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CONTROL OF CONTINUOUS-TIME SYSTEMS WITH DISCRETE JUMPS

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Abstract

In this paper, the design of output feedback controllers for linear systems under sampled measurements is investigated. The performance we use is the worst-case gain from disturbances to the controlled output, which comprises both a continuous-time and a discrete-time signal to be controlled. Control problems in both the finite and infinite horizon are addressed. Necessary and sufficient conditions for the existence of a suitable sampled-data output feedback controller are given in terms of two Riccati differential equations with finite discrete jumps. A numerical example is given to show the potential of the proposed technique.

1. Introduction

A sampled-data system is defined as one that operates on information or data obtained only at discrete-time points which we call sampling points. Although the use of sampled-data in control systems was recognized quite early in the history of feedback control systems, it was only from the early 1950s, when digital computers were first used in control systems, that significant attention was given to the development of analysis and design techniques of sampled-data control systems, see [1] and the references therein. The increasing use of digital computers in control systems has led to considerable activity in the field of sampled-data and digital control systems.

There are usually two approaches to the design of a digital controller for a continuous-time system. The first approach is to design a controller in continuous time and then to discretize it. The second approach is to discretize the continuous time plant in some way to obtain a discrete-time model and then to design a discrete-time controller using this discrete-time model. Note that these two approaches are more or

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less traditional [1]. The former is only an approximation whereas the latter considers the behavior of the system only at the sampling instants, and thus the intersample behavior is lost in the process of discretization [2, 23].

In sampled-data control, as the plant is continuous, it is highly desirable to have design performances in terms of continuous-time signals. The motivation for this is that it allows intersample behaviour to be directly taken into account in the control design. It is also desirable that the control design technique directly uses a continuous-time model of the plant without any transformation. A lot of attention has been recently paid to problems involving the intersample behavior of sampled-data systems [2–4, 6, 8–10, 22, 24, 25]. The stability issue of sampled-data systems has been addressed in [7] whereas the sampled-data control problems have been tackled using different performance measures. The H_2 optimal sampled-data control problem has been studied in [6, 8, 10] whereas the sampled-data control design with an H_∞ performance measure has been investigated by a number of researchers; see [2–4, 16–18, 21, 24, 25]. More recently, robust control of continuous-time systems with both discrete and continuous measurements has been considered in [14]. In particular, in [24, 25], a Riccati equation approach, similar to the one for continuous-time systems, has been proposed to solve the H_∞ control problem for sampled-data systems. It is observed that in [24, 25] only a continuous controlled output signal is taken into consideration. However, in many applications, it is important to include both a continuous-time and a discrete-time controlled output in the performance measure. This leads to a more general performance measure for sampled-data systems.

In this paper we consider the design of output feedback controllers for sampled-data systems. The performance measure we will use is the worst-case gain from the disturbances, which includes the disturbance input, measurement noise and unknown initial state, to the controlled output. This output comprises both a continuous-time and a discrete-time signal to be controlled. Control problems in both a finite and infinite horizon will be addressed. Necessary and sufficient conditions for the existence of a suitable H_∞ sampled-data output feedback controller are given in terms of two Riccati differential equations (RDEs) with finite discrete jumps. Our results extend the work of [24, 25] to consider a more general situation which allows us to handle both continuous-time and discrete-time controlled outputs. They can also be viewed as a unified treatment of continuous and discrete-time H_∞ control problems.

NOTATION. Most of the notation used in this paper is fairly standard. $\mathscr{R}^n$ and $\mathscr{R}^{n\times m}$ denote the n dimensional Euclidean space and the set of all $n \times m$ real matrices respectively. The superscript "T" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $L_2[0, T]$ stands for the space of square integrable vector functions over $[0, T]$, $l_2(0, T)$ is the space of square summable

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vector sequences over $(0, T)$, $\|\cdot\|_{[0,T]}$ will refer to the $L_2[0, T]$ norm over $[0, T]$ and $\|\cdot\|_{(0,T)}$ is the $l_2(0, T)$ norm over $(0, T)$. T is allowed to be ∞ and in this case by the notation $[0, T]$ we mean $[0, \infty)$. $F(\theta^-)$ and $F(\theta^+)$ stand for the left limit and right limit of a function $F(\theta)$, respectively.

2. Problem formulation and preliminaries

In this paper, we will deal with the output feedback H_∞ control of the following class of linear time-varying sampled-data systems:

$$(\Sigma): \quad \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \quad \forall t \in [0, T], \quad x(0) = x_0, \tag{2.1}$$

$$z(t) = C_1 x(t) + D_{12} u(t), \quad \forall t \in [0, T], \tag{2.2}$$

$$z_d(ih) = C_d x(ih), \quad \forall ih \in (0, T), \tag{2.3}$$

$$y(ih) = C_2 x(ih) + D_{21} v(ih), \quad \forall ih \in (0, T), \tag{2.4}$$

where $x(t) \in \mathscr{R}^n$ is the state of the system, x_0 is an unknown initial state, $w(t) \in \mathscr{R}^p$ is the disturbance input, $u \in \mathscr{R}^\ell$ is the control input, $y \in \mathscr{R}^m$ is the sampled measurement, $v \in \mathscr{R}^q$ is the measurement noise, $z \in \mathscr{R}^r$ is the controlled continuous output, $z_d \in \mathscr{R}^s$ is the controlled discrete output, $0 < h \in \mathscr{R}$ is the sampling period, i is a positive integer and A, B_1, B_2, C_1, C_2, C_d, D_{12} and D_{21} are known real time-varying bounded matrices of appropriate dimensions with A, B_1, B_2, C_1 and D_{12} being piecewise continuous.

We are concerned with designing a linear causal dynamic feedback control law for (2.1)–(2.4), based on the sampled output measurements of (2.4), that is, $u(ih + \tau) = \mathscr{G}\{y(kh), k = 0, 1, \ldots, i\}$, $0 \le \tau < h$, $i = 1, 2, \ldots$, such that the controller $\mathscr{G}$ reduces z uniformly for any w, v and x_0 in the sense that given a scalar $\gamma > 0$, the worst-case performance measure of a closed-loop system of (2.1)–(2.4) with the controller $\mathscr{G}$, defined by:

$$J_{\mathscr{K}}(\Sigma, R, T) = \sup_{w,v,x_0} \left\{ \left[\frac{\|z\|^2_{[0,T]} + \|z_d\|^2_{(0,T)}}{\|w\|^2_{[0,T]} + \|v\|^2_{(0,T)} + x_0^T R x_0} \right]^{1/2} \quad \text{and} \right.$$
$$\left. (w, v, x_0) \in L_2[0, T] \oplus \ell_2(0, T) \oplus \mathscr{R}^n : \|w\|^2_{[0,T]} + \|v\|^2_{(0,T)} + x_0^T R x_0 \neq 0 \right\}, \tag{2.5}$$

satisfies $J_{\mathscr{K}}(\Sigma, R, T) < \gamma$. In the above, $R = R^T > 0$ is a given weighting matrix for the initial state x_0. In this situation, the closed-loop system of (2.1)–(2.4) with $\mathscr{G}$ is said to have an H_∞ *performance* γ *over the horizon* $[0, T]$.

The control problem we address in this paper is as follows: *given a scalar* $\gamma > 0$, *design a linear causal controller* ($\mathscr{G}$) *based on the sampled measurements,* $y(ih)$, *such that*:

- *in the finite horizon case, the closed-loop system of* (2.1)–(2.4) *with* $\mathcal{G}$ *has an* H_∞ *performance* γ *over a given horizon* $[0, T]$;
- *in the infinite horizon case, that is,* $T \to \infty$, *the closed-loop system of* (2.1)–(2.4) *with* $\mathcal{G}$ *is uniformly exponentially stable and has an* H_∞ *performance* γ *over* $[0, \infty)$.

In the infinite horizon case, the controller is required to ensure the uniform exponential stability of the closed-loop system.

Note that the performance measure in (2.5) is in terms not only of the controlled signals at the sampling instants but also of the continuous-time controlled output between the sampling instants. This allows the intersampling behaviour to be taken into account in the control design. When only the controlled continuous output is considered, (2.5) will reduce to the performance measure used in [25].

REMARK 2.1. It should be remarked that (2.3)–(2.4) can be viewed as "mixed L_2/ℓ_2" output signals. In real environmental systems, we always face continuous-time systems, discrete-time systems, sampled-data systems and hybrid systems, that is, systems with both continuous- and discrete-time states. The study of this kind of system is motivated by robust sampled-data control, filtering and loop transfer recovery of sampled-data systems [20].

We shall make the following assumption for the system (Σ).

ASSUMPTION 2.1.

(a) $R_d = D_{12}^T D_{12} > 0$ *on* $[0, T]$.
(b) $R_D = D_{21} D_{21}^T > 0$ *on* $[0, T]$.

Note that Assumption 2.1(b) means that the sampled-data H_∞ control problem is "non-singular".

In the infinite horizon control problem, the system (2.1)–(2.4) is assumed to be time-invariant and we shall adopt the following assumption.

ASSUMPTION 2.2.

(a) *The continuous-time system* (A, B_2, C_1) *is stabilizable and detectable.*
(b) *The discrete-time system* $(e^{Ah}, \tilde{B}, C_2)$ *is stabilizable and detectable, where*

$$\tilde{B}\tilde{B}^T := \int_0^h e^{At} B_1 B_1^T e^{A^T t} dt.$$

Assumption 2.2 can be seen as the sampled-data systems counterpart of the standard detectability and stabilizability assumptions in the H_∞ control problems for continuous-time and discrete-time systems.

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In the remainder of this section we shall recall a version of the strict bounded real lemma for linear time-varying systems with finite discrete jumps which will be fundamental in the derivation of our main results.

