COMPOSITIO MATHEMATICA

Kähler groups, real hyperbolic spaces and the Cremona group. With an appendix by Serge Cantat

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Abstract

Generalizing a classical theorem of Carlson and Toledo, we prove that *any* Zariski dense isometric action of a Kähler group on the real hyperbolic space of dimension at least three factors through a homomorphism onto a cocompact discrete subgroup of $PSL_2(\mathbb{R})$. We also study actions of Kähler groups on infinite-dimensional real hyperbolic spaces, describe some exotic actions of $PSL_2(\mathbb{R})$ on these spaces, and give an application to the study of the Cremona group.

Contents

1. Introduction

1.1 Kähler groups and real hyperbolic spaces

In this text, we study actions of fundamental groups of compact Kähler manifolds (referred to as *Kähler groups*) on finite or infinite-dimensional real hyperbolic spaces. For an introduction to the study of Kähler groups, the reader can consult [ABCKT96] as well as [Bur10, CS08, KKM11, Kli10, NR08, Rez02] for more recent developments.

Throughout the text, we denote by X a compact Kähler manifold and by Γ its fundamental group.

Firstly, it is a classical result of Carlson and Toledo [CT89] that a cocompact lattice of the group $\mathrm{Isom}(\mathbb{H}^n)$ of isometries of the real hyperbolic space $\mathbb{H}^n$ of dimension $n \geqslant 3$ cannot be isomorphic to a Kähler group (see also [ABCKT96, §6.4], for a proof in the non-uniform case, as well as [JY91, YiH00, YiH02]). Their result is actually more precise. Let us first make the following classical definitions.

DEFINITION 1. A hyperbolic 2-orbifold Σ is a quotient of the unit disc Δ in $\mathbb{C}$ by a discrete cocompact subgroup (denoted by $\pi_1^{\mathrm{orb}}(\Sigma)$) of $\mathrm{PSL}_2(\mathbb{R})$. A map from a complex manifold X to Σ is holomorphic if it lifts to a holomorphic map from the universal cover of X to the unit disc.

Note that we could have given an equivalent definition of a hyperbolic 2-orbifold in terms of Riemann surfaces with marked points (see [Delz08]); however, Definition 1 will be sufficient for our purpose.

DEFINITION 2. A fibration of a compact complex manifold X onto a hyperbolic 2-orbifold Σ is a holomorphic surjective map $f : X \to \Sigma$ with connected fibers.

We say that a linear representation $\rho : \Gamma \to \mathrm{GL}_N(\mathbb{C})$ factors through a fibration $f : X \to \Sigma$ onto a hyperbolic 2-orbifold if there exists a linear representation

$$\hat{\rho} : \pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{GL}_N(\mathbb{C})$$

such that $\rho = \hat{\rho} \circ f_*$, where $f_* : \Gamma \to \pi_1^{\mathrm{orb}}(\Sigma)$ is the homomorphism induced by f. In [CT89], Carlson and Toledo, relying on a result of Sampson [Sam86], proved that any homomorphism from a Kähler group onto a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n)$ ($n \geqslant 3$) factors through a fibration as above. Note that Carlson and Toledo's proof still applies and yields the same result for homomorphisms $\Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ with Zariski dense and discrete image. We will prove that the same result remains true for any homomorphism into $\mathrm{Isom}(\mathbb{H}^n)$ ($n \geqslant 3$) with Zariski dense image, discrete or not.

We now want to study isometric actions of Γ on infinite-dimensional real hyperbolic spaces. For an introduction to these spaces, see [Gro93, §6], and [BIM05]. Let $\mathscr{H}$ be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. We always assume that $\mathscr{H}$ is separable. Let u be a unit vector in $\mathscr{H}$, and define a quadratic form B on $\mathscr{H}$ by

$$B(v_1 + t_1 u, v_2 + t_2 u) = t_1 t_2 - \langle v_1, v_2 \rangle,$$

where the v_i are orthogonal to u and $t_i \in \mathbb{R}$. Copying the construction of finite-dimensional hyperbolic spaces, one defines the hyperbolic space $\mathbb{H}^\infty$ associated to $\mathscr{H}$ by

$$\mathbb{H}^\infty := \{v \in \mathscr{H}, B(v, v) = 1, B(v, u) > 0\}.$$

The formula

$$\cosh(d(x, y)) = B(x, y) \quad (x, y \in \mathbb{H}^\infty),$$

defines a distance on $\mathbb{H}^\infty$, which turns it into a complete CAT(-1) metric space. We denote by $\mathrm{Isom}(\mathbb{H}^\infty)$ the group of isometries of $\mathbb{H}^\infty$ and by $\partial \mathbb{H}^\infty$ its boundary. We say that an isometric action of a finitely generated group on $\mathbb{H}^\infty$ is *elementary* if it fixes a geodesic in $\mathbb{H}^\infty$ or a point in $\mathbb{H}^\infty \cup \partial \mathbb{H}^\infty$; *non-elementary* otherwise. On the other hand we say that a group of isometries of $\mathbb{H}^\infty$ is *elliptic* if it fixes a point in $\mathbb{H}^\infty$. We can then prove the following theorem.

THEOREM 1. *Let Γ be a Kähler group. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^\infty)$ be a non-elementary action of Γ on $\mathbb{H}^\infty$. Assume that ρ is minimal, i.e. that $\mathbb{H}^\infty$ contains no non-trivial closed ρ-invariant totally geodesic subspace. Then, one of the following two cases happens.*

(i) *The representation ρ factors through a fibration onto a hyperbolic 2-orbifold.*

(ii) *The representation ρ can be written as $\rho = \Psi \circ \theta$, where θ is a homomorphism from Γ to $\mathrm{PSL}_2(\mathbb{R})$ with dense image and $\Psi : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{Isom}(\mathbb{H}^\infty)$ is a continuous homomorphism.*

Let us make some comments about this result. First, according to [BIM05, Proposition 4.3], if $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^\infty)$ is a non-elementary homomorphism, there exists a unique closed totally geodesic subspace $\mathbb{H}^\infty_\rho$ of $\mathbb{H}^\infty$ which is Γ-invariant and minimal with respect to this property. Hence, if ρ is non-minimal, one can always apply the previous theorem to the induced action on $\mathbb{H}^\infty_\rho$.

Secondly, it will be clear from the proof that Theorem 1 remains true when $\mathbb{H}^\infty$ is replaced by a finite-dimensional hyperbolic space $\mathbb{H}^n$. However, the second possibility in Theorem 1 never occurs for finite-dimensional spaces (and if $n \geqslant 3$). This is a consequence of the fact that any continuous action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^n$ has to preserve a totally geodesic plane (see [GP91, p. 122], as well as [Kar53, Mos55]), and therefore cannot be minimal. Hence, we obtain the following corollary.

COROLLARY 1. *If $n \geqslant 3$, any homomorphism $\Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ with Zariski dense image factors through a fibration onto a hyperbolic 2-orbifold.*

Thirdly, let us recall that in [CS08], Corlette and Simpson classified Zariski dense representations of Kähler groups into $\mathrm{PSL}_2(\mathbb{C})$ (see [Delz08] for the non-Zariski dense case): any such representation either factors through a fibration onto a hyperbolic 2-orbifold or factors through a holomorphic map to a compact manifold whose universal cover is a product of discs. Observe also that when $n = 3$ the group $\mathrm{Isom}(\mathbb{H}^3)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$ (up to a 2-sheeted cover), and the corollary above contains only one case, as opposed to the result of [CS08]. This is because we ask that the *real* Zariski closure of ρ is $\mathrm{PSL}_2(\mathbb{C})$, instead of merely asking that its *complex* Zariski closure is $\mathrm{PSL}_2(\mathbb{C})$. Observe also that when the second case of Theorem 1 occurs, we can apply the result of Corlette and Simpson to the representation θ.

The key fact we just used (namely, that any action of $\mathrm{PSL}_2(\mathbb{R})$ on a finite-dimensional hyperbolic space preserves a totally geodesic plane) is not true for actions on $\mathbb{H}^\infty$: there exist irreducible continuous linear representations of $\mathrm{PSL}_2(\mathbb{R})$ on a Hilbert space, which are not unitary but which preserve a quadratic form of signature $(\infty, 1)$. The associated actions on $\mathbb{H}^\infty$ are minimal, and hence do not have an invariant totally geodesic plane. As a consequence, the second possibility from Theorem 1 actually occurs in some examples. Although this might be well known to experts in representation theory, we will describe these representations in §2. There, we will prove the following theorem.

THEOREM 2. *There exists a one parameter family of representations*

$$\rho_t : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{Isom}(\mathbb{H}^\infty)$$

where $t \in (0, 1)$ with the following properties.

(i) *The action ρ_t is non-elementary and has no non-trivial closed totally geodesic invariant subspace.*

(ii) *There exists a ρ_t-equivariant harmonic map $f_t : \Delta \to \mathbb{H}^\infty$ whose image is a minimal surface with curvature $-2/(t(t+1))$.*

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1.2 Cremona group and Picard–Manin space

The main example of a group acting on an infinite-dimensional hyperbolic space comes from algebraic geometry. Recall that the Cremona group, denoted by $\mathrm{Bir}(\mathbb{P}^2)$, is the group of birational transformations of the complex projective plane $\mathbb{P}^2$. A birational transformation of $\mathbb{P}^2$ is a transformation of the form

$$[x : y : z] \mapsto [P(x, y, z) : Q(x, y, z) : R(x, y, z)],$$

where P, Q, R are three homogeneous polynomials of the same degree, and which admits an inverse of the same form. It was proved by Noether that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by the standard quadratic involution

$$\sigma : [x : y : z] \mapsto \left[\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right] = [yz : xz : xy],$$

and the group $\mathrm{PGL}(3, \mathbb{C})$ of automorphisms of $\mathbb{P}^2$. This group was studied a lot by algebraic geometers, and we refer the reader to [Can11] for some references. Recently, Cantat and Deserti started the study of infinite finitely generated subgroups of $\mathrm{Bir}(\mathbb{P}^2)$, using ideas from geometric group theory; see [Can11, CL10, Des06, Des07] as well as [Fav10]. In particular, using ideas of Zariski and Manin, Cantat recently proved that the group $\mathrm{Bir}(\mathbb{P}^2)$ can be embedded into the isometry group of an infinite-dimensional hyperbolic space $\mathbb{H}_{\mathbb{P}^2}$, called the *Picard–Manin space*. We will briefly recall its construction in § 5. Using this embedding he proved the following result: *any homomorphism with infinite image from a discrete Kazhdan group into the Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ *is conjugated to a homomorphism into* $\mathrm{PGL}(3, \mathbb{C})$. In particular, this result applies to any lattice Λ in a connected simple Lie group with property T. This left open the problem of classifying homomorphisms from lattices in the groups $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$ into the Cremona group. Note that this discussion is motivated by the fact that (for some values of n) there exist injective homomorphisms from lattices in $\mathrm{SO}(n, 1)$ to the Cremona group (see [CL10, DZ01] and the references there).

