  $\sigma_u$  is precisely  $j_u$ . So  $\sigma_u$  has the property (i). The assertion that  $\sigma_u$  is unique for these properties is obvious.

If  $u, v \in U_\Gamma(F)$ , then

$$v(\sigma_u(x')) = \sigma_u(x') + k_v(\sigma_u(x')) = \sigma_u(x') + k_v j_u(x''),$$

and so

$$\begin{aligned} vu(x) &= v(x + \sigma_u(x') + \frac{1}{2}k_u j_u(x'')) \\ &= x + \sigma_v(x') + \frac{1}{2}k_v j_v(x') + \sigma_u(x') + k_v j_u(x'') + \frac{1}{2}k_u j_u(x'') \\ &= x + (\sigma_u + \sigma_v)(x') + \frac{1}{2}(k_u j_u + 2k_v j_u + k_v j_v)(x''). \end{aligned}$$

Since  $vu = uv$ , the symmetry property (ii) follows. As  $\sigma_{vu}$  is characterized by

$$vu(x) = x + \sigma_{vu}(x') + \frac{1}{2}(k_u + k_v)(j_u + j_v)(x'')$$

this also yields the linearity of  $\sigma$ . The same formula shows that for  $r \in \mathbb{Z}$ ,

$$u^r(x) = x + r\sigma_u(x') + \frac{1}{2}r^2 k_u j_u(x'').$$

Property (iii) follows from this. □

A case of interest is when  $\Gamma$  stabilizes a *proper* face  $F$  of  $C_+$  (i.e.,  $\{0\} \subsetneq F \subsetneq C_+$ ). Then Theorem 5.1 shows that  $\Gamma$  contains an extension of  $\Gamma(F) \times \Gamma(F^\dagger)$  by the abelian unipotent group  $U_\Gamma(F)$  (of rank equal to  $\dim T_F$ ) as a subgroup of finite index. If we take  $F$  minimal for this property, then  $\Gamma(F)$  leaves no proper face of  $F$  invariant and if we take  $F$  maximal for this property, then  $\Gamma(F^\dagger)$  leaves no proper face of  $F^\dagger$  invariant. In this way we can often reduce our discussion to the case when no proper face  $F$  of  $C_+$  is preserved by  $\Gamma$ .

We can take this one step further by reducing to the irreducible case, by which we mean that  $\Gamma$  does not leave invariant any proper subspace of  $V$  defined over  $\mathbb{Q}$ . In fact, if  $W \subset V$  is a proper  $\Gamma$ -invariant subspace defined over  $\mathbb{Q}$ , then we can distinguish three cases:

- (a) if  $W \cap \bar{C} = \{0\}$ , then the projection  $\pi_W : V \rightarrow V/W$  maps  $C$  onto a nondegenerate open cone and the triple  $(V/W, \pi_W C, \Gamma)$  is polyhedral;
- (b) if dually,  $W \cap C \neq \emptyset$ , then  $(W, C \cap W, \Gamma)$  is polyhedral;
- (c) if  $W$  meets  $C_+$  in a proper face, then  $\Gamma$  stabilizes this face, a case we discussed above.

Observe that in the first two cases  $\Gamma$  acts with finite kernel on  $V/W$ , respectively  $W$ . In case (b) this is clear, because  $W$  meets the locus where  $\Gamma$  acts properly discontinuously and case (a) then follows by duality.

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