Consider the linear time-varying system with finite discrete jumps:

$$(\Sigma^*): \qquad \dot{x}(t) = Ax(t) + Bw(t), \quad t \neq ih;\ x(0) = x_0 \tag{2.6}$$

$$x(ih) = A_d x(ih^-) + B_d v(ih), \quad \forall ih \in (0, T) \tag{2.7}$$

$$z(t) = Cx(t), \quad \forall t \in [0, T], \tag{2.8}$$

$$z_d(ih) = C_d x(ih^-), \quad \forall ih \in (0, T) \tag{2.9}$$

where $x(t) \in \mathscr{R}^n$ is the state of the system, x_0 is an unknown initial state, $w(t) \in \mathscr{R}^p$ and $v(ih) \in \mathscr{R}^q$ are the continuous and discrete inputs which belong to $L_2[0, T]$ and $\ell_2(0, T)$ respectively, $z(t) \in \mathscr{R}^r$ and $z_d(ih) \in \mathscr{R}^s$ are the continuous and discrete outputs, respectively, $0 < h \in \mathscr{R}$ is the sampling period, i is a positive integer and A, B, A_d, B_d, C and C_d are known real time-varying bounded matrices of appropriate dimensions with A, B and C being piecewise continuous.

Next, we introduce the following worst-case performance measure:

$$J(\Sigma^*, R, T) = \sup_{w,v,x_0} \left\{ \left[\frac{\|z\|^2_{[0,T]} + \|z_d\|^2_{(0,T)}}{\|w\|^2_{[0,T]} + \|v\|^2_{(0,T)} + x_0^T R x_0} \right]^{1/2} \quad \text{and} \right.$$
$$\left. (w, v, x_0) \in L_2[0, T] \oplus \ell_2(0, T) \oplus \mathscr{R}^n : \|w\|^2_{[0,T]} + \|v\|^2_{(0,T)} + x_0^T R x_0 \neq 0 \right\}, \tag{2.10}$$

where $R = R^T > 0$ is a given weighting matrix for x_0.

We now present a version of the strict bounded real lemma on the finite horizon for a system of the form of (Σ^*) and with performance measure (2.10).

THEOREM 2.1 ([19]). *Consider the system* (2.6)–(2.9) *and let* $\gamma > 0$ *be a given scalar. Then the following statements are equivalent*:

(a) $J(\Sigma^*, R, T) < \gamma$;

(b) *there exists a bounded matrix function* $P(t) = P^T(t) \geq 0, \forall t \in [0, T]$, *such that*

$$-\dot{P} = A^T P + PA + \gamma^{-2} PBB^T P + C^T C, \quad t \neq ih; \quad P(T) = 0, \tag{2.11}$$

$$\gamma^2 I - B_d^T P(ih^+) B_d > 0, \tag{2.12}$$

$$P(ih) = A_d^T P(ih^+) A_d + A_d^T P(ih^+) B_d \times [\gamma^2 I - B_d^T P(ih^+) B_d]^{-1} B_d^T P(ih^+) A_d + C_d^T C_d, \tag{2.13}$$

$$P(0^+) < \gamma^2 R; \tag{2.14}$$

(c) *there exists a bounded matrix function* $Q(t) = Q^T(t) > 0, \forall t \in [0, T)$, *such that*

$$-\dot{Q} > A^T Q + QA + \gamma^{-2} QBB^T Q + C^T C, \quad t \neq ih, \ Q(T) = 0, \tag{2.15}$$

$$\gamma^2 I - B_d^T Q(ih^+) B_d > 0, \tag{2.16}$$

$$Q(ih) > A_d^T Q(ih^+) A_d + A_d^T Q(ih^+) B_d \times [\gamma^2 I - B_d^T Q(ih^+) B_d]^{-1} B_d^T Q(ih^+) A_d + C_d^T C_d, \tag{2.17}$$

$$Q(0^+) < \gamma^2 R. \tag{2.18}$$

The next result presents an alternative version of the strict bounded real lemma on the finite horizon in terms of either an RDE, or inequality, of the "filtering form".

THEOREM 2.2 ([19]). *Consider the system* (2.6)–(2.9) *and let* $\gamma > 0$ *be a given scalar. Then the following statements are equivalent*:

(a) $J(\Sigma^*, R, T) < \gamma$;

(b) *there exists a bounded matrix function* $P(t) = P^T(t) \geq 0, \forall t \in [0, T]$, *such that*

$$\dot{P} = AP + PA^T + \gamma^{-2} PC^T CP + BB^T, \quad t \neq ih; \tag{2.19}$$

$$\gamma^2 I - C_d P(ih^-) C_d^T > 0, \tag{2.20}$$

$$P(ih) = A_d P(ih^-) A_d^T + A_d P(ih^-) C_d^T \times [\gamma^2 I - C_d P(ih^-) C_d^T]^{-1} C_d P(ih^-) A_d^T + B_d B_d^T, \tag{2.21}$$

$$P(0) = R^{-1}; \tag{2.22}$$

(c) *there exists a bounded matrix function* $Q(t) = Q^T(t) > 0, \forall t \in [0, T]$, *such that*

$$\dot{Q} > AQ + QA^T + \gamma^{-2} QC^T CQ + BB^T, \quad t \neq ih; \tag{2.23}$$

$$\gamma^2 I - C_d Q(ih^-) C_d^T > 0, \tag{2.24}$$

$$Q(ih) > A_d Q(ih^-) A_d^T + A_d Q(ih^-) C_d^T \times [\gamma^2 I - C_d Q(ih^-) C_d^T]^{-1} C_d Q(ih^-) A_d^T + B_d B_d^T, \tag{2.25}$$

$$Q(0) > R^{-1}. \tag{2.26}$$

In the next theorem we will deal with the infinite horizon case. We first present the infinite horizon counterpart of Theorem 2.1.

THEOREM 2.3 ([19]). *Consider the system* (2.6)–(2.9) *and let* $\gamma > 0$ *be a given scalar. Then the following statements are equivalent*:

(a) *the system* (2.6)–(2.9) *is stable and* $J(\Sigma^*, R, \infty) < \gamma$;

(b) *there exists a stabilizing solution* $P(t) = P^T(t) \geq 0, \forall t \in [0, \infty)$, *to* (2.11)–(2.13) *satisfying* (2.14);

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(c) *there exists a bounded time-varying matrix* $Q(t) = Q^T(t) > 0,\ t \in [0, \infty)$, *satisfying* (2.15)–(2.18) *over* $[0, \infty)$.

The infinite horizon version of Theorem 2.2 is provided in the following theorem.

THEOREM 2.4 ([19]). *Consider the system* (2.6)–(2.9) *and let* $\gamma > 0$ *be a given scalar. Then the following statements are equivalent*:

(a) *the system* (2.6)–(2.9) *is stable and* $J(\Sigma^*, R, \infty) < \gamma$;
(b) *there exists a stabilizing solution* $P(t) = P^T(t) \geq 0, \forall t \in [0, \infty)$, *to* (2.19)–(2.22);
(c) *there exists a bounded time-varying matrix* $Q(t) = Q^T(t) > 0, \forall t \in [0, \infty)$, *satisfying* (2.23)–(2.26) *over* $[0, \infty)$.

We end this section by introducing two technical lemmas that will be used to prove the main results of this paper.

LEMMA 2.1.

(a) *Suppose that for a given* $\tau > 0$, *the RDE*

$$-\dot{P}(t) = \bar{A}^T P + P\bar{A} + P(\gamma^{-2}B_1B_1^T - B_2B_2^T)P + C_1^T\left(I - D_{12}R_d^{-1}D_{12}\right)C_1, \quad t \neq ih,\ P(\tau) = S \geq 0, \tag{2.27}$$

$$P(ih) = P(ih^+) + C_d^T C_d, \tag{2.28}$$

where

$$\bar{A} = A - B_2R_d^{-1}D_{12}^TC_1, \tag{2.29}$$

has a bounded symmetric solution $P(t)$ *on* $[0, \tau]$ *satisfying* $P(0^+) < \gamma^2 R$ *for a given matrix* $R = R^T > 0$. *Then the RDE with jumps*

$$\dot{Q}(t) = AQ + QA^T + \gamma^{-2}QC_1^TC_1Q + B_1B_1^T, \quad t \neq ih;\ Q(0) = R, \tag{2.30}$$

$$Q(ih) = \left[Q^{-1}(ih^-) + C_2^TR_D^{-1}C_2 - \gamma^{-2}C_d^TC_d\right]^{-1}, \tag{2.31}$$

has a bounded symmetric positive definite solution $Q(t)$ *on* $[0, \tau]$ *and* $\rho(P(t)Q(t)) < \gamma^2, \forall t \in [0, \tau]$, *if and only if the RDE with jumps*

$$\dot{Z}(t) = A_zZ(t) + Z(t)A_z^T + \gamma^{-2}Z(t)C_z^TR_d^{-1}C_zZ(t) + B_1B_1^T, \quad t \neq ih,\ Z(0) = R[I - \gamma^{-2}P(0^+)R]^{-1}, \tag{2.32}$$

$$Z(ih) = \left[Z^{-1}(ih^-) + C_2^TR_D^{-1}C_2 - \gamma^{-2}C_d^TC_d\right]^{-1}, \tag{2.33}$$

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where

$$A_z := A + \gamma^{-2} B_1 B_1^T P, \qquad C_z = B_2^T P + D_{12}^T C_1,$$

has a bounded symmetric positive definite solution $Z(t)$ *on* $[0, \tau]$. *Furthermore,* $Z(t) = Q(t)\left[I - \gamma^{-2} P(t) Q(t)\right]^{-1}$, $\forall t \in [0, \tau]$.