Remark 1. We will see that the space $\mathbb{H}_{\mathbb{P}^2}$ is not separable; however, any finitely generated group acting on it preserves a closed, separable, totally geodesic subspace.

In the following, we say that a homomorphism from a finitely generated group into the group $\mathrm{Bir}(\mathbb{P}^2)$ is non-elementary if the associated action on $\mathbb{H}_{\mathbb{P}^2}$ is non-elementary. In the same way, we say that a subgroup of the Cremona group is elliptic if it fixes a point in the space $\mathbb{H}_{\mathbb{P}^2}$. Using our previous theorem, and the fact that the action of $\mathrm{Bir}(\mathbb{P}^2)$ on $\mathbb{H}_{\mathbb{P}^2}$ is, in some sense, discrete, we can prove the following theorem.

THEOREM 3. *Let Γ be the fundamental group of a compact Kähler manifold X. Let*

$$\rho : \Gamma \to \mathrm{Bir}(\mathbb{P}^2)$$

be a non-elementary homomorphism. Then there exists a fibration $X \to \Sigma$ onto a hyperbolic 2-orbifold such that the kernel H of the map $\Gamma \to \pi_1^{\mathrm{orb}}(\Sigma)$ has the following property. The group $\rho(H)$ fixes pointwise a closed totally geodesic subspace of $\mathbb{H}_{\mathbb{P}^2}$ of dimension at least two.

Observe that if the group $\rho(H)$ appearing in the theorem above is not trivial, it provides an example of an elliptic subgroup of the Cremona group for which the action of the normalizer is non-elementary (since the action of Γ itself is non-elementary). To study further homomorphisms from Kähler groups to the Cremona group, one is thus led to study elliptic subgroups of the Cremona group whose normalizer is large enough. Such an elliptic subgroup can be infinite as the following example shows.

Example 1. In $\mathbb{P}^2$, with the homogeneous coordinates $[x:y:z]$, consider the complement of the union of the three lines $\{x=0\}$, $\{y=0\}$ and $\{z=0\}$. This is an open set O isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Thinking of O as a group, we get an action of the group $\mathbb{C}^* \times \mathbb{C}^*$ on O by translation. On the other hand, the group $\mathrm{GL}(2,\mathbb{Z})$ acts on O by monomial transformations: the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts by the transformation $(x,y) \mapsto (x^a y^b, x^c y^d)$. One obtains in this way an injective homomorphism of the group $(\mathbb{C}^* \times \mathbb{C}^*) \rtimes \mathrm{GL}(2,\mathbb{Z})$ into the Cremona group. We will denote by G_{toric} its image. One can see that the group $\mathbb{C}^* \times \mathbb{C}^*$ fixes pointwise an infinite-dimensional totally geodesic subspace of $\mathbb{H}_{\mathbb{P}^2}$. We will explain this in §5, after constructing the space $\mathbb{H}_{\mathbb{P}^2}$.

This is essentially the only example of an elliptic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ with large normalizer as shown by the following theorem proved by Serge Cantat in the appendix.

THEOREM 4. *Let N be a subgroup of the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$. Assume that there exists a short exact sequence*

$$1 \to A \to N \to B \to 1$$

where N contains at least one hyperbolic element, and A is infinite elliptic. Then N is conjugate to a subgroup of the group G_{toric} of automorphisms of $(\mathbb{C}^*)^2$.

Note that, as in the finite-dimensional case, isometries of infinite-dimensional hyperbolic spaces fall into three types: elliptic, hyperbolic and parabolic (see [BIM05]). The word *hyperbolic* in the theorem above refers to the type of the action of a birational map on the space $\mathbb{H}_{\mathbb{P}^2}$.

Combining Theorem 3 with Cantat's theorem, we obtain a more precise description of non-elementary homomorphisms from Kähler groups to the Cremona group.

COROLLARY 2. *Let $\rho: \Gamma \to \mathrm{Bir}(\mathbb{P}^2)$ be a non-elementary homomorphism. One of the following two cases occurs.*

(i) *The homomorphism ρ is conjugated to a homomorphism with values in the group G_{toric}.*

(ii) *After passing to a finite index subgroup of Γ (equivalently, after taking a finite cover of X), ρ factors through a fibration onto a hyperbolic 2-orbifold.*

Proof. We keep the notation of Theorem 3. If the group $\rho(H)$ is infinite, Cantat's theorem implies that $\rho(\Gamma)$ is conjugated to a subgroup of the group G_{toric}. If $\rho(H)$ is finite, we argue as follows. We consider the homomorphism $\Gamma \to \mathrm{Aut}(\rho(H))$ given by the action of Γ on $\rho(H)$ by conjugation. Let Γ_1 be its kernel, H_1 be the intersection of Γ_1 with H and Λ_1 be the image of Γ_1 in $\pi_1^{\mathrm{orb}}(\Sigma)$. Up to replacing Γ_1 by another finite index subgroup we also assume that Λ_1 is torsion free, and hence a surface group.

Let $V = \rho(H_1)$; this is a finite abelian subgroup of the Cremona group, and the *central* extension

$$1 \to V \to \rho(\Gamma_1) \to \rho(\Gamma_1)/V \to 1$$

determines a class in $H^2(\rho(\Gamma_1)/V, V)$. Let e be the pull-back of this class in $H^2(\Lambda_1, V)$. The class e becomes trivial on a finite index subgroup Λ_2 of Λ_1. Let Γ_2 be the inverse image of Λ_2 in Γ_1. The fact that the class e is trivial on Λ_2 says that the homomorphism $\rho: \Gamma_2 \to \mathrm{Bir}(\mathbb{P}^2)$ factors through a homomorphism to the direct product $\Lambda_2 \times V$. Hence, up to taking once again a finite index subgroup, it factors through the projection on Λ_2. $\square$

The following corollary answers partially a question raised by Cantat [Can11].

COROLLARY 3. *Let Γ_1 be a cocompact lattice in the group* $\mathrm{SU}(n, 1)$ *with* $n \geqslant 2$. *If*

$$\rho : \Gamma_1 \to \mathrm{Bir}(\mathbb{P}^2)$$

is an injective homomorphism, then one of the following two possibilities holds.

  (i) *The group $\rho(\Gamma_1)$ fixes a point in the Picard–Manin space $\mathbb{H}_{\mathbb{P}^2}$.*

  (ii) *The group $\rho(\Gamma_1)$ fixes a unique point in the boundary of the Picard–Manin space $\mathbb{H}_{\mathbb{P}^2}$.*

The proof of Corollary 3 will be given in § 5.

A natural problem left open by this work is the study of homomorphisms from Kähler groups to the group G_{toric} as well as homomorphisms into the Cremona group for which the associated action on $\mathbb{H}_{\mathbb{P}^2}$ has a unique fixed point at infinity (see [Can11] for an interpretation of these morphisms in terms of algebraic geometry). It might also be possible to extend these results to non-cocompact lattices in $\mathrm{SU}(n, 1)$ (or more generally fundamental groups of quasiprojective varieties) using techniques similar to the ones used in [CS08, KM08], but we have not tried to establish such results.

1.3 About the proofs

We now briefly discuss the (classical) strategy of the proof of Theorem 1. We denote by $\widetilde{X}$ the universal cover of X. We use the fact that there is an equivariant harmonic map f from $\widetilde{X}$ to $\mathbb{H}^\infty$; this is due to Korevaar and Schoen [KS97]. It is well known [Sam86] (at least in the finite-dimensional case) that this map has real rank at most two. This is the main reason why actions of Kähler groups on real hyperbolic spaces are so constrained. We can then prove a *factorization theorem* in the spirit of many factorization theorems in the literature (see [CT89, GS92, JY91, JZ00]...), i.e. we prove that f can be written as the composition of a holomorphic map from $\widetilde{X}$ to the unit disc $\Delta \subset \mathbb{C}$ and a harmonic map from Δ to $\mathbb{H}^\infty$. A similar result appears in [JY91]; however, we give a detailed account of the proof here. This decomposition gives rise to a homomorphism θ from Γ to $\mathrm{PSL}_2(\mathbb{R})$. Our main observation, inspired by the reading of [GS92] and which is essentially the only new remark in the proof, is that if the group $\theta(\Gamma) \subset \mathrm{PSL}_2(\mathbb{R})$ is not discrete the natural action of $\theta(\Gamma)$ on $\mathbb{H}^\infty$ extends to a continuous action of $\mathrm{PSL}_2(\mathbb{R})$. This allows to prove easily Theorem 1.

Remark 2. The same arguments show that a Zariski dense representation of a Kähler group Γ in a non-compact simple Lie group G (with trivial center and not isomorphic to $\mathrm{PSL}_2(\mathbb{R})$) will factor through a fibration onto a hyperbolic 2-orbifold as soon as the associated harmonic map f to the symmetric space of G has the following property: the complex linear part $df^{1,0}$ of the differential of f has complex rank one (see the end of § 4.2).