(b) *Suppose the RDE with jumps* (2.27)–(2.28) *has a positive definite stabilizing solution* $P(t)$, *satisfying* $P(0^+) < \gamma^2 R$ *for a given matrix* $R = R^T > 0$. *Then the RDE with jumps* (2.30)–(2.31) *has a stabilizing solution* $Q(t) = Q^T(t) > 0$ *and* $\rho(P(t)Q(t)) < \gamma^2$, $\forall t \in [0, \infty)$, *if and only if the RDE with jumps* (2.32)–(2.33) *has a positive definite stabilizing solution* $Z(t)$. *Furthermore,* $Z(t) = Q(t)\left[I - \gamma^{-2} P(t) Q(t)\right]^{-1}$, $\forall t \in [0, \infty)$.

PROOF. When $t \neq ih$, the result follows directly from [12, 13]. For $t = ih$, using standard matrix manipulations it is easy to show that

$$\left[Z^{-1}(ih^-) + C_2^T R_D^{-1} C_2 - \gamma^{-2} C_d^T C_d\right]^{-1} = Q(ih)\left[I - \gamma^{-2} P(ih) Q(ih)\right]^{-1}, \quad \forall ih \in (0, \tau).$$

Hence we can conclude that results (a) and (b) hold.

LEMMA 2.2. *Consider the system* (2.1)–(2.4) *and let* $\gamma > 0$ *be a given scalar. Then we have the following results.*

(a) *Suppose that for a given* $\tau > 0$, *the RDE*

$$-\dot{P}(t) = \bar{A}^T P + P\bar{A} + P(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)P + C_1^T\left(I - D_{12} R_d^{-1} D_{12}^T\right) C_1, \quad t \neq ih, \; P(\tau) = S \geq 0, \tag{2.34}$$

$$P(ih) = P(ih^+) + C_d^T C_d, \tag{2.35}$$

where $\bar{A}$ *is as in* (2.29), *has a bounded solution* $P(t)$ *on* $[0, \tau]$ *satisfying* $P(0^+) < \gamma^2 R$ *for a given matrix* $R = R^T > 0$. *If there exists a linear causal controller* $\mathscr{K}$ *for* (Σ) *such that* $J_{\mathscr{K}}(\Sigma, R, \tau) < \gamma$ *then there exists a linear causal* H_∞ *filter* $\mathscr{F}$ *for estimating the signals* $z(t)$ *and* $z_d(ih)$ *of the system*

$$(\Sigma^a): \quad \begin{aligned} \dot{x}(t) &= (\bar{A} + \gamma^{-2} B_1 B_1^T P)x(t) + B_1 w(t), \; x(0) = x_0, \\ z(t) &= R_d^{-1/2}\left(B_2^T P + D_{12}^T C_1\right) x(t), \\ z_d(ih) &= C_d x(ih), \\ y(ih) &= C_2 x(ih) + D_{21} v(ih), \end{aligned}$$

such that $J_{\mathscr{F}}(\Sigma^a, \bar{R}, \tau) < \gamma$, *where* $\bar{R} = R - \gamma^{-2} P(0) > 0$ *and* $\bar{A}$ *is as in* (2.29).

(b) *Suppose that the RDE with jumps* (2.34)–(2.35) *has a non-negative definite stabilizing solution* $P(t)$ *satisfying* $P(0^+) < \gamma^2 R$ *for a given matrix* $R = R^T > 0$. *If there exists a linear causal stabilizing controller* $\mathscr{K}$ *for* (Σ) *such that* $J_{\mathscr{K}}(\Sigma, R, \infty) < \gamma$ *then there exists a linear causal stable* H_∞ *filter* $\mathscr{F}$ *for estimating the signals* $z(t)$ *and* $z_d(ih)$ *of the system* (Σ^a) *such that* $J_{\mathscr{F}}(\Sigma^a, \bar{R}, \infty) < \gamma$.

PROOF. The desired results in (a) and (b) can be established by using the same technique as that used in the proof of Theorem 6.2 of [24] with [12, 13], except that now we should add a discrete-time controlled output $z_d(ih)$ to the system.

3. H_∞ control of sampled-data systems

Our first result deals with the output feedback H_∞ control of the system (2.1)–(2.4) in the finite horizon case.

THEOREM 3.1. *Consider the system* (2.1)–(2.4) *and let* $\gamma > 0$ *be a given scalar. Then there exists a linear causal controller* $\mathscr{K}$ *such that the closed-loop system of* (2.1)–(2.4) *with* $\mathscr{K}$ *satisfies* $J_{\mathscr{K}}(\Sigma, R, T) < \gamma$ *if and only if the following conditions hold*:

(1) *there exists a bounded solution* $P(t)$ *over* $[0, T]$ *to the following RDE*

$$\begin{aligned} -\dot{P}(t) &= (A - B_2 R_d^{-1} D_{12}^T C_1)^T P(t) + P(t)(A - B_2 R_d^{-1} D_{12}^T C_1) \\ &\quad + P(t)(\gamma^{-2} B_1 B_1^T - B_2 R_d^{-1} B_2^T) P(t) \\ &\quad + C_1^T (I - D_{12} R_d^{-1} D_{12}^T) C_1, \quad t \neq ih, \end{aligned} \tag{3.1}$$

$$P(T) = 0, \tag{3.2}$$

$$P(ih) = P(ih^+) + C_d^T C_d, \quad \forall\, ih \in (0, T), \tag{3.3}$$

such that $P(0^+) < \gamma^2 R$;

(2) *there exists a bounded solution* $Q(t) = Q^T(t) > 0$ *over* $[0, T]$ *to the following RDE*

$$\dot{Q}(t) = A Q(t) + Q(t) A^T + \gamma^{-2} Q(t) C_1^T C_1 Q(t) + B_1 B_1^T, \quad t \neq ih, \tag{3.4}$$

$$Q(0) = R^{-1}, \tag{3.5}$$

$$Q(ih) = [Q^{-1}(ih^-) + C_2^T R_D^{-1} C_2 - \gamma^{-2} C_d^T C_d]^{-1}, \quad \forall\, ih \in (0, T); \tag{3.6}$$

(3) $\rho(P(t)Q(t)) < \gamma^2$ *for all* $t \in [0, T]$.

Moreover, if the above conditions hold, a suitable controller $\mathscr{K}$ *is as follows*:

$$\dot{x}_c(t) = [A + \gamma^{-2} B_1 B_1^T P(t) - B_2 K(t)] x_c(t), \quad t \neq ih, \quad x_c(0) = 0, \tag{3.7}$$

$$x_c(ih) = x_c(ih^-) + L(ih)[y(ih) - C_2 x_c(ih^-)], \quad \forall\, ih \in (0, T), \tag{3.8}$$

$$u(t) = -K(t) x_c(t), \quad \forall\, t \in [0, T], \tag{3.9}$$

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where

$$K(t) = R_d^{-1}\left(B_2^T P(t) + D_{12}^T C_1\right) \quad \textit{and}$$
$$L(ih) = Q(ih)\left[I - \gamma^{-2} P(ih) Q(ih)\right]^{-1} C_2^T R_D^{-1}, \quad \forall\, ih \in (0, T).$$

PROOF. For simplicity of presentation, we shall assume $\gamma = 1$ in the following.

Sufficiency.

We first note that by Lemma 2.1 (a), it follows from conditions (1)–(3) that there exists a bounded piecewise-differentiable matrix function $Z(t) = Z^T(t) > 0$, $\forall t \in [0, T]$, satisfying

$$\dot{Z}(t) = \hat{A}Z + Z\hat{A}^T + Z\hat{B}\hat{B}^T Z + B_1 B_1^T, \quad t \neq ih,$$
$$Z(0) = [R - P(0)]^{-1},$$
$$Z(ih) = \left[Z^{-1}(ih^-) + C_2^T R_D^{-1} C_2 Z(ih^-) - C_d^T C_d\right]^{-1}, \quad \forall ih \in (0, T).$$

By using standard matrix manipulations, we can rewrite the last equation as

$$Z(ih) = A_d(ih) Z(ih^-) A_d^T(ih) + B_d(ih) B_d^T(ih), \quad \forall\, ih \in (0, T)$$

where

$$\hat{A} \triangleq A + B_1 B_1^T P, \qquad \hat{B} \triangleq \gamma^{-1}(B_2^T P + D_{12}^T C_1)^T R_d^{-1/2},$$
$$A_d(ih) \triangleq I - L(ih) C_2, \qquad B_d(ih) \triangleq L(ih) D_{21}. \tag{3.10}$$

In view of Theorem 2.2, this implies that the linear system with jumps

$$\dot{\xi}(t) = \hat{A}\xi(t) + B_1 \hat{w}(t), \quad t \neq ih;\ \xi(0) = \xi_0,$$
$$\xi(ih) = A_d(ih)\xi(ih^-) + B_d(ih)\hat{v}(ih),$$
$$\nu(t) = \hat{B}^T \xi(t),$$

satisfies

$$\sup\left\{\left[\frac{\|\nu\|^2_{[0,T]}}{\|\hat{w}\|^2_{[0,T]} + \|\hat{v}\|^2_{(0,T)} + \xi_0^T [R - P(0^+)]\xi_0}\right]^{1/2}\right\} < 1, \tag{3.11}$$

where the supremum is taken over all $\hat{w} \in L_2[0, T]$, $\hat{v} \in \ell_2(0, T)$, $\xi_0 \in \mathscr{R}^n$ such that $\|\hat{w}\|^2_{[0,T]} + \|\hat{v}\|^2_{(0,T)} + \xi_0^T [R - P(0^+)]\xi_0 \neq 0$.

Using Theorem 2.1, it follows that there exists a bounded piecewise differentiable matrix function $W(t) = W^T(t) \geq 0, \forall t \in [0, T]$, satisfying

$$- \dot{W}(t) = \hat{A}^T W + W\hat{A} + WB_1B_1^T W + \hat{B}\hat{B}^T, \quad t \neq ih;\ W(T) = 0, \tag{3.12}$$

$$I - B_d^T W(ih^+)B_d > 0, \tag{3.13}$$

$$W(ih) = A_d^T W(ih^+)A_d + A_d^T W(ih^+)B_d \left[I - B_d^T W(ih^+)B_d\right]^{-1} \times B_d^T W(ih^+)A_d, \tag{3.14}$$

$$W(0^+) < R - P(0^+). \tag{3.15}$$

We now consider the closed-loop system associated with system (2.1)–(2.4) and controller (3.7)–(3.9). This system is described by the state equations

$$\dot{\eta}(t) = \bar{A}(t)\eta(t) + \bar{B}w(t), \quad t \neq ih;\ \eta(0) = \eta_0, \tag{3.16}$$

$$\eta(ih) = \bar{A}_d(ih)\eta(ih^-) + \bar{B}_d(ih)v(ih), \tag{3.17}$$

$$z(t) = \bar{C}\eta(t), \tag{3.18}$$

$$z_d(ih) = \bar{C}_d\eta(ih), \tag{3.19}$$

where

$$\eta := \begin{bmatrix} x \\ x - x_c \end{bmatrix}, \quad \eta_0 := \begin{bmatrix} x_0 \\ x_0 \end{bmatrix},$$

$$\bar{A} := \begin{bmatrix} A - B_2K(t) & B_2K(t) \\ -B_1B_1^TP(t) & A + B_1B_1^TP(t) \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} B_1 \\ B_1 \end{bmatrix}, \tag{3.20}$$

$$\bar{A}_d(ih) := \begin{bmatrix} I & 0 \\ 0 & I - L(ih)C_2 \end{bmatrix}, \quad \bar{B}_d(ih) := \begin{bmatrix} 0 \\ -L(ih)D_{21} \end{bmatrix} \quad \text{and} \tag{3.21}$$

$$\bar{C}(t) := \begin{bmatrix} C_1 - D_{12}K(t) & D_{12}K(t) \end{bmatrix}, \quad \bar{C}_d := \begin{bmatrix} C_d & 0 \end{bmatrix}. \tag{3.22}$$

Next we define the time-varying matrix $X(t) = X^T(t) \geq 0, \forall t \in [0, T]$, that is,

$$X(t) = \begin{bmatrix} P(t) & 0 \\ 0 & W(t) \end{bmatrix}. \tag{3.23}$$

Note that $X(t)$ is bounded and piecewise-differentiable on $[0, T]$. Using (3.1)–(3.3), (3.12)–(3.14) and (3.20)–(3.22), it can be easily verified that $X(t)$, $\forall t \in [0, T]$, satisfies

$$- \dot{X}(t) = \bar{A}^T X + X\bar{A} + X\bar{B}\bar{B}^T X + \bar{C}^T\bar{C}, \quad t \neq ih;\ X(T) = 0, \tag{3.24}$$

$$I - \bar{B}_d^T(ih)X(ih^+)\bar{B}_d(ih) > 0, \tag{3.25}$$

$$X(ih) = \bar{A}_d^T(ih)X(ih^+)\bar{A}_d(ih) + \bar{A}_d^T X(ih^+)\bar{B}_d \times \left[I - \bar{B}_d^T(ih)X(ih^+)\bar{B}_d(ih)\right]^{-1} \bar{B}_d^T(ih)X(ih^+)\bar{A}_d(ih) + \bar{C}_d^T\bar{C}_d. \tag{3.26}$$

As $W(0^+) < R - P(0^+)$, there exists a sufficiently small scalar $\delta > 0$ such that

$$X(0^+) < \bar{R} = \begin{bmatrix} P(0^+) + \delta I & 0 \\ 0 & R - P(0^+) - \delta I \end{bmatrix}. \tag{3.27}$$

Finally, using Theorem 2.1 and considering that $\eta_0^T \bar{R} \eta_0 = x_0^T R x_0$, we can conclude that the closed-loop system (3.16)–(3.19) satisfies $J_{\mathscr{K}}(\Sigma, R, T) < 1$.

Necessity.

In order to establish the existence of a bounded solution $Q(t)$ to (3.4)–(3.6) over $[0, T]$, we first consider the case when $t \in [0, h)$. The existence of a linear causal controller $\mathscr{K}$ which guarantees that the closed-loop system of (2.1)–(2.4) achieves $J_{\mathscr{K}}(\Sigma, R, T) < 1$ implies

$$\sup_{w, x_0} \left\{ \left[\frac{\|z\|^2_{[0,h)}}{\|w\|^2_{[0,h)} + x_0^T R x_0} \right], (w, x_0) \in L_2[0, h) \oplus \mathscr{R}^n : \|w\|^2_{[0,h)} + x_0^T R x_0 \neq 0 \right\} < 1.$$

Hence, by Theorem 4.1 in [24], it follows that there exists a bounded matrix function $Q(t) = Q^T(t) > 0, \forall t \in [0, h)$, satisfying (3.4)–(3.5) over $[0, h)$.

We now consider $t \in [h, 2h)$. Choosing $v(h) = -D_{21}^T R_D^{-1} C_2 x(h)$, we have $y(h) = 0$, which implies that $u(t) = 0$ over $[0, 2h)$. Then from $J_{\mathscr{K}}(\Sigma, R, T) < 1$ we obtain that for any $(x_0, w) \in \mathscr{R}^n \oplus L_2[0, h)$ such that $\|w\|^2_{[0,h]} + x_0^T R x_0 \neq 0$,

$$\|w\|^2_{[0,h)} + x_0^T R x_0 + x(h)^T C_2^T R_D^{-1} C_2 x(h) - \|z\|^2_{[0,h)} - x(h)^T C_d^T C_d x(h) > 0. \tag{3.28}$$

Next, choosing $w(t) = B_1^T Q^{-1}(t) x(t), \forall t \in [0, h)$, and using (3.4)–(3.5), completing the squares gives us

$$\begin{aligned} 0 &= \int_0^{h^-} \frac{d}{dt} \left[x^T(t) Q^{-1}(t) x(t) \right] dt - x^T(h) Q^{-1}(h^-) x(h) + x_0^T R x_0 \\ &= \|w\|^2_{[0,h)} + x_0^T R x_0 - \|z\|^2_{[0,h)} - x^T(h) Q^{-1}(h^-) x(h). \end{aligned} \tag{3.29}$$

By combining (3.28) and (3.29) we get

$$x^T(h) \left[Q^{-1}(h^-) + C_2^T R_D^{-1} C_2 - C_d^T C_d \right] x(h) > 0.$$

Since x_0 is arbitrary, it follows that under the above conditions $x(h)$ is arbitrary as well. Then we have that

$$Q^{-1}(h^-) + C_2^T R_D^{-1} C_2 - C_d^T C_d > 0.$$

Therefore the matrix $Q(h)$ as below is well defined and

$$Q(h) = \left[Q^{-1}(h^-) + C_2^T R_D^{-1} C_2 - C_d^T C_d \right]^{-1} > 0. \tag{3.30}$$

Next, $J_{\mathscr{K}}(\Sigma, R, T) < 1$ leads to

$$\sup\left\{\|w\|^2_{[0,2h)} + \|v(h)\|^2 + x_0^T R x_0 - \|z\|^2_{(0,2h)} - \|z_d(h)\|^2\right\} > 0,$$

where the supremum is taken over all $w \in L_2[0, 2h)$, $v(h) \in \mathscr{R}^q$, $x_0 \in \mathscr{R}^n$ such that $\|w\|^2_{[0,2h)} + \|v(h)\|^2 + x_0^T R x_0 \neq 0$.

Using the above $v(h)$ and $w(t)$, $\forall\, t \in [0, h)$, and considering (3.29) and (3.30), we obtain that

$$\sup_{x(h),w}\left\{\|w\|^2_{[h,2h)} + x^T(h)Q(h)x(h) - \|z\|^2_{[h,2h)}\right\} > 0,$$

where the supremum is taken over all $x(h) \in \mathscr{R}^n$ and $w \in L_2[h, 2h)$ such that $|w\|^2_{[h,2h)} + x^T(h)Q(h)x(h) \neq 0$. Hence by Theorem 4.1 in [24] it follows that there exists a bounded positive definite solution $Q(t)$ to (3.4) over $[h, 2h)$ with initial condition $Q(h)$.

By repeating the above procedure for $t \in [2h, T]$, we conclude that there exists a bounded piecewise differentiable matrix function $Q(t) = Q^T(t) > 0$, $\forall\, t \in [0, T]$, which satisfies (3.4)–(3.6).

We will now show the existence of a bounded solution $P(t) = P^T(t) \geq 0$, $\forall t \in [0, T]$, to (3.1)–(3.3) over $[0, T]$. To this end, initially, we shall set $x_0 = 0$ and consider the case when $t \in (Nh, T]$, where N is the largest integer such that $Nh < T$. The existence of a linear causal controller $\mathscr{K}$ such that $J_{\mathscr{K}}(\Sigma, R, T) < 1$ implies

$$\sup_{w}\left\{\frac{\|z\|^2_{(Nh,T]}}{\|w\|^2_{(Nh,T]}},\ w \in L_2(Nh, T];\ \|w\|_{(Nh,T]} \neq 0\right\} < 1.$$

By Theorem 4.1 in [24], and [12, 13], there exists a bounded non-negative definite solution $P(t)$ to (3.1)–(3.2) over $(Nh, T]$. In addition, we define

$$P(Nh) := P(Nh^+) + C_d^T C_d. \tag{3.31}$$

We now consider the interval $(Nh-h, Nh]$. Since there exists a linear causal controller $\mathscr{K}$ such that $J_{\mathscr{K}}(\Sigma, R, T) < 1$, taking $w(t) \equiv 0$ over $[0, Nh - h]$ and $[Nh, T]$, and setting $v(ih) = 0$, for $i = 1, 2, \ldots, Nh - h$, implies that

$$\frac{\|z\|^2_{(Nh-h,Nh]} + \|z_d(Nh)\|^2}{\|w\|^2_{(Nh-h,Nh]} + \|v(Nh)\|^2} < 1.$$

Again, by Theorem 4.1 in [24], and [12, 13], there exists a bounded non-negative definite solution $P(t)$ to (3.1)–(3.2) over $(Nh - h, Nh]$. We can define

$$P[(N-1)h] := P[(N-1)h^+] + C_d^T C_d. \tag{3.32}$$

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By repeating the above procedure for $t \in [0, Nh - h)$, we conclude that there exists a bounded matrix function $P(t) = P^T(t) \geq 0, \forall t \in [0, T]$, satisfying (3.1)–(3.3).