The text is organized as follows. Section 2 contains the description of exotic actions of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^\infty$ and the proof of Theorem 2. In § 3, we recall a few facts concerning harmonic maps, for the reader interested in the application to the Cremona group and not familiar with these techniques. We also check that classical results (such as the Bochner–Siu–Sampson formula) still apply when the target of the harmonic map is an infinite-dimensional hyperbolic space. In § 4, we prove Theorem 1. The application to the Cremona group is described in § 5. In § 6, we check that the classical Koszul–Malgrange theorem concerning integrability of certain almost complex structures on complex vector bundles can be adapted to the infinite-dimensional setting. This is needed in § 4. Finally, § 7 contains the proof of a technical proposition needed in § 3.

2. Exotic actions of $\mathrm{PSL}_2(\mathbb{R})$

In this section we describe some infinite-dimensional linear representations of $\mathrm{PSL}_2(\mathbb{R})$ which lead to a family of exotic actions of $\mathrm{PSL}_2(\mathbb{R})$ on the infinite-dimensional real hyperbolic space. This will lead to the proof of Theorem 2.

2.1 A glimpse at the representation theory of $\mathrm{PSL}_2(\mathbb{R})$

Let Δ be the unit disc in $\mathbb{C}$. We will consider the usual action of $\mathrm{PSL}_2(\mathbb{R})$ on Δ obtained by conjugating the homographic action of $\mathrm{PSL}_2(\mathbb{R})$ on the upper half plane by the map $z \mapsto (z - i)/(z + i)$. All along this section, we will denote by $\mathscr{H}_{\mathbb{C}}$ the Hilbert space of complex-valued square integrable functions on $\mathrm{S}^1 = \partial\Delta$, endowed with the angular measure $d\theta$ and for $n \in \mathbf{Z}$ by e_n the function $z \mapsto z^n$ on the circle. For each real number s, we define a representation π_s of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathscr{H}_{\mathbb{C}}$ by

$$\pi_s(g) \cdot f = \mathrm{Jac}(g^{-1})^{\frac{1}{2}+s} f \circ g^{-1} \quad (g \in \mathrm{PSL}_2(\mathbb{R}), f \in \mathscr{H}_{\mathbb{C}}).$$

Here $\mathrm{Jac}(g)$ is the Jacobian of an element g with respect to the measure $d\theta$ on the circle. For each $g \in \mathrm{PSL}_2(\mathbb{R})$, the operator $\pi_s(g) : \mathscr{H}_{\mathbb{C}} \to \mathscr{H}_{\mathbb{C}}$ is a bounded linear operator (although not unitary in general). This defines a continuous representation of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathscr{H}_{\mathbb{C}}$. It is well known that the representation π_s is irreducible if and only if $s \notin \frac{1}{2} + \mathbb{Z}$ (see [Kna01] or [Wal71] for instance).

Remark 3. Note that the formula above still defines a representation when s is a complex number. When s is purely imaginary the representation is unitary and one obtains the so-called *principal series* of unitary representations of $\mathrm{PSL}_2(\mathbb{R})$ (although our description of these representations might differ from the usual one by a translation in the parameter s).

If $f_1, f_2 \in \mathscr{H}_{\mathbb{C}}$, we will write $(f_1, f_2) := \int_{\mathrm{S}^1} f_1 \overline{f_2} \, d\theta$. From the definition of the Jacobian, we obtain

$$(\pi_s(g)(f_1), \pi_{-s}(g)(f_2)) = (f_1, f_2) \quad (f_1, f_2 \in \mathscr{H}_{\mathbb{C}}, g \in \mathrm{PSL}_2(\mathbb{R})).$$

Hence, if one can construct a continuous operator A_s from $\mathscr{H}_{\mathbb{C}}$ to itself which intertwines the representations π_s and π_{-s} (i.e. satisfies $\pi_{-s}(g) \circ A_s = A_s \circ \pi_s(g)$), this will imply that there is a hermitian pairing on $\mathscr{H}_{\mathbb{C}}$, invariant by the representation π_s. It turns out that such an operator exists (for the right values of s) and has been much studied in the literature on representation theory; see for instance [Kna01, Sal67, Sal70], as well as [JW77, KS71] for a study of intertwining operators for representations of more general simple Lie groups than $\mathrm{PSL}_2(\mathbb{R})$. Since this material might not be so well known outside the representation theory community, we will include a brief exposition of the case of interest to us. Namely, we will prove the following proposition.

PROPOSITION 1. *Let $s > 0$ and let $A_s : \mathscr{H}_{\mathbb{C}} \to \mathscr{H}_{\mathbb{C}}$ be the operator which maps e_n to $\lambda_n e_n$, where λ_n is given by $\lambda_0 = 1$ and, for $n \neq 0$,*

$$\lambda_n = \prod_{i=0}^{|n|-1} \frac{i + \frac{1}{2} - s}{i + \frac{1}{2} + s}.$$

Then A_s defines a bounded linear operator from $\mathscr{H}_{\mathbb{C}}$ to $\mathscr{H}_{\mathbb{C}}$ which satisfies $\pi_{-s}(g) \circ A_s = A_s \circ \pi_s(g)$ for all $g \in \mathrm{PSL}_2(\mathbb{R})$.

The reader will notice that the formula for A_s given above does not define a bounded linear operator on $\mathscr{H}_{\mathbb{C}}$ if $s < 0$. Indeed if A_s was bounded for some negative s not in $\frac{1}{2} + \mathbb{Z}$, there would

exist a constant $C > 0$ such that

$$\left| \prod_{i=0}^{|n|-1} \frac{i + \frac{1}{2} - s}{i + \frac{1}{2} + s} \right| \leqslant C.$$

This would imply the convergence of the series

$$\sum_{i \geqslant 0} \log\left(1 + \frac{-2s}{i + \frac{1}{2} + s} \right),$$

which is a contradiction. This implies that, for $s > 0$ and $s \notin \frac{1}{2} + \mathbb{Z}$, A_s has a dense, *non-closed* image: if A_s was surjective, it would be an isomorphism by the open mapping theorem. Its inverse $A_s^{-1} = A_{-s}$ would then be bounded.

Once the above proposition is known we can define, for $f_1, f_2 \in \mathscr{H}_{\mathbb{C}}$,

$$\langle f_1, f_2 \rangle_s = \int_{S^1} f_1 \overline{A_s(f_2)} \, d\theta.$$

The bilinear form $\langle \cdot, \cdot \rangle_s$ on $\mathscr{H}_{\mathbb{C}}$ is invariant by the representation π_s. If $s \in (0, \frac{1}{2})$, it is positive definite. This implies that the representations $(\pi_s)_{0 < s < \frac{1}{2}}$ are unitary (although the invariant inner product is not the standard one on $\mathscr{H}_{\mathbb{C}}$). This is the so-called *complementary series* which arises in the classification of irreducible unitary representations of $\mathrm{PSL}_2(\mathbb{R})$. For $s > \frac{1}{2}$, the pairing $\langle \cdot, \cdot \rangle_s$ is not positive definite anymore, but this does not make it less interesting! If $s \in (p - \frac{1}{2}, p + \frac{1}{2})$ where $p \geqslant 1$ is an integer, it is easily checked that the bilinear form $\langle \cdot, \cdot \rangle_s$ has index p (here *index* should be understood as the dimension of a maximal isotropic subspace).

- If $s \in (p - \frac{1}{2}, p + \frac{1}{2})$ and p is odd, p of the coefficients $(\lambda_n)_{n \in \mathbb{Z}}$ are positive, all the others being negative.
- If $s \in (p - \frac{1}{2}, p + \frac{1}{2})$ and p is even, p of the coefficients $(\lambda_n)_{n \in \mathbb{Z}}$ are negative, all the others being positive.

In particular, for each $s \in (\frac{1}{2}, \frac{3}{2})$, we can look at the restriction of the representation π_s to the subspace $\mathscr{H}_{\mathbb{R}} \subset \mathscr{H}_{\mathbb{C}}$ of real-valued functions. However, we are not yet in the situation described in the introduction to construct the space $\mathbb{H}^\infty$. Indeed, the space of functions with zero mean, endowed with the scalar product $-\langle \cdot, \cdot \rangle_s$, is not complete. This follows from the fact that the sequence $(\lambda_n)_{n \geqslant 0}$ which appears in Proposition 1 above tends to zero as n goes to $+\infty$.

If $f \in \mathscr{H}_{\mathbb{R}}$, we will denote by f_0 the function

$$f - \frac{1}{2\pi} \int_{S^1} f \, d\theta,$$

and by $m(f)$ the number $\int_{S^1} f \, d\theta$. Let $\overline{\mathscr{H}_{\mathbb{R}}}$ be the completion of the space $\mathscr{H}_{\mathbb{R}}$ for the scalar product associated to the norm

$$\|f\| = \sqrt{\left(\int_{S^1} f \, d\theta \right)^2 - \langle f_0, f_0 \rangle_s}.$$

The bilinear form $\langle \cdot, \cdot \rangle_s$ extends to $\overline{\mathscr{H}_{\mathbb{R}}}$ and its extension is still denoted by the same symbol. We then have the following proposition.

PROPOSITION 2. *The action of* $\mathrm{PSL}_2(\mathbb{R})$ *on* $\mathscr{H}_{\mathbb{R}}$ *extends to a continuous, irreducible, linear representation* $\overline{\pi_s}$ *on* $\overline{\mathscr{H}_{\mathbb{R}}}$. *The action of* $\mathrm{PSL}_2(\mathbb{R})$ *on* $\overline{\mathscr{H}_{\mathbb{R}}}$ *preserves the extended quadratic form* $\langle \cdot, \cdot \rangle_s$.