The proof of $P(0^+) < R$ can be carried out using the same arguments as those in the proof of a similar result in Theorem 2.1.

Finally, the condition $\rho(P(t)Q(t)) < 1$ for all $t \in [0, T]$ follows from Lemma 2.2(a), Theorem 3.1 [15] and Lemma 2.1(a).

Theorem 3.1 shows that as in standard output feedback H_∞ control for continuous-time systems, two RDEs are needed to solve the sampled-data output feedback H_∞ control problem. However, due to the sampled measurements and the existence of a discrete-time output to be controlled along with a continuous-time output, here the RDEs have finite discrete jumps. One of the RDEs is used for the state estimation and another is used for the controller design. The proposed controller is a linear time-varying system with jumps. It has the structure of an observer-based controller, where the observer is an H_∞ sampled-data filter as in [15], and with a control law $u = -K(t)x_c(t)$.

The next theorem deals with the sampled-data H_∞ control problem on the infinite horizon.

THEOREM 3.2. *Consider the system* (2.1)–(2.4) *and let* $\gamma > 0$ *be a given scalar. Then there exists a linear causal controller* $\mathscr{K}$ *such that the closed-loop system of* (2.1)–(2.4) *with* $\mathscr{K}$ *is uniformly exponentially stable and satisfies* $J_{\mathscr{K}}(\Sigma, R, \infty) < \gamma$ *if and only if the following conditions hold.*

(1) *There exists a stabilizing solution* $P(t) = P^T(t) \geq 0, \forall t \in [0, \infty)$, *to* (3.1) *and* (3.3) *such that* $P(0^+) < \gamma^2 R$.
(2) *There exists a stabilizing solution* $Q(t) = Q^T(t) > 0, \forall t \in [0, \infty)$, *to* (3.4)–(3.6).
(3) $\rho(P(t)Q(t)) < \gamma^2$ *for all* $t \in [0, \infty)$.

Moreover, if the above conditions hold, a suitable controller $\mathscr{K}$ *is given by* (3.7)–(3.9) *with* $T \to \infty$.

PROOF. *Sufficiency.*

The proof can be carried out using the same arguments as those in the sufficiency proof of Theorem 3.1, except that Lemma 2.1(b) and Theorems 2.3 and 2.4 need to be utilized instead of Lemma 2.1(a) and Theorems 2.1 and 2.2, respectively. Furthermore, here we also need to assert the exponential stability of the closed-loop system (3.16)–(3.19).

Following the proof of Theorem 3.1, it can be seen that there exists a stabilizing solution $W(t) = W^T(t) \geq 0, \forall t \in [0, \infty)$, to (3.12)–(3.14) satisfying (3.15). Moreover, the matrix function $X(t)$ of (3.23) is bounded, non-negative definite and piecewise differentiable on $[0, \infty)$, and satisfies (3.24)–(3.27) with $T \to \infty$.

We now show that $X(t)$ is the stabilizing solution to (3.24)–(3.26). Using (3.10), (3.20) and (3.21), it is easy to show that

$$\begin{aligned}\tilde{A} &:= \bar{A} + \bar{B}\bar{B}^T X \\ &= \begin{bmatrix} A + B_1 B_1^T P(t) - B_2 K(t) & B_2 K(t) + B_1 B_1^T W \\ 0 & \hat{A} + B_1 B_1^T W \end{bmatrix} \quad \text{and} \\ \tilde{A}_d(ih) &:= \bar{A}_d(ih) + \bar{B}_d(ih)\left[I - \bar{B}_d^T(ih) X(ih^+) \bar{B}_d(ih)\right]^{-1} \bar{B}_d^T(ih) X(ih^+) \bar{A}_d(ih) \\ &= \begin{bmatrix} I & 0 \\ 0 & A_d + B_d\left[I - B_d^T W(ih^+) B_d\right]^{-1} B_d^T W(ih^+) A_d \end{bmatrix}.\end{aligned}$$

Since $P(t)$ and $W(t)$ are the stabilizing solutions to (3.1)–(3.3) and (3.12)–(3.15), respectively, it follows that the linear system with jumps

$$\begin{aligned}\dot{x} &= \tilde{A}(t)x(t), \quad t \neq ih \\ x(ih) &= \tilde{A}_d(ih)x(ih^-)\end{aligned}$$

is uniformly exponentially stable. This implies that $X(t)$ is the stabilizing solution to (3.24)–(3.26). Therefore, using Theorem 2.3 we conclude that the closed-loop system (3.16)–(3.19) associated with the system (2.1)–(2.4) and controller (3.7)–(3.9) is uniformly exponentially stable and satisfies $J_{\mathscr{K}}(\Sigma, R, \infty) < 1$.

Necessity.

Since there exists a linear causal controller $\mathscr{K}$ such that $J_{\mathscr{K}}(\Sigma, R, \infty) < 1$, by Theorem 4.2 in [24] and [12,13], there exists a nonnegative definite stabilizing solution P satisfying (3.1). Then by Theorem 4.1 in [24], and [12, 13], for any given $T > 0$, there exists a bounded nonnegative definite $P(t)$ on $[0, T]$ that satisfies (3.1). Hence using Theorem 2.8 of [11], $\lim_{t\to\infty} P(t) = P$, that is, $P(t)$ is bounded on $[0, \infty)$.

Hence the existence of a bounded $P(t) \geq 0$, $t \in [0, \infty)$, to (3.1)–(3.3) such that $P(0^+) < R$ follows from the proof of Theorem 3.1.

In the following, we will show that $P(t)$, $t \in [0, \infty)$, is the stabilizing solution to (3.1)–(3.3).

Since $\mathscr{K}$ is a causal admissible controller over $[0, \infty)$, it is also admissible over $[0, T]$ for any $T > 0$. Hence we have that

$$J_{\mathscr{K}}(\Sigma, R, T) = \sup_{w,v,x_0} \left\{ \left[\frac{\|z\|^2_{[0,T]} + \|z_d\|^2_{(0,T)}}{\|w\|^2_{[0,T]} + \|v\|^2_{(0,T)} + x_0^T R x_0} \right]^{1/2} \quad \text{and} \right.$$
$$\left. (w, v, x_0) \in L_2[0, T] \oplus \ell_2(0, T) \oplus \mathscr{R}^n : \ \|w\|^2_{[0,T]} + \|v\|^2_{(0,T)} + \|x_0\|^2 \neq 0 \right\} < 1. \tag{3.33}$$

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We define

$$S_T z(t) := \begin{cases} z(t) & 0 \le t \le T \\ 0 & t > T, \end{cases}$$

$$S_T z_d(ih) := \begin{cases} z_d(ih) & 0 < ih < T \\ 0 & ih \ge T. \end{cases}$$

When $x_0 = 0$, (3.33) implies that there exists an $\varepsilon > 0$ such that

$$\begin{aligned} &\|S_T z\|^2_{[0,T]} + \|S_T z_d\|^2_{(0,T)} - \|S_T w\|^2_{[0,T]} - \|S_T v\|^2_{(0,T)} \\ &\quad \le -\varepsilon \left[\|S_T w\|^2_{[0,T]} + \|S_T v\|^2_{(0,T)}\right]. \end{aligned} \tag{3.34}$$

When $x_0 \ne 0$, we can set the initial state of the controller to be zero. As $\mathscr{K}$ is linear, the output $z(t)$ and $z_d(ih)$ to any inputs w and v can be written as

$$z(t) = z_1(t) + z_2(t), \quad t \ne ih, \tag{3.35}$$

$$z_d(ih) = z_{d1}(ih) + z_{d2}(ih), \quad ih \in (0, T), \tag{3.36}$$

where $z_1(t)$ and $z_{d1}(ih)$ are the homogeneous parts of $z(t)$ and $z_d(ih)$, respectively (depending only on x_0) and $z_2(t)$ and $z_{d2}(ih)$ are the forced parts of $z(t)$ and $z_d(ih)$, respectively (depending only on w and v). It is easy to see that

$$\|S_T z\|^2_{[0,T]} \le \|S_T z_1\|^2_{[0,T]} + \|S_T z_2\|^2_{[0,T]} + 2\|S_T z_1\|_{[0,T]}\|S_T z_2\|_{[0,T]} \quad \text{and} \tag{3.37}$$

$$\|S_T z_d\|^2_{(0,T)} \le \|S_T z_{d1}\|^2_{(0,T)} + \|S_T z_{d2}\|^2_{(0,T)} + 2\|S_T z_{d1}\|_{(0,T)}\|S_T z_{d2}\|_{(0,T)}. \tag{3.38}$$