Proof. Let g be in $\mathrm{PSL}_2(\mathbb{R})$. We first look for a constant C such that

$$\|\pi_s(g)(f)\| \leqslant C\|f\|,$$

for $f \in \mathscr{H}_{\mathbb{R}}$. Write $\pi_s(g)(f_0) = c + u_0$ where c is constant and u_0 is a function with zero mean. We have

$$|c| = |\langle \pi_s(g)(f_0), (2\pi)^{-1}\rangle_s| = |\langle f_0, \pi_s(g^{-1})((2\pi)^{-1})\rangle_s| \leqslant \|f_0\| \cdot C_1 \leqslant \|f\| \cdot C_1,$$

where $C_1 = \|(\pi_s(g^{-1})((2\pi)^{-1}))_0\|$. Since $\pi_s(g)$ preserves $\langle \cdot, \cdot \rangle_s$, we have

$$-\langle u_0, u_0\rangle_s = c^2 - \langle f_0, f_0\rangle_s \leqslant c^2 + \|f\|^2 \leqslant (1 + C_1^2)\|f\|^2.$$

Finally we have

$$\|\pi_s(g)(f)\| \leqslant \|\pi_s(g)(m(f))\| + \|c\| + \|u_0\| \leqslant C\|f\|,$$

for $C = \|\pi_s(g)(1)\| + C_1 + \sqrt{1 + C_1^2}$. This implies that $\pi_s(g)$ extends to a bounded operator

$$\overline{\pi_s}(g) : \overline{\mathscr{H}_{\mathbb{R}}} \to \overline{\mathscr{H}_{\mathbb{R}}}.$$

One deduces easily from this that $\overline{\pi_s}$ is continuous and preserves the (extension of the) form $\langle \cdot, \cdot \rangle_s$ on $\overline{\mathscr{H}_{\mathbb{R}}}$. As for the irreducibility of $\overline{\pi_s}$, we argue as follows. Let $V \subset \overline{\mathscr{H}_{\mathbb{R}}}$ be a non-zero closed $\mathrm{PSL}_2(\mathbb{R})$-invariant subspace. It is, in particular, invariant under the group $\mathrm{SO}(2)$, which preserves the norm $\|\cdot\|$. Hence V is a direct sum of one-dimensional subspaces invariant by $\mathrm{SO}(2)$. From this, one sees that V must contain a non-zero vector in $\mathscr{H}_{\mathbb{R}}$. The irreducibility of π_s then implies that V contains $\mathscr{H}_{\mathbb{R}}$. Hence $V = \overline{\mathscr{H}_{\mathbb{R}}}$ since it is closed. □

As a consequence of the previous proposition, for each $s \in (\frac{1}{2}, \frac{3}{2})$, the representation $\overline{\pi_s}$ defines a non-elementary action of $\mathrm{PSL}_2(\mathbb{R})$ on the hyperbolic space $\mathbb{H}^\infty$, with no non-trivial closed invariant totally geodesic subspace. When the parameter s if greater than $\frac{3}{2}$, one obtains similarly actions of $\mathrm{PSL}_2(\mathbb{R})$ on the infinite-dimensional symmetric spaces associated to the (naturally defined) groups $\mathrm{O}(p, \infty)$ (see [Gro93, §6] for a description of these spaces).

Remark 4. The article [JW77] contains similar results for all classical simple Lie groups of rank one (namely $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ and $\mathrm{Sp}(n, 1)$). For $\mathrm{SO}(n, 1)$ ($n \geqslant 3$) one obtains invariant quadratic forms of finite index on certain representations, the index depending on the dimensions of the space of spherical harmonics of degree k. Note that the index 1 can be obtained. For $\mathrm{SU}(n, 1)$ and $\mathrm{Sp}(n, 1)$, however, the quadratic forms that one obtains are of infinite index (when they are not unitary); see the formulas in [JW77, p. 156].

Proof of Proposition 1. To prove the proposition, we use a different model for the representation π_s, which we now describe. Let $\xi = 1 \in \partial\Delta$ be the unique point fixed by the affine group of matrices of the form

$$\begin{pmatrix} * & *' \\ 0 & *^{-1} \end{pmatrix}.$$

We will use the fact that any element $g \in \mathrm{SL}_2(\mathbb{R})$ can be written uniquely as

$$g = k a_\lambda n_t,$$

where $k \in \mathrm{SO}(2)$, $a_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Let X^s be the space of measurable functions $F : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ which satisfy the following properties.

- The function F descends to $\mathrm{PSL}_2(\mathbb{R})$: $F(-g) = F(g)$.
- The restriction of F to $\mathrm{SO}(2)$ is square integrable.
- Induction rule: $F(g a_\lambda n_t) = F(g)\lambda^{-1-2s}$.

In the following, we identify $\mathrm{SO}(2)/\pm\,\mathrm{Id}$ to $\partial\Delta$ via the map $k\mapsto k(\xi)$. Now, if $F\in X^s$, its restriction to $\mathrm{SO}(2)$ defines an element of $\mathscr{H}_{\mathbb{C}}$, and conversely, if $f\in\mathscr{H}_{\mathbb{C}}$, one can extend it to a function $F\in X^s$ by declaring that $F(ka_\lambda n_t)=f(k(\xi))\lambda^{-1-2s}$. One verifies that under the previous identification of X^s with $\mathscr{H}_{\mathbb{C}}$ the action of $\pi_s(g)$ becomes simply the precomposition with the left translation by g^{-1}. Consider now the operator $F\mapsto LF$ defined by

$$LF(x)=\int_{\mathbf{R}}F(xw\underline{n}(u))\,du,$$

where $\underline{n}_u=\begin{pmatrix}1&0\\u&1\end{pmatrix}$ and $w=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. It is easy to check that if $F\in X^s$ is continuous the previous integral is well defined and defines a continuous function on $\mathrm{SL}_2(\mathbb{R})$. We will check that $LF\in X^{-s}$. Observe that one has the following identity:

$$xa_\lambda n_t w\underline{n}(u)=xw\underline{n}(\lambda^2(u-t))a_{\lambda^{-1}};$$

hence

$$F(xa_\lambda n_t w\underline{n}(u))=F(xw\underline{n}(\lambda^2(u-t)))\lambda^{1+2s},$$

and a change of variable gives $LF(xa_\lambda n_t)=LF(x)\lambda^{-(1-2s)}$. To finish the proof of the proposition, we thus have to check that the operator L extends continuously to all of X^s and coincides (up to a scalar factor) with the operator A_s defined by the formula in the statement of the proposition. Note that, by its very definition, the operator L intertwines the actions of $\mathrm{SL}(2,\mathbf{R})$ on X^s and X^{-s}. Let $k_\theta\in\mathrm{SO}(2)$ be the rotation of angle θ. A simple calculation shows that

$$w\underline{n}(u)=k_{\theta_u}a_{\sqrt{1+u^2}}n_{u/(1+u^2)}$$

where the angle θ_u is defined by the relation $e^{i\theta_u}=(i-u)/\sqrt{1+u^2}$. From this we deduce that $e_k(k_\theta w\underline{n}(u))=e_k(k_\theta)((i-u)/\sqrt{1+u^2})^{2k}(1+u^2)^{-(1/2)-s}$. This implies that $Le_k=\lambda_k e_k$, where λ_k is given by

$$\lambda_k=\int_{\mathbf{R}}\left(\frac{i-u}{\sqrt{1+u^2}}\right)^{2k}\frac{du}{(1+u^2)^{\frac{1}{2}+s}}=\int_{\mathbf{R}}\frac{(i-u)^{2k}\,du}{(1+u^2)^{k+\frac{1}{2}+s}}.$$

All λ_k having modulus bounded by $\int_{\mathbf{R}}du/(1+u^2)^{(1/2)+s}$, L extends continuously to X^s. Since $\lambda_k=\lambda_{-k}$ it is now enough to check the equality $\lambda_{k+1}=(k+\frac{1}{2}-s)/(k+\frac{1}{2}+s)\lambda_k$ for $k\geqslant 0$. Up to dividing by λ_0, this will show that the operator L has the form given in the statement of the proposition. We check this equality in the next lemma. □

Lemma 3. *For any positive integer k, one has $\lambda_{k+1}=((k+\frac{1}{2}-s)/(k+\frac{1}{2}+s))\lambda_k$.*

Proof. We start from the equality $\lambda_k=\int_{\mathbf{R}}((i-u)^{2k}/(1+u^2)^{k+(1/2)+s})\,du$ and integrate by parts (integrating $(i-u)^{2k}$) to obtain

$$\lambda_k=\frac{2(k+\frac{1}{2}+s)}{2k+1}\left(\lambda_{k+1}-\int_{\mathbf{R}}\frac{(i+u)(i-u)^{2k+1}}{(1+u^2)^{k+\frac{3}{2}+s}}\,du+\int_{\mathbf{R}}\frac{u(i-u)^{2k+1}}{(1+u^2)^{k+\frac{3}{2}+s}}\,du\right).$$

The second term in the parenthesis above equals λ_k; the third term is proportional to λ_k (by an integration by parts which is inverse to the one we just performed). Hence we get

$$\lambda_k=\frac{2(k+\frac{1}{2}+s)}{2k+1}\left(\lambda_{k+1}+\lambda_k-\frac{2k+1}{2(k+\frac{1}{2}+s)}\lambda_k\right).$$

The proof is now finished by a simple calculation. □

2.2 Some properties of actions of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^\infty$

In this section we prove a proposition which describes how the family $(\overline{\pi_s})_{s\in(\frac{1}{2},\frac{3}{2})}$ of actions of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^\infty$ varies with s. This will complete the proof of Theorem 2 (setting $t = s - \frac{1}{2}$, one recovers the notation of the theorem).

We denote here by $\mathbb{H}_s^\infty$ the hyperbolic space constructed from the pair $(\overline{\mathscr{H}_\mathbb{R}}, \langle \cdot, \cdot \rangle_s)$, where $\frac{1}{2} < s < \frac{3}{2}$ (all these spaces are isometric but we will keep this convenient notation). Observe that the group $\mathrm{SO}(2)$ has a unique fixed point in $\mathbb{H}_s^\infty$; hence there exists a unique $\overline{\pi_s}$-equivariant map $f_s : \Delta \to \mathbb{H}_s^\infty$. This map is automatically harmonic: indeed if the tension field of f_s was non-zero at a point $\xi \in \Delta$ the stabilizer of ξ in $\mathrm{PSL}_2(\mathbb{R})$ would fix the geodesic going through $f_s(\xi)$ and tangent to the tension field there, which is a contradiction. For the definition of the tension field of a map, or of its harmonicity, we refer the reader to paragraph 3.1.