Since the closed-loop system of (2.1)–(2.4) with the controller (3.7)–(3.9) is exponentially stable, this implies that both z_1 and z_2 are in $L_2[0, T]$ and both z_{d1} and z_{d2} are in $\ell_2(0, T)$ for any $(w, v, x_0) \in L_2[0, T] \oplus \ell_2(0, T) \oplus \mathscr{R}^n$. This means there exist constants $\alpha, \varepsilon > 0$ which are independent of T, such that

$$\|S_T z_1\|^2_{[0,T]} \le \alpha\|x_0\|, \quad \|S_T z_{d1}\|^2_{(0,T)} \le \alpha\|x_0\|$$

and

$$\begin{aligned} &\|S_T z_2\|^2_{[0,T]} + \|S_T z_{d2}\|^2_{(0,T)} - \|S_T w\|^2_{[0,T]} - \|S_T v\|^2_{(0,T)} \\ &\quad \le -\varepsilon \left[\|S_T w\|^2_{[0,T]} + \|S_T v\|^2_{(0,T)}\right]. \end{aligned} \tag{3.39}$$

Hence we have that

$$\begin{aligned} &\|S_T z\|^2_{[0,T]} + \|S_T z_d\|^2_{(0,T)} - \|S_T w\|^2_{[0,T]} - \|S_T v\|^2_{(0,T)} \\ &\quad \le 2\alpha^2\|x_0\| - \varepsilon\left(\|S_T w\|^2_{[0,T]} + \|S_T v\|^2_{(0,T)}\right) \\ &\qquad + 2\alpha\|S_T z_2\|_{[0,T]}\|x_0\| + 2\alpha\|S_T z_{d2}\|_{(0,T)}\|x_0\| \\ &\quad \le 2\alpha^2\|x_0\| - \left(\|S_T w\|^2_{[0,T]} + \|S_T v\|^2_{(0,T)}\right)^{1/2} \\ &\qquad \times \left[\varepsilon\left(\|S_T w\|^2_{[0,T]} + \|S_T v\|^2_{(0,T)}\right)^{1/2} - 4\alpha\|x_0\|\right]. \end{aligned} \tag{3.40}$$

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We next consider the following system

$$\dot{\eta}(t) = \left[A + B_1 B_1^T P(t) - B_2 K(t)\right] \eta(t). \tag{3.41}$$

If we set $w(t) = B_1^T P(t) x(t)$ then the system (3.41) can be rewritten as

$$\dot{x}(t) = A_c x(t) + B_1 w(t), \tag{3.42}$$

where $A_c = A - B_2 K(t)$.

Now we claim that there exists $\zeta > 0$ such that

$$\|w\|^2_{[0,T]} \le \zeta \|x_0\|^2. \tag{3.43}$$

By contradiction, suppose that for any $\kappa > 0$ there exist x_0 and $T > 0$ such that

$$\|w\|^2_{[0,T]} > \kappa^2 \|x_0\|^2. \tag{3.44}$$

Let

$$\kappa \ge \frac{2\alpha}{\varepsilon}\left[1 + \left(1 + \frac{\sqrt{\varepsilon}}{2}\right)\right]. \tag{3.45}$$

Then it follows that from (3.40) (setting $v(ih) \equiv 0$) that

$$\|S_T z\|^2_{[0,T]} + \|S_T z_d\|^2_{(0,T)} - \|S_T w\|^2_{[0,T]} < \left[2\alpha^2 - \kappa^2\left(\varepsilon - \frac{4\alpha}{\kappa}\right)\right] \|x_0\|^2. \tag{3.46}$$

Note that since κ satisfies (3.45), we obtain

$$\|S_T z\|^2_{[0,T]} + \|S_T z_d\|^2_{(0,T)} - \|S_T w\|^2_{[0,T]} < 0.$$

By Corollary 2.1 [15], we have that

$$x_0^T P(0^+) x_0 = \|S_T z\|^2_{[0,T]} + \|S_T z_d\|^2_{(0,T)} - \|S_T w\|^2_{[0,T]} < 0, \tag{3.47}$$

which contradicts the fact that $P(t) \ge 0$, $\forall t \in [0, T]$. Therefore, (3.43) holds.

Since the closed-loop system of (2.1)–(2.4) with the controller (3.7)–(3.9) is stable, it follows that there exists $\delta_1 > 0$ such that

$$\|x\|_{[0,T]} \le \delta_1 \|w\|_{[0,T]} \le \delta_1 \sqrt{\zeta} \|x_0\|, \tag{3.48}$$

where the bound is independent of T. The exponential stability of system (3.41) follows immediately from [5]. Hence we conclude that $P(t)$ is a stabilizing solution to (3.1)–(3.3).

The existence of a stabilizing solution $Q(t) > 0$, $t \in [0, \infty)$, to (3.4)–(3.6) can be carried out by Lemma 2.2(b).

Finally, the condition $\rho(P(t)Q(t)) < 1$, $t \in [0, \infty)$, follows from Lemma 2.2(b), Theorem 3.2 [15] and Lemma 2.1(b).

Inspired by the result in [25], when the initial state of system (2.1)–(2.4) is zero and the infinite horizon is concerned, we may have a stationary controller with constant filter and control gains which is formulated in the following theorem.

THEOREM 3.3. *Consider the system* (2.1)–(2.4) *and let* $\gamma > 0$ *be a given scalar. Then there exists a linear causal controller* $\mathscr{K}$ *such that the closed-loop system of* (2.1)–(2.4) *with* $\mathscr{K}$ *is exponentially stable and satisfies* $J_{\mathscr{K}}(\Sigma, \infty) < \gamma$ *if and only if the following conditions hold.*

(1) *There exists a symmetric matrix* $P \geq 0$ *satisfying*

$$\Pi_{21}(h) + \Pi_{22}(h)P = \left[P + C_d^T C_d\right][\Pi_{11}(h) + \Pi_{12}(h)P], \tag{3.49}$$

where

$$\Pi(t) = \begin{bmatrix} \Pi_{11}(t) & \Pi_{12}(t) \\ \Pi_{21}(t) & \Pi_{22}(t) \end{bmatrix} = \exp\left\{\begin{bmatrix} -A & -(\gamma^{-2}B_1B_1^T - B_2B_2^T) \\ C_1^T C_1 & A^T \end{bmatrix} t\right\}$$

such that

$$\left[P + C_d^T C_d\right][\Pi_{11}(h) + \Pi_{12}(h)P]$$

is stable and $\Pi_{11}(t) + \Pi_{12}(t)P$ *is nonsingular on* $t \in [0, h]$.

(2) *There exists a symmetric matrix* $Q \geq 0$ *satisfying*

$$\Psi_{21}(h) + \Psi_{22}(h)Q = Q\left[I - \gamma^{-2}C_d^T C_d Q + C_2^T R_D^{-1} C_2 Q\right]^{-1}[\Psi_{11}(h) + \Psi_{12}(h)Q], \tag{3.50}$$

where

$$\Psi(t) = \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{bmatrix} = \exp\left\{\begin{bmatrix} (A + \gamma^{-2}B_1B_1^T P)^T & \gamma^{-2}B_2B_2^T) \\ -B_1B_1^T & -(A + \gamma^{-2}B_1B_1^T P) \end{bmatrix} t\right\}$$

such that

$$\left[I - \gamma^{-2}C_d^T C_d Q + C_2^T R_d^{-1} C_2 Q\right]^{-1}[\Psi_{11}(h) + \Psi_{12}(h)Q]$$

is stable and $\Psi_{11}(t) + \Psi_{12}(t)Q$ *is nonsingular on* $t \in [0, h]$.

Moreover, if the above conditions hold, a suitable controller $\mathscr{K}$ *given by*

$$\begin{aligned} \hat{x}(ih) &= \left[I - \gamma^{-2}QC_d^T C_d + QC_2^T R_d^{-1} C_2\right]^{-1} \\ &\quad \times \left[(I - \gamma^{-2}QC_d^T C_d)e^{(A+\gamma^{-2}B_1B_1^T P - B_2B_2^T P)h}\hat{x}(ih - h)\right. \\ &\quad \left. + QC^T R_D^{-1} y(ih)\right], \quad \hat{x}(0) = 0, && (3.51) \\ u(t) &= -B_2^T P e^{(A+\gamma^{-2}B_1B_1^T P - B_2B_2^T P)(t-ih)}\hat{x}(ih), \quad t \in [ih, ih + h), && (3.52) \end{aligned}$$

is stable and achieves $J_{\mathscr{K}}(\Sigma, \infty) < \gamma$.

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PROOF. It can be established using a similar technique to that used in [25], except that now the discrete-time output is to be considered.

REMARK 3.1. It can be observed that when $C_d = 0$ and $D_{21} = I$, the results of Theorems 3.1 and 3.2 will reduce to the sampled-data H_∞ control results of [24].

REMARK 3.2. In Theorem 3.3, it is shown that the controller (3.52) can be easily computed by solving two algebraic Riccati equations (3.49) and (3.50) which are given directly in terms of the problem data. The generalized hold function depends on the system matrices as well as the H_∞ disturbance attenuation γ. To be more precise, the controller (3.52) can be assumed to be a discrete-time linear time-invariant system followed by the hold function as given in (3.52).

4. An example

We consider the problem of stabilization of the sampled-data system

$$\dot{x}_1(t) = x_2(t), \tag{4.1}$$

$$\dot{x}_2(t) = (-3+a)x_1 + x_2 + b\sin[x_2(t)] \cdot x_1(t) + (a+2)u, \tag{4.2}$$

$$y(ih) = x(ih), \tag{4.3}$$

where $a \in [-1, 1]$ and $b \in [-1, 1]$. The sampling period h is 0.2 seconds and we shall consider the design of a sampled-data control law along with a zero-order hold function.