PROPOSITION 4. *If g_{hyp} denotes the hyperbolic metric with curvature -1 both on $\mathbb{H}_s^\infty$ and on Δ, we have $f_s^* g_{\mathrm{hyp}} = c_s g_{\mathrm{hyp}}$ where*

$$c_s = \frac{(1+2s)(s-\frac{1}{2})}{4}.$$

As a consequence of this proposition, the image of f_s is a minimal surface in $\mathbb{H}_s^\infty$ with curvature $-c_s^{-1}$ (which goes to $-\infty$ or -1 as s goes to $\frac{1}{2}$ or $\frac{3}{2}$ respectively).

Proof. Since there is only one $\mathrm{PSL}_2(\mathbb{R})$-invariant metric on the unit disc, up to a scalar, we can write $f_s^* g_{\mathrm{hyp}} = c_s g_{\mathrm{hyp}}$, for some constant c_s. We want to compute the value of c_s. It will be convenient to identify $\mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{SU}(1,1)$ (up to a 2-sheeted cover). So let $g_t : \Delta \to \Delta$ be the isometry induced by the matrix

$$\begin{pmatrix} \mathrm{ch}(t) & \mathrm{sh}(t) \\ \mathrm{sh}(t) & \mathrm{ch}(t) \end{pmatrix}.$$

We will also denote by o the origin of the disc. We will prove the following estimate:

$$d_{\mathbb{H}_s^\infty}(f_s(g_t(o)), f_s(o)) = \sqrt{(1+2s)(s-\tfrac{1}{2})}\, t + o(t). \tag{$*$}$$

This allows us to draw a conclusion easily. Denote by v the derivative of the curve $g_t \cdot o$ (which is nothing other than $1 \in \mathbb{C}$). Since the hyperbolic metric on the disc is $4\,dz\,d\bar{z}/(1-|z|^2)^2$ we have $g_{\mathrm{hyp}}(v,v) = 4$. On the other hand, the equality above says that $g_{\mathrm{hyp}}(Df_s(v), Df_s(v)) = (1+2s)(s-\frac{1}{2})$. Hence $c_s = (1+2s)(s-\frac{1}{2})/4$.

We now prove the equality $(*)$. Observe that the constant function equal to $1/\sqrt{2\pi}$ is the unique fixed point of the group $\mathrm{SO}(2)$ in $\mathbb{H}_s^\infty$; hence $f_s(o) = 1/\sqrt{2\pi}$ and $f_s(g_t(o))$ is the function sending a point $e^{i\theta}$ to $(2\pi)^{-\frac{1}{2}}|\mathrm{sh}(t)e^{i(\theta+\pi)} + \mathrm{ch}(t)|^{-1-2s}$. Define

$$u(s,t) := \langle \pi_s(g_t)(1), 1 \rangle_s = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|\mathrm{sh}(t)e^{i\theta} + \mathrm{ch}(t)|^{1+2s}}.$$

A calculation shows that

$$\frac{1}{|\mathrm{sh}(t)e^{i\theta} + \mathrm{ch}(t)|^{1+2s}} = 1 - \left(\frac{1}{2} + s\right) 2t\cos(\theta)$$
$$+ \left((1+2s)\left(\frac{3}{2} + s\right)\cos^2(\theta) - (1+2s)\right) t^2 + o(t^2)$$

and then that

$$u(s,t) = 1 + (1 + 2s)\left(\frac{s}{2} - \frac{1}{4}\right)t^2 + o(t^2).$$

Remembering that $u(s,t) = \mathrm{ch}(d(\pi_s(g_t) \cdot 1, 1))$, one easily deduces the formula $(*)$. □

3. Preliminaries on harmonic maps

3.1 The classical case

We start by recalling a few properties of *harmonic maps*. Some references for a more detailed presentation of the material in this section are [CMP03, GS92, KS93, Nis02].

A map $f : M \to N$ between two Riemannian manifolds is harmonic if it is a critical point of the energy functional $E(f) := \int_M e(f)(x)\,dx$, where dx is the Riemannian measure on M and where $e(f)(x) = \frac{1}{2}\|df_x\|^2$ is the energy density of f. The *tension field* of a map $f : M \to N$ is the vector field along the image of f defined as follows. Let ∇ be the connection on the vector bundle $\mathrm{T}^*M \otimes f^*\mathrm{T}N$ constructed from the Levi-Civita connections of M and N. The differential df of f can be thought of as a section of the bundle $\mathrm{T}^*M \otimes f^*\mathrm{T}N$, and the tension field $\tau(f)$ of f is the trace of the 2-tensor $\nabla\,df$:

$$\tau(f) = \sum_{i=1}^{m} \nabla df(e_i, e_i),$$

where $(e_i)_{1 \leqslant i \leqslant m}$ is a local orthonormal frame of $\mathrm{T}M$. One can show that the harmonicity of f is equivalent to the vanishing of the tension field $\tau(f)$ of f (see for instance [Nis02]).

A more general context for studying harmonic maps is the following. Assume we are given an isometric action of the fundamental group $\pi_1(M)$ of M on a Riemannian manifold $\widetilde{N}$ (which plays the role of the universal cover of N in the previous discussion):

$$\rho : \pi_1(M) \to \mathrm{Isom}(\widetilde{N}),$$

where $\mathrm{Isom}(\widetilde{N})$ is the isometry group of $\widetilde{N}$. One can consider maps $f : \widetilde{M} \to \widetilde{N}$ which are equivariant with respect to the natural action of $\pi_1(M)$ on $\widetilde{M}$ and the action ρ on $\widetilde{N}$ (here $\widetilde{M}$ is the universal cover of M). Such maps exist when $\widetilde{N}$ is non-positively curved and simply connected. The energy density

$$e(f) : \widetilde{M} \to \mathbb{R}$$

of such an f is invariant under the action of $\pi_1(M)$, and can be thought of as a function on M. Hence one can still define the energy of f, and once again f is said to be harmonic if it is a critical point of the energy functional. If $\widetilde{N} = G/K$ is a symmetric space of non-compact type, it is known that a harmonic equivariant map exists when the Zariski closure of the image of ρ is a reductive subgroup of G. This is due to Corlette; see [ABCKT96, CMP03].

When M is Kähler, one can say a lot more about harmonic maps (this is the content of the so-called *non-abelian Hodge theory*). In this context the equation of harmonic maps takes the following form. Let ω be the Kähler form on M. The Hodge star operator applied to the tension field gives rise to a $2n$-form $*\tau(f)$ on M ($n = \dim_{\mathbb{C}}(M)$) with values in the bundle $f^*\mathrm{T}\widetilde{N}$ which has the following expression:

$$*\tau(f) = \frac{-1}{(n-1)!}\omega^{n-1} \wedge d_\nabla(df \circ i),$$

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where d_∇ is the natural operator on $f^*T\widetilde{N}$-valued differential forms induced by the pull-back of the Levi-Civita connection of $\widetilde{N}$ (see [CMP03, §14.2, p. 363] for a proof). Hence the harmonicity of f is equivalent to the equation

$$\omega^{n-1} \wedge d_\nabla(df \circ i) = 0.$$

We now denote again by X a compact Kähler manifold, with fundamental group Γ. We fix a Zariski dense representation $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$. According to the previous discussion, there exists a map $f : \widetilde{X} \to \mathbb{H}^n$ which is Γ-equivariant and harmonic. Moreover, f satisfies the following properties.

(i) It is pluriharmonic, i.e. it satisfies $d_\nabla(df \circ i) = 0$.

(ii) The $(0, 1)$ part of d_∇, denoted by $d_\nabla^{0,1}$, satisfies

$$(d_\nabla^{0,1})^2 = 0.$$

According to the Koszul–Malgrange integrability theorem [KM58], this implies that there exists a holomorphic structure on the bundle $f^*T\mathbb{H}^n \otimes \mathbb{C}$ for which $\overline{\partial} = d_\nabla^{0,1}$.

(iii) The complex linear part $\alpha := df^{1,0}$ of the differential of f is a holomorphic 1-form with values in $f^*T\mathbb{H}^n \otimes \mathbb{C}$ (for the previous holomorphic structure).

(iv) The complex rank of α is everywhere less than or equal to 1, and consequently the real rank of f is everywhere less than or equal to 2.

The first three results are proved using the classical Bochner–Siu–Sampson formula (see [ABCKT96] or [CMP03]). The last one is due to Sampson [Sam86]. Note that the first three results hold more generally when the real hyperbolic space is replaced by any symmetric space of non-compact type. The last one, on the other hand, is very specific to $\mathbb{H}^n$.

All the notions that we have just described still make sense when the representation $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ is replaced by a non-elementary action of Γ on the space $\mathbb{H}^\infty$ and f is an equivariant map from $\widetilde{X}$ to $\mathbb{H}^\infty$. The existence of a harmonic map in this case is a result of Korevaar and Schoen [KS93, KS97]. In the next two subsections, we will describe this result and explain why the four properties listed above still hold in this context.

3.2 Korevaar and Schoen's harmonic map

We start with a remark. We need to do some differential geometric calculations in $\mathbb{H}^\infty$. To this end, observe that the Levi-Civita connection of $\mathbb{H}^\infty$ can be defined in the usual way as the unique torsion-free connection on the tangent bundle $T\mathbb{H}^\infty$ of $\mathbb{H}^\infty$, which is compatible with the metric. When we use the Poincaré ball model for $\mathbb{H}^\infty$, i.e. we identify $\mathbb{H}^\infty$ with the unit ball $\mathscr{B} \subset \mathscr{H}$, endowed with the metric $4\langle \cdot, \cdot \rangle/(1 - |x|^2)^2$, we see easily that if $X : \mathscr{B} \to \mathscr{H}$ is a smooth map thought of as a vector field on $\mathbb{H}^\infty$ then

$$\nabla X = dX + A(X),$$

where A is a smooth 1-form with values in the space of endomorphisms of $\mathscr{H}$. One can compute easily the precise expression of A in terms of the function $4/(1 - |x|^2)^2$ and of its gradient, but we will not need it. In the following, if v is a vector in $T\mathbb{H}^\infty$, we will write A_v for the value of the 1-form A on v, so that $A_v : \mathscr{H} \to \mathscr{H}$ is a continuous linear map.

We now fix a non-elementary homomorphism $\rho : \Gamma \to \text{Isom}(\mathbb{H}^\infty)$, and we want to study ρ via an associated harmonic map. The existence of an equivariant harmonic map in this context is a consequence of the work of Korevaar and Schoen [KS93, KS97]; see [KS97, Theorem 2.3.1].

The minimizing map constructed by Korevaar and Schoen is Lipschitz and belongs to a certain Sobolev space defined in metric terms; see [KS93] for the definition. Since in our case the target is an infinite-dimensional manifold, we will see that the harmonic map is in fact smooth. It is well known that harmonic maps are smooth once we know their continuity (see for instance [BG80] for a proof for maps from the Euclidean space into the sphere, as well as [SU87] for a discussion of the regularity of harmonic maps). In our case the stronger fact that f is Lipschitz was established in [KS97]; deducing the smoothness from this is easy. Since we were not able to find a reference for this in the literature, we now prove this fact.

In the following, B will be a closed ball in $\widetilde{X}$ contained in some coordinate chart, g will denote the Riemannian metric on $\widetilde{X}$ and Δ_g the Laplacian associated to g. We will need to deal with various Lebesgue and Sobolev spaces of maps with values in the real separable Hilbert space $\mathscr{H}$. One defines the space $L^p(B, \mathscr{H})$ (for $p \geqslant 1$), which we will simply denote by L^p, as the space of all measurable maps $u : B \to \mathscr{H}$ such that $\int_B \|u(x)\|^p \, dx < +\infty$ (where dx stands for the Riemannian measure associated to g and $\|\cdot\|$ is the norm on $\mathscr{H}$). We define the Sobolev spaces $W^{k,p}(B, \mathscr{H})$ (or $W^{k,p}$ for short) in the usual way, using the scalar product on $\mathscr{H}$. Finally, we will write $u \in L^p_{\mathrm{loc}}$ (or $u \in W^{1,p}_{\mathrm{loc}}$ or $u \in W^{2,p}_{\mathrm{loc}} \ldots$) if the restriction of u to any closed ball contained in the interior of B is in L^p. See [DU77] for a detailed study of Lebesgue spaces of vector-valued functions. Note that in the present case, since we are dealing with functions with values in a Hilbert space, the integral of a measurable map $u : B \to \mathscr{H}$ such that $\int_B \|u\| \, dx < +\infty$ can be defined easily by duality.

We start with an easy proposition.

PROPOSITION 5. *Assume that the measurable map $u : B \to \mathscr{H}$ is in $W^{k,2}$ for any integer k. Then u is of class C^∞.*

Proof. Writing $(u_i)_{i \geqslant 0}$ for the coordinates of u in a Hilbert basis of $\mathscr{H}$, the hypothesis implies that each u_i is in the usual Sobolev space $W^{k,2}$ and for each fixed k, the series $\sum_i |u_i|^2_{W^{k,2}}$ is convergent. Using the classical regularity theorems, each u_i is smooth, and for each r, and each k large enough compared to r, there exists a constant $D_{r,k}$ such that $|u_i|_{C^r} \leqslant D_{r,k} |u_i|_{W^{k,2}}$. Thus, the series $\sum_i |u_i|^2_{C^r}$ is convergent, which implies that u is C^∞. $\square$

The following proposition is classical, at least for real-valued functions. We will briefly recall its proof in §7, to explain why it extends to the case of vector-valued maps.

PROPOSITION 6. *If a map u is in $W^{1,p}_{\mathrm{loc}}$ for all p and if $\Delta_g u$ is in L^p_{loc} for all p, then u is in $W^{2,p}_{\mathrm{loc}}$ for all p.*

We are now ready to prove the following proposition.

PROPOSITION 7. *The harmonic map $f : \widetilde{X} \to \mathbb{H}^\infty$ is of class C^∞.*

Proof. We fix a point $x_0 \in \widetilde{X}$ and work in a ball B around x_0. We denote by $x_1, \ldots, x_{2n}$ the coordinates on B. We also use the Poincaré ball model for $\mathbb{H}^\infty$, i.e. we identify $\mathbb{H}^\infty$ with the unit ball $\mathscr{B}$ of $\mathscr{H}$ endowed with the metric $\langle \cdot, \cdot \rangle_{\mathrm{hyp}} := 4\langle \cdot, \cdot \rangle/(1 - |x|^2)^2$. Since f is continuous, we can assume, by shrinking B if necessary, that $f(B)$ is contained in a small ball of $\mathbb{H}^\infty$ where the hyperbolic and the flat metric are comparable. This implies that f (restricted to B) is in the Sobolev space defined by Korevaar and Schoen for any of the two metrics. However, for the flat metric on $\mathscr{H}$, Korevaar and Schoen's Sobolev space coincides with the usual space $W^{1,2}$ defined above. This is [KS93, Theorem 1.6.2] (that theorem deals with real-valued functions but can be adapted easily to the case where the target is a Hilbert space).

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Observe that, since f is Lipschitz, the partial derivatives of order one of f are in fact L^∞. As in [BG80], we are going to show that for any integer k and any multiindex $I = \{\alpha_1, \ldots, \alpha_k\}$ of length k, the partial derivative $\partial_I f := \partial_{\alpha_1} \cdots \partial_{\alpha_k} f$ is in L^p_{loc} for any $p > 1$. Together with Proposition 5, this will prove the proposition.

First observe that the fact that f minimizes energy implies (as in the smooth case) that for any smooth compactly supported map $V : B \to \mathscr{H}$ (thought of as a vector field along f) one has

$$\int \langle \nabla V, df \rangle_{\mathrm{hyp}} \, dx = 0.$$

Here the symbol dx represents the integration on B with respect to the volume form induced by g, and the scalar product of the two 1-forms is defined in the usual way. When f is smooth this last integral equals $-\int \langle V, \tau(f) \rangle_{\mathrm{hyp}} \, dx$ where $\tau(f)$ is the tension field of f (see [Nis02, p. 102]). Remembering that $\nabla V = dV + A(V)$, it is easy to deduce from the previous equality that f satisfies (in the sense of distributions) the equation

$$\Delta_g f + \sum_{i,j=1}^{2n} g^{ij} A_{\partial f / \partial x_i} \left(\frac{\partial f}{\partial x_j} \right) = 0. \qquad (*)$$

We now prove by induction on k that $\partial_I f \in L^p$ for any p and for any multiindex I of length k. This is true for $k = 1$. Assuming it is true for a certain integer $k \geqslant 1$ we prove it for $k + 1$. Let I be a multi-index of length $k - 1$. According to $(*)$, the function $\Delta_g \partial_I f$ is a linear combination of products of partial derivatives of length at most k of $A_i(f)$, g^{ij} and f and of the term $\Delta_g \partial_I f - \partial_I \Delta_g f$. This last term involves derivatives of f up to the order k. By induction, all these terms are in L^p for any p, and hence by the Hölder inequality $\Delta_g \partial_I f$ is in L^p for any p. By Proposition 6, $\partial_I f \in W^{2,p}$, and hence $f \in W^{k+1,p}$. $\qquad \square$

Once f is known to be smooth, we can define, as before, the operator d_∇ acting on differential forms with values in the bundle $f^* T\mathbb{H}^\infty \to \widetilde{X}$, and one checks, as in the finite-dimensional case, that the harmonicity of f is equivalent to the usual equation

$$\omega^{n-1} \wedge d_\nabla(df \circ i) = 0.$$

3.3 The Bochner–Siu–Sampson formula

The next step is to check that the usual Bochner–Siu–Sampson formula still holds in this context, showing that f is pluriharmonic and that the curvature $(d_\nabla)^2$ is purely of type $(1,1)$.

PROPOSITION 8. *The following hold.*

 (i) *The map f is pluriharmonic, i.e. $d_\nabla(df \circ i) = 0$.*

 (ii) *The pull-back of the curvature of $\mathbb{H}^\infty$ (thought of as a 2-form with values in $\mathrm{End}(T\mathbb{H}^\infty)$) is of type $(1,1)$, which implies that $D := d_\nabla^{1,0}$ satisfies $D^2 = 0$.*

 (iii) *The 1-form $df^{1,0}$ satisfies $D(d^{1,0} f) = 0$.*

Proof. The proof of this proposition in the finite-dimensional case is an application of the classical Bochner–Siu–Sampson formula; see [ABCKT96] or [CMP03, §14.2]. Here we only remind the reader of the strategy of the proof and explain why it goes through without changes when the target is an infinite-dimensional hyperbolic space.

We first introduce the following notation. If β and δ are differential forms on $\widetilde{X}$ with values in the bundle $f^* T\mathbb{H}^\infty \otimes \mathbb{C}$, one defines naturally the $k + l$ form $\langle \beta \wedge \delta \rangle$: if $\beta = \sum \beta_i s_i$

and $\delta = \sum_j \delta_j v_j$ (where the β_i and the δ_j are complex valued forms and the s_i and the t_j are sections of $f^* T\mathbb{H}^\infty \otimes \mathbb{C}$) we define

$$\langle \beta \wedge \delta \rangle := \sum_{i,j} \beta_i \wedge \overline{\delta_j} \langle s_i, t_j \rangle$$

where $\langle s_i, t_j \rangle$ is computed using the natural hermitian metric on $f^* T\mathbb{H}^\infty \otimes \mathbb{C}$.

The proof consists in showing that the real-valued $2n$-form (which is well defined on X)

$$\langle d_\nabla^2 (df \circ i) \wedge df \circ i \rangle \wedge \omega^{n-2} + \langle d_\nabla (df \circ i) \wedge d_\nabla (df \circ i) \rangle \wedge \omega^{n-2}$$

is at the same time exact and pointwise non-positive. Stokes' theorem then implies that it is identically zero. The exactness of this form is proved as in the finite-dimensional case. As for the sign, the harmonic equation tells us that for any section s of $f^* T\mathbb{H}^\infty$, the 2-form $\langle d_\nabla (df \circ i), s \rangle \wedge \omega^{n-1}$ is primitive. This implies that the second term in the formula above is non-positive and vanishes only if $d_\nabla (df \circ i) = 0$ identically. The first term is also non-positive and vanishes precisely if the curvature d_∇^2 is of type $(1,1)$. The proof of this fact goes through exactly as in the finite-dimensional case; see [ABCKT96, p. 74].