Note that the origin is not a stable equilibrium point of the unforced system of (4.1)–(4.3). By Theorem 3.3, an H_∞ sampled-data controller for the above stabilization problem can be designed with a zero-order hold function to achieve a unitary disturbance attenuation. The corresponding zero-order hold control law is given by:

$$u(t) = \begin{bmatrix} -0.1966 & -2.0311 \end{bmatrix} x(ih), \quad \forall t \in [ih, ih+h). \tag{4.4}$$

Figures 1–4 show the time responses of the control signal and state variables of the closed-loop system for the parameters a and b at the extreme values.

5. Conclusions

This paper has studied the design of output feedback controllers for linear continuous-time systems with sampled measurements. Control problems on both the finite and infinite horizon have been considered. Necessary and sufficient conditions for the existence of a suitable H_∞ sampled-data output feedback controller are given in terms of two RDEs with finite discrete jumps.

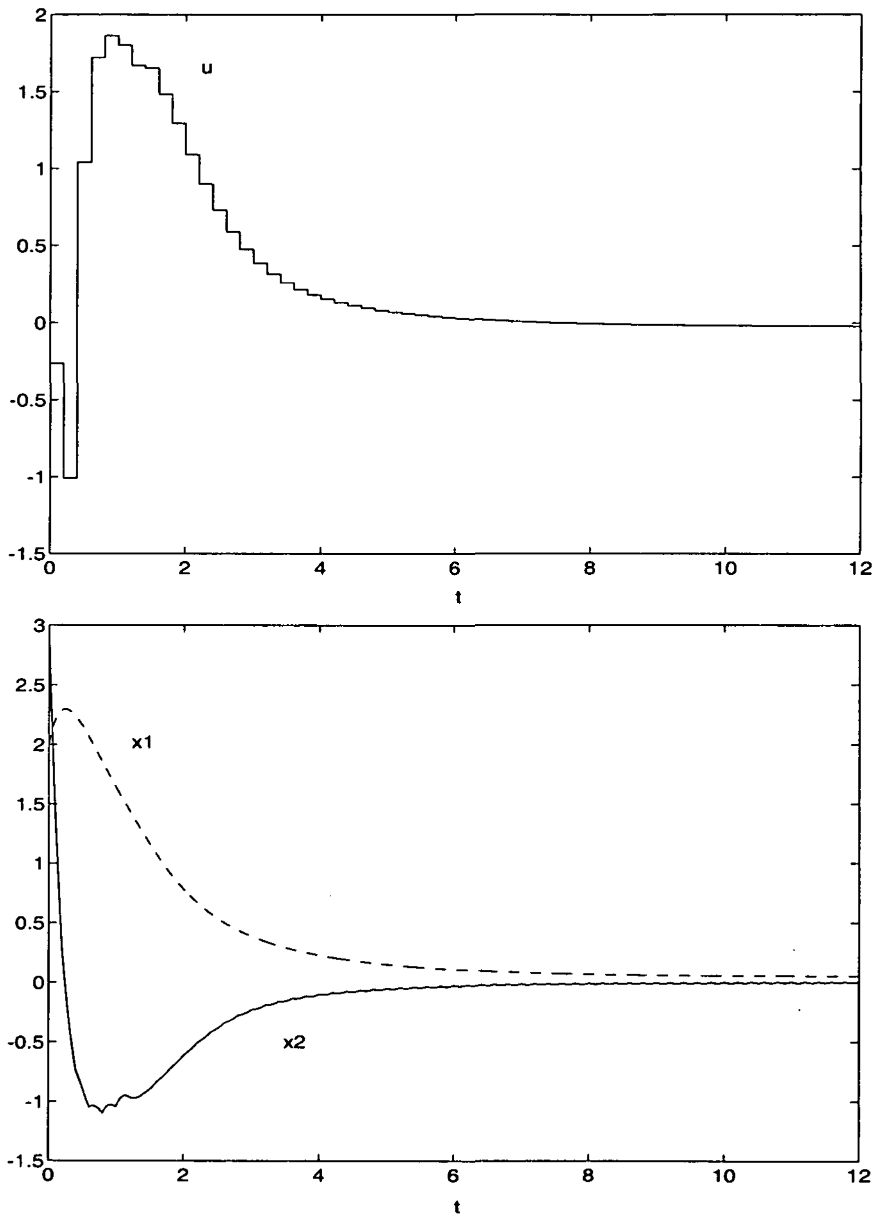

FIGURE 1. The time response of the closed-loop system of (4.1)–(4.3) with (4.4) for $a = 1$ and $b = 1$.

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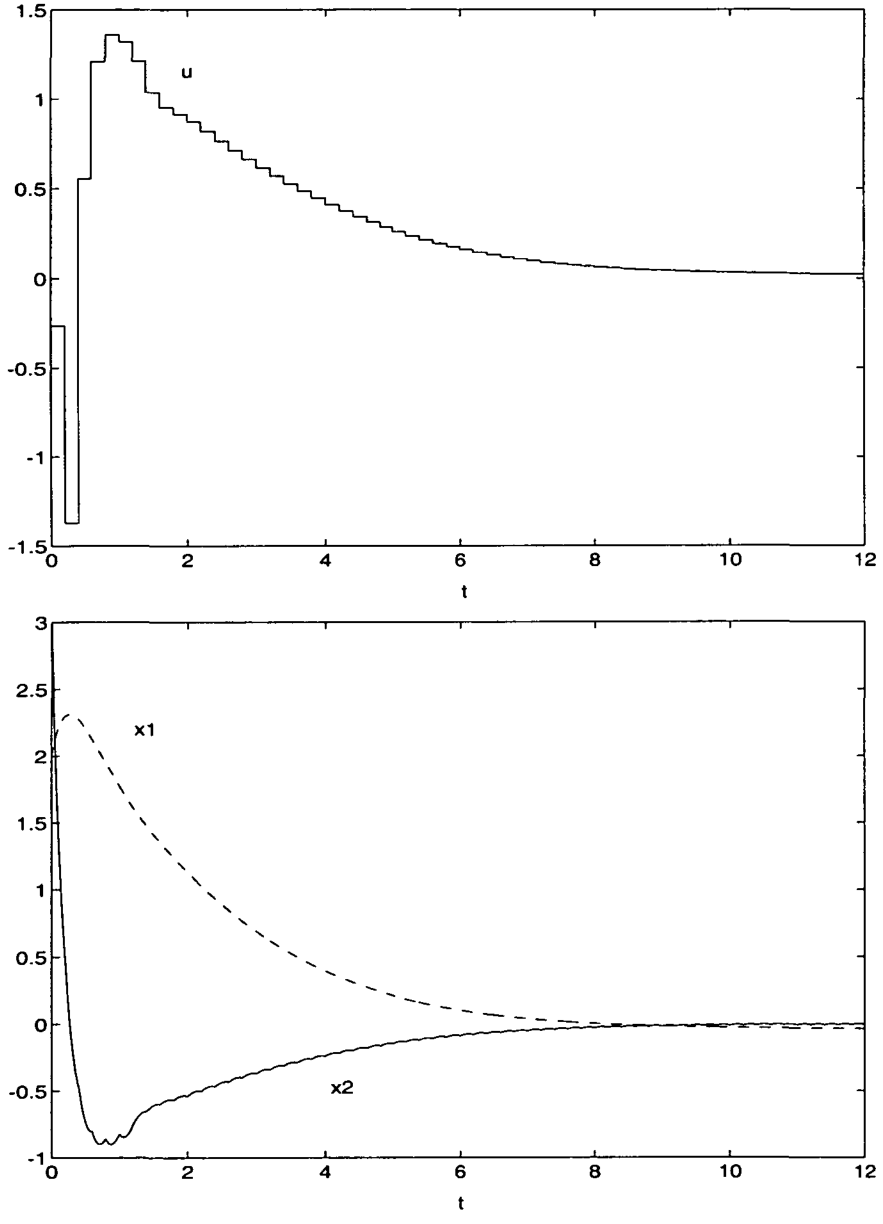

FIGURE 2. The time response of the closed-loop system of (4.1)–(4.3) with (4.4) for $a = 1$ and $b = -1$.

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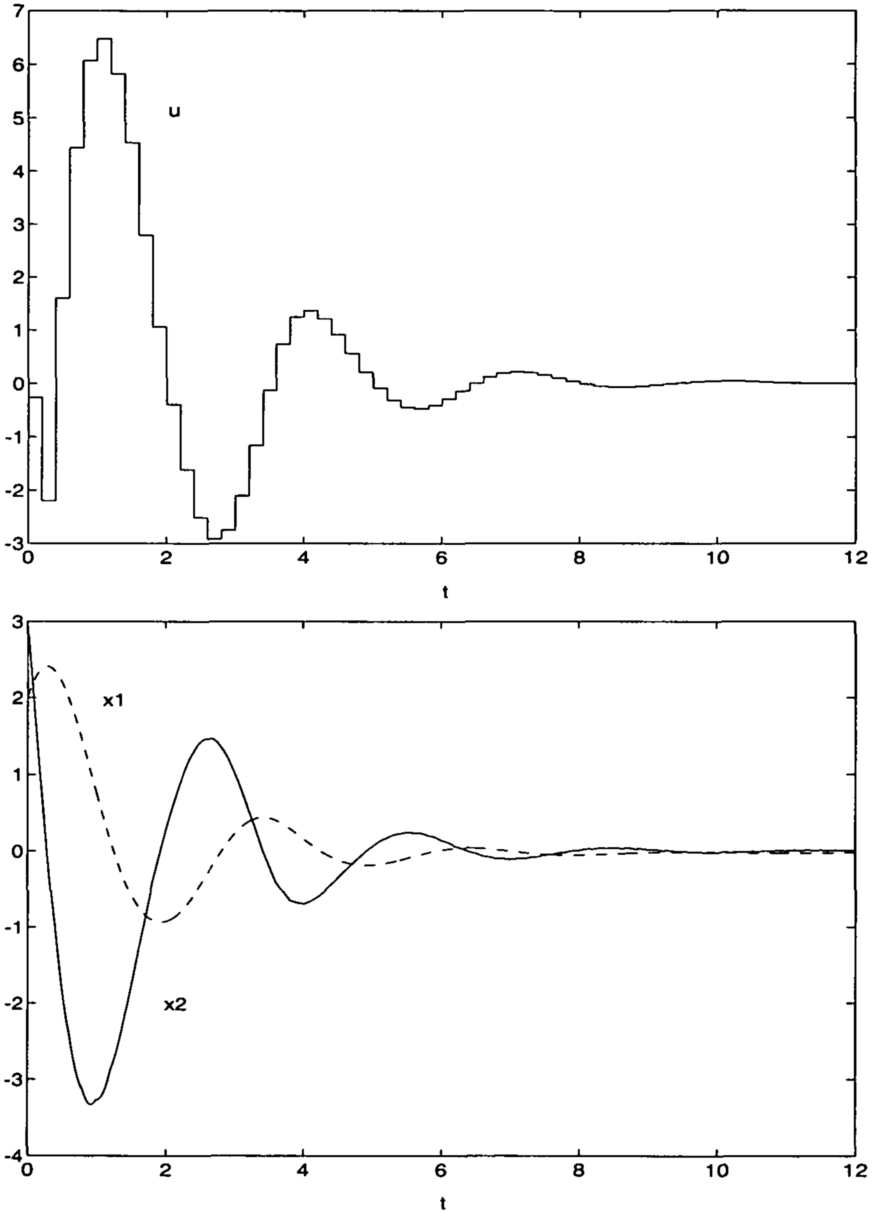