The proof of the third point of the proposition is now easy. The 1-form

$$\alpha := df^{1,0} = df - i \, df \circ i$$

is d_∇-closed since both df and $df \circ i$ are. Since α is of type $(1,0)$, this implies that both $d_\nabla^{1,0} \alpha$ and $d_\nabla^{0,1} \alpha$ are 0. $\qquad\square$

In the following, we will continue to denote by α the complex linear part of the differential of f:

$$\alpha := df^{1,0}.$$

Exactly as in [Sam86], the previous proposition implies that the complex rank of α is at most one. We will briefly summarize the proof of this fact here. See [ABCKT96, ch. 6, p. 77], for more details. We denote by R the curvature tensor of $\mathbb{H}^\infty$ and by $R_\mathbb{C}$ its complex linear extension to $(T\mathbb{H}^\infty \otimes \mathbb{C})^{\otimes 4}$. Denoting by $\langle \cdot, \cdot \rangle$ the complex bilinear extension of the Riemannian metric of $\mathbb{H}^\infty$ to $T\mathbb{H}^\infty \otimes \mathbb{C}$, one has

$$R_\mathbb{C}(X, Y, Z, W) = \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle. \qquad (*)$$

Now, one can show (see [ABCKT96]) that the second item of the previous proposition, namely, the fact that the pull-back by f of the curvature is of type $(1,1)$, is equivalent to the fact that the *Hermitian sectional curvature* is zero on the image of α: for all X, Y in the image of α we have

$$R_\mathbb{C}(X, Y, \overline{X}, \overline{Y}) = 0.$$

From the expression $(*)$ for $R_\mathbb{C}$, one sees easily that this implies that the image of α has complex dimension at most one. This implies that the real rank of f is not greater than two.

The next step is to check that the 1-form α is actually holomorphic for a suitable holomorphic structure on the bundle $f^* T\mathbb{H}^\infty \otimes \mathbb{C}$. To achieve this, we will need an analogue of the Koszul–Malgrange theorem in this infinite-dimensional context. The statement of this theorem for a Hilbert space bundle does not present any difficulty. Knowing that the operator D acting on sections of the bundle $f^* T\mathbb{H}^\infty \otimes \mathbb{C} \to \widetilde{X}$ satisfies $D^2 = 0$, we want to prove that there exist local trivializations of this bundle in which the operator D becomes the usual $\overline{\partial}$ operator. We will see in § 6 that the proof in the finite-dimensional case can be adapted without difficulty to this

context. Hence we assume from now on that such trivializations exist. Observe, however, that around any point x such that $\alpha_x \neq 0$ the image of α is a rank one subbundle of $f^*T\mathbb{H}^\infty$ which is stable by the operator D. Therefore, we can apply the classical Koszul–Malgrange theorem to this rank one bundle around any point where α is not zero. So we only need the infinite-dimensional version of the theorem to study points where α vanishes.

4. Actions on infinite-dimensional real hyperbolic spaces

In this section, we prove Theorem 1. We consider a minimal non-elementary action $\Gamma \to \mathrm{Isom}(\mathbb{H}^\infty)$ and work with the associated equivariant harmonic map $f : \widetilde{X} \to \mathbb{H}^\infty$. First, let us observe that one can assume that the rank of f equals two on a dense open set. Otherwise the image of f would be a geodesic or a point (see [Sam78, Theorem 3]; the proof works just as well in our context) and the representation ρ would be elementary.

4.1 Factorization

The goal of this subsection is to prove the following theorem. It is classical in spirit and similar to many results in the literature. Although a very similar statement appears in [JY91], we include a complete proof here.

THEOREM 5. *There exists a holomorphic map $\pi : \widetilde{X} \to \Delta$ and a harmonic map $u : \Delta \to \mathbb{H}^\infty$ such that*

$$f = u \circ \pi.$$

Moreover, there exists a homomorphism $\theta : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ and a homomorphism

$$\Psi : \Lambda := \theta(\Gamma) \to \mathrm{Isom}(\mathbb{H}^\infty)$$

such that $\rho = \Psi \circ \theta$. The map π is θ-equivariant and the map u is Ψ-equivariant.

To prove this theorem, we follow an idea from [Mok88, § 2.2]. In the course of the proof, we will use the fact that the map f is real-analytic (as a solution of an elliptic equation with real-analytic coefficients), as well as the following lemma, whose proof we omit.

LEMMA 9. *Let O be a manifold and let $R \subset O \times O$ be the graph of an equivalence relation on O. If R is closed and if the two projections $p_1, p_2 : R \to O$ are open maps (R being endowed with the topology induced from $O \times O$), then the quotient space O_R is Hausdorff and the map $O \to O_R$ is an open map.*

Let U be the open set of $\widetilde{X}$ where the holomorphic 1-form α is non-zero. Note that its kernel defines a holomorphic foliation $\mathscr{F}$ on U (for instance because the kernel of α coincides with that of df on a dense open set). The foliation $\mathscr{F}$ extends to $O := \widetilde{X} - Z$ where Z is an analytic set of codimension at least two in $\widetilde{X}$, called the singular set of $\mathscr{F}$. To prove Theorem 5, it is enough to construct a holomorphic equivariant map $O \to \Delta$, since such a map will automatically extend to $\widetilde{X}$.

We define a subset R of $O \times O$ as follows. The pair $(x, y) \in O \times O$ is in R if we can find two germs of holomorphic maps $\varphi_x : (\mathbb{C}, 0) \to (O, x)$ and $\varphi_y : (\mathbb{C}, 0) \to (O, y)$ whose images are not contained in a leaf of $\mathscr{F}$ and such that $f \circ \varphi_x = f \circ \varphi_y$. Let us make some comments about this definition.

(i) If one takes coordinates $z = (z_1, z') \in \Delta(0, \varepsilon) \times \Delta(0, \delta) \subset \mathbb{C} \times \mathbb{C}^{n-1}$ and $w = (w_1, w') \in \Delta(0, \varepsilon) \times \Delta(0, \delta) \subset \mathbb{C} \times \mathbb{C}^{n-1}$ centered at x and y respectively, and such that $\mathscr{F}$ is locally defined

by the equations $dz_1 = 0$ and $dw_1 = 0$, one sees that R is locally the 'pull-back' by the map $(z, w) \mapsto (z_1, w_1)$ of a subset R_0 of $\Delta(0, \varepsilon) \times \Delta(0, \varepsilon)$. Hence, by projecting onto the z_1 (or w_1) coordinate we can think of the two maps φ_x and φ_y as maps $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$. We will always do this in what follows, and work with R_0. Observe also that in such coordinates one has $f(z_1, z') = f(z_1)$ (and similarly $f(w_1, w') = f(w_1)$).

(ii) We observe that R defines an equivalence relation on O. The only non-trivial point is the fact that R defines a transitive relation, but this essentially follows from the fact that two (germs of) ramified covers of $(\mathbb{C}, 0)$ share a common cover.

(iii) If $(x, y) \in R$, Let $t > 0$ be such that φ_x and φ_y are both defined on the disc $\Delta(0, t)$. Then the germ of complex analytic set

$$\{(\varphi_x(z), \varphi_y(z)), z \in \Delta(0, t)\} \subset R_0 \subset \mathbb{C}^2$$

is contained in the real analytic set $R_1 := \{(z, w), f(z) = f(w)\} \subset \mathbb{C}^2$. Since f is not constant on any open set, R_1 cannot contain a germ of analytic curve whose projection to one of the two factors of $\mathbb{C}^2$ is constant. Hence, we see that R_0 is exactly the set of points of R_1 which are contained in a positive dimensional germ A of complex analytic set with $A \subset R_1$. According to [DM08] this set is closed. Hence the graph R of the equivalence relation we just defined is closed. The projection maps $p_1, p_2 : R \subset O \times O \to O$ are open. According to the lemma mentioned above, the quotient O_R of O by the equivalence relation R is Hausdorff and the projection map $\pi : O \to O_R$ is open.

We now prove that O_R is a Riemann surface. We start with the following proposition.

PROPOSITION 10. *For any point p in O, there is a neighborhood U of (p, p) in $O \times O$ such that $R \cap U$ is a complex analytic set.*

Proof. We choose coordinates $(z_1, \ldots, z_n) = (z_1, z')$ in a polydisc

$$V = \Delta(0, \varepsilon) \times \Delta(0, \delta) \subset \mathbb{C} \times \mathbb{C}^{n-1}$$

centered at p in such a way that the foliation $\mathscr{F}$ is defined on V by the equation $dz_1 = 0$. Let D be the disc $\Delta(0, \varepsilon) \times \{0\}$. In these coordinates, the harmonic map depends on z_1 only. After choosing a trivialization of the bundle $f^* T\mathbb{H}^\infty \otimes \mathbb{C}$ near p, the form α can be thought of as a 1-form depending on z_1 only, with values in a fixed complex Hilbert space $\mathbf{H}$. Let $(x, y) \in V \times V \cap R$ and let $\varphi_x, \varphi_y : (\mathbb{C}, 0) \to D$ be the associated maps as in the definition of R. From the equation $f \circ \varphi_x = f \circ \varphi_y$ one gets $\varphi_x^* \alpha = \varphi_y^* \alpha$. By choosing a basis $(e_1, e_2, \ldots)$ of $\mathbf{H}$ and maybe making a change of coordinate on z_1, we can assume that

$$\alpha = z_1^k dz_1 e_1 + \sum_{j \geqslant 2} a_j(z_1) \, dz_1 e_j,$$

for some integer k. By projecting the equation $\varphi_x^* \alpha = \varphi_y^* \alpha$ on the first coordinate, we get

$$\varphi_x^k \varphi_x' = \varphi_y^k \varphi_y'.$$

Hence the set $R_0 \subset D \times D$ (which was defined to be the image of R by the map $(z, w) \mapsto (z_1, w_1)$) is contained in the set

$$\bigcup_{c \in \mathbb{C}} A_c,$$

where $A_c := \{(z, w) \in D \times D, z^{k+1} - w^{k+1} = c\}$. We are going to see that, up to replacing D by a smaller disc, one has $R_0 \subset A_0$. This will prove the proposition. If this is false, there is a