FIGURE 3. The time response of the closed-loop system of (4.1)–(4.3) with (4.4) for $a = -1$ and $b = -1$.

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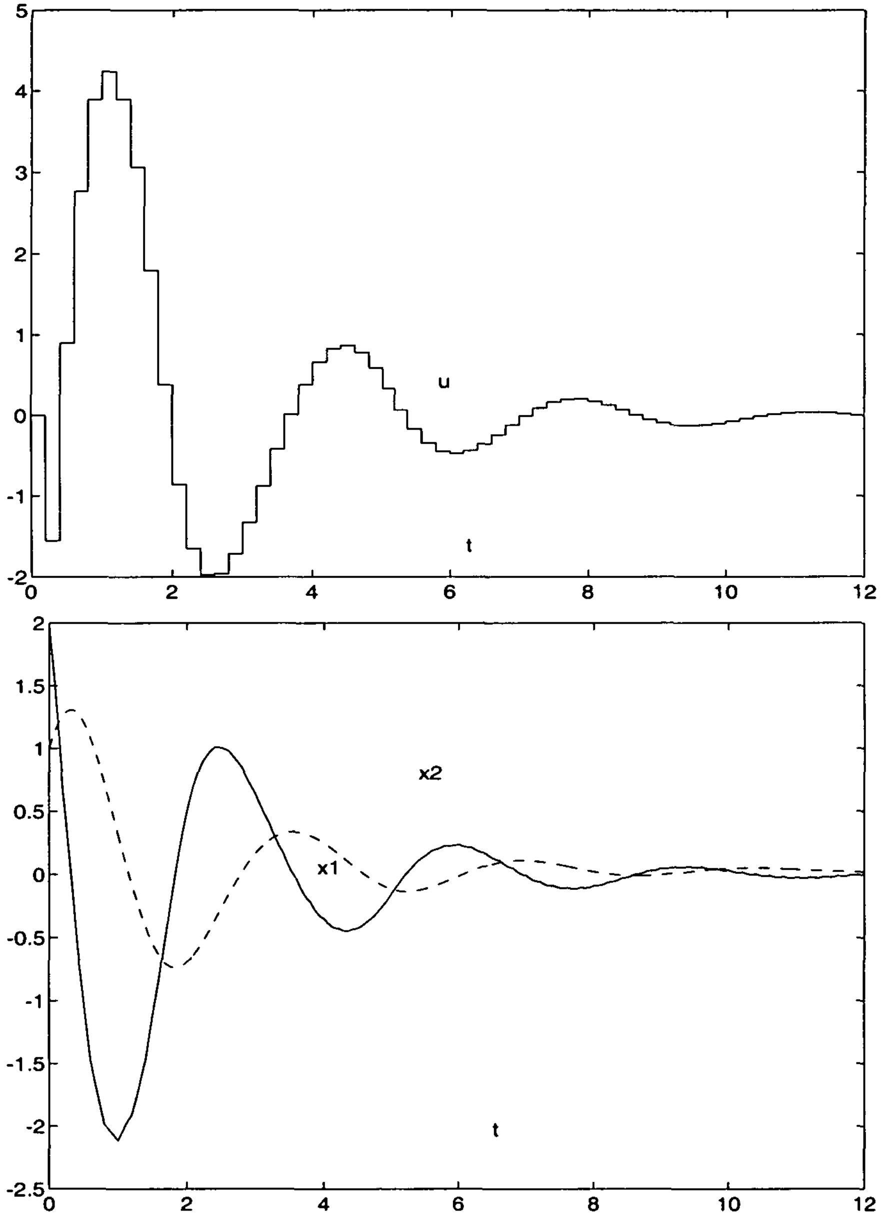

FIGURE 4. The time response of the closed-loop system of (4.1)–(4.3) with (4.4) for $a = -1$ and $b = 1$.

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References

[1] K. J. Astrom and B. Wittenmark, *Computer-controlled systems – theory and design* (Englewood Cliffs, N. J., 1984).

[2] B. A. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with applications to H_∞ sampled-data control", *IEEE Trans. Automat. Control* **37** (1992) 418–435.

[3] T. Basar, "Optimum H_∞ design under sampled state measurements", *System & Control Letters* **16** (1991) 399–409.

[4] P. Bernhard, "Application of the min-max certainty equivalence principle to the sampled-data output feedback H_∞ control problem", *System & Control Letters* **17** (1991) 229–234.

[5] R. W. Brockett, *Finite dimensional linear systems* (John Wiley and Sons, New York, 1970).

[6] T. Chen and B. A. Francis, "H_2– optimal sampled-data control", *IEEE Trans. Automat. Control* **36** (1991) 387–397.

[7] T. Chen and B. A. Francis, "Input-output stability of sampled-data systems", *IEEE Trans. Automat. Control* **36** (1991) 50–58.

[8] T. Chen and B. A. Francis, "Linear time-varying H_2-optimal control of sampled-data systems", *Automatica* **27** (1991) 963–974.

[9] P. T. Kabamba and S. Hara, "Worst-case analysis and design of sampled-data control systems", *IEEE Trans. Automat. Control* **38** (1993) 1337–1357.

[10] P. P. Khargonekar and N. Sivashankar, "H_2 optimal control for sampled–data systems", *System & Control Letters* **17** (1991) 425–436.

[11] D. J. N. Limebeer, B. D. O. Anderson, P. P. Khargonekar and M. Green, "A game theoretic approach to H_∞ control for time–varying systems", *SIAM J. Control and Optimiz.* **30** (1992) 262–283.

[12] I. R. Petersen, B. D. O. Andersen and E. A. Jonckheere, "A first principles solution to the non-singular H_∞ control problem", *Int. J. Robust & Nonlinear Control* **1** (1991) 171–185.

[13] M. G. Safonov, D. J. N. Limebeer and R. Y. Chiang, "Simplifying the H_∞ theory via loop-shifting, matrix-pencil and descriptor concepts", *Int. J. Control* **50** (1989) 2467–2488.

[14] A. V. Savkin and I. R. Petersen, "Model validation for robust control uncertain continuous-time systems with discrete and continuous measurements", in *Proc. of 13th IFAC World Congress* (San Francisco, USA, 1996).

[15] P. Shi, "Issues in robust filtering and control of sampled-data systems", Ph. D. Thesis, The University of Newcastle, Australia, 1994.

[16] P. Shi, "Robust filtering for uncertain systems with sampled measurements", *Int. J. Systems Science* **27** (1996) 1403–1415.

[17] P. Shi, "H_∞ filtering for linear continuous-time systems with sampled measurements", *IMA J. Mathe. Contr. Inform.* (1998), to appear.

[18] P. Shi, "Filtering on sampled-data systems with parametric uncertainty", *IEEE Trans. Automat. Control* **43** (1998) 1022–1027.

[19] P. Shi, C. E. de Souza and L. Xie, "Bounded real lemma for linear systems with finite discrete jumps", *Int. J. Control* **66** (1997) 145–159.

[20] P. Shi, M. Fu and C. E. de Souza, "Loop transfer recovery for systems under sampled measurements", *IEEE Proc.-Control Theory Appl.* **143** (1996) 333–337.

[21] P. Shi, Y. Wang and L. Xie, "Robust filtering for interconnected uncertain systems under sampled measurements", *J. Dynamic Systems, Measurement, and Control, Transactions of the ASME* **199** (1997) 337–340.

[22] N. Sivashankar and P. P. Khargonekar, "Characterization of the L_2-induced norm for linear systems with jumps with applications to sampled-data systems", *SIAM J. Control Optim.* **32** (1994) 1128–1150.

[23] C. E. de Souza and G. C. Goodwin, "Intersample variances in discrete minimum variance control", *IEEE Trans. Automat. Control* **29** (1984) 759–761.

[24] W. Sun, K. M. Nagpal and P. P. Khargonekar, "H_∞ control and filtering for sampled-data systems", preprint, 1991.

[25] W. Sun, K. M. Nagpal and P. P. Khargonekar, "H_∞ control and filtering for sampled-data systems", *IEEE Trans. Automat. Control* **38** (1993) 1162–1175.