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sequence c_n of non-zero complex numbers converging to 0 such that

$$A_{c_n} \cap R_0 \neq \emptyset. \qquad (*)$$

We will prove that this implies that the rank of f equals one on an open set, which is a contradiction. For n large enough, the set A_{c_n} is a connected Riemann surface in $D \times D$. If it intersects R_0, the two harmonic maps $f \circ p_1$ and $f \circ p_2$ (where the p_i are the projections $D \times D \to D$) coincide on an open set of A_{c_n}. Hence they are equal on A_{c_n}, and A_{c_n} is entirely contained in R_0. We thus assume that

$$A_{c_n} \subset R_0 \subset R_1,$$

for all n. Now, for any point $x \in D$ one can find a sequence of points x_n such that

$$x^{k+1} - x_n^{k+1} = c_n$$

and $x_n \to x$. Since $A_{c_n} \subset R_1 := \{(a, b), f(a) = f(b)\}$ we have $f(x) = f(x_n)$, and by taking a limit point of the sequence $(x - x_n)/|x - x_n|$ one sees that rank $df_x \leqslant 1$ for $x \in D$. This is the contradiction we were looking for. $\qquad \square$

With the help of the previous proposition, we can complete the proof. The quotient of a polydisc V, as in the proposition above, naturally has the structure of a Riemann surface. Since π is an open map, this endows the open set $\pi(V) \subset O_R$ with a complex structure. It is then easy to check that the various complex structures on the open sets $\pi(V)$ (for various choices of V) are compatible and make the map π holomorphic.

Remark 5. The idea of choosing a complex analytic equivalence relation whose graph is contained in the real analytic set $\{(a, b), f(a) = f(b)\}$ is due to Mok; see [Mok88]. However, we use it in a slightly different way. Note also that we cannot prove immediately that R is complex analytic everywhere; this is why in Proposition 10 we restrict to a neighborhood of the diagonal. Indeed, if we start with a point $(p, q) \in R$ $(p \neq q)$, there is no way to identify (*a priori*) the fibers of the bundle $f^* T\mathbb{H}^\infty \otimes \mathbb{C}$ near p and near q.

From now on we will denote by Σ the Riemann surface O_R. At this stage we have a holomorphic map $\pi : \widetilde{X} \to \Sigma$ and a continuous map $u : \Sigma \to \mathbb{H}^\infty$ such that $f = u \circ \pi$. We still have to justify that u is harmonic. However, this is clear in a neighborhood of any point which is not a critical value of π since f is pluriharmonic. To establish the smoothness and the harmonicity of u near a critical value of π, we borrow an argument from [Mok92, p. 577]. Let p be a critical value of π and let $x \in \widetilde{X}$ be such that $\pi(x) = p$. We can assume that x is a regular point of the foliation defined by the kernel of α. We choose coordinates $z = (z_1, \ldots, z_n)$ centered at x such that π is locally represented by the map $z \mapsto z_1^k$ for some integer k. From the equation

$$f(z) = u(z_1^k),$$

we want to deduce that u is smooth and harmonic. We will denote by D_0 a small disc centered at the origin in the complex plane (corresponding to the z_1 coordinate) and by D its image under the map $z_1 \mapsto z_1^k$. We also write $f(z) = f(z_1)$ in what follows. Let $v : D \to \mathbb{H}^\infty$ be the harmonic map whose boundary value is equal to u. It exists according to [KS93, Theorem 2.2]. Now the map $\widetilde{v}$ defined on D_0 by $\widetilde{v}(z_1) = v(z_1^k)$ is harmonic and equals f on the boundary of D_0. Hence it must equals f on D_0. This implies that $u = v$ on D and proves the result.

We now explain why we can assume that Σ is the unit disc Δ. First, by the maximum principle, the Riemann surface Σ cannot be compact. Second, we observe that the map $\pi : \widetilde{X} \to \Sigma$

is equivariant with respect to a homomorphism $\theta : \Gamma \to \mathrm{Aut}(\Sigma)$. The group $\mathrm{Aut}(\Sigma)$ cannot be solvable as a consequence of the following lemma.

LEMMA 11. *Let G be a finitely generated solvable subgroup of $\mathrm{Isom}(\mathbb{H}^\infty)$. Then either G stabilizes a geodesic of $\mathbb{H}^\infty$ or G fixes a point in $\mathbb{H}^\infty \cup \partial\mathbb{H}^\infty$.*

Note that this lemma is very general and remains true for groups of isometries of δ-hyperbolic spaces. A proof in the case of $\mathbb{H}^\infty$ can be found in [Can11, §6.3], for instance. It implies that the universal cover of Σ is isomorphic to the unit disc. Although Σ might not be simply connected at the beginning, one can always lift the map π (as well as the representation $\Gamma \to \mathrm{Aut}(\Sigma)$) to the unit disc. To be also able to lift the map $u : \Sigma \to \mathbb{H}^\infty$ to a map defined on Δ, it is enough to check that the lift $\widetilde{\pi} : \widetilde{X} \to \Delta$ of π is surjective. Since the image of $\widetilde{\pi}$ is open we need to check that it is also closed. If the automorphism group of Σ is discrete, it is easy to see that the image of $\widetilde{\pi}$ is closed. We finish by observing that the only hyperbolic Riemann surface whose automorphism group is not discrete and not solvable is the unit disc itself; if $\Sigma = \Delta$, $\widetilde{\pi} = \pi$ and we know from the beginning that this map is surjective. Hence, up to replacing Σ by its universal cover and π by a lift, we now assume that $\Sigma = \Delta$.

Before concluding the proof of Theorem 5, we state the following lemma.

LEMMA 12. *If an isometry $g : \mathbb{H}^\infty \to \mathbb{H}^\infty$ fixes pointwise the image of the map f then g is the identity.*

Proof. Let W be the intersection of all closed totally geodesic subspaces of $\mathbb{H}^\infty$ containing the image of f. It is itself a closed totally geodesic subspace of $\mathbb{H}^\infty$, which is invariant by the action of Γ. Since we assumed that the action was minimal, we have $W = \mathbb{H}^\infty$. We complete the proof by observing that the fixed point set of g contains W. □

We can now finish. Let Λ be the image of θ. According to Lemma 12, the kernel of θ is contained in the kernel of ρ. Hence we can write $\rho = \Psi \circ \theta$, where $\Psi : \Lambda \to \mathrm{Isom}(\mathbb{H}^\infty)$ is a homomorphism. This completes the proof of Theorem 5.

4.2 Conclusion of the proof of Theorem 1

If the group Λ introduced above is discrete in $\mathrm{PSL}_2(\mathbb{R})$, the representation factors through a fibration onto a hyperbolic 2-orbifold. Indeed one gets a map $X \to \Delta/\Lambda$, which might not have connected fibers. However, we can apply the Stein factorization theorem to it to obtain a fibration of X onto a hyperbolic 2-orbifold. The obvious, but crucial observation, which appears already in [GS92, Lemma 9.4], is now the following: *if Λ is not discrete, it is topologically dense in $\mathrm{PSL}_2(\mathbb{R})$.* Assuming that Λ is not discrete, we now prove that the homomorphism $\Psi : \Lambda \to \mathrm{Isom}(\mathbb{H}^\infty)$ extends continuously to $\mathrm{PSL}_2(\mathbb{R})$. Let (γ_n) be a sequence in Λ which converges to $g \in \mathrm{PSL}_2(\mathbb{R})$. We are going to prove that $\rho(\gamma_n)$ converges to an isometry of $\mathbb{H}^\infty$. First observe the following lemma.

LEMMA 13. *Let (g_n) be a sequence of isometries of $\mathbb{H}^\infty$. Then the set*

$$\{x \in \mathbb{H}^\infty, \, (g_n(x)) \text{ is convergent}\}$$

is a closed totally geodesic space.

Proof. The fact that this set is closed follows from the completeness of $\mathbb{H}^\infty$ and the equicontinuity of the sequence (g_n). The fact that it is totally geodesic follows from the fact that the geodesic ray passing through two given points of $\mathbb{H}^\infty$ depends continuously on these two points. □

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The two sequences $\rho(\gamma_n)$ and $\rho(\gamma_n)^{-1}$ are pointwise convergent on the image of the harmonic map. According to the lemma and the fact that the smallest closed totally geodesic subspace of $\mathbb{H}^\infty$ containing $\text{Image}(f)$ is $\mathbb{H}^\infty$, they are pointwise convergent on $\mathbb{H}^\infty$. This proves that the homomorphism Ψ extends to a continuous homomorphism $\text{PSL}_2(\mathbb{R}) \to \text{Isom}(\mathbb{H}^\infty)$. This completes the proof of Theorem 1: if the representation $\rho : \Gamma \to \text{Isom}(\mathbb{H}^\infty)$ does not factor through a fibration onto a hyperbolic 2-orbifold, it factors through a continuous homomorphism from $\text{PSL}_2(\mathbb{R})$ to $\text{Isom}(\mathbb{H}^\infty)$.

As mentioned in Remark 2, the same arguments prove that a Zariski dense representation of Γ in a non-compact simple Lie group with trivial center and different from $\text{PSL}_2(\mathbb{R})$ factors through a hyperbolic 2-orbifold as soon as the associated harmonic map

$$f : \widetilde{X} \to G/K$$

satisfies $\text{rank}_{\mathbb{C}} \, df^{1,0} = 1$. Indeed this hypothesis ensures that there is a holomorphic foliation of codimension 1 on $\widetilde{X}$, and we can argue as before. We use the fact that there is no non-trivial subsymmetric space containing the image of f; as a consequence any sequence in G which converges pointwise on the image of f has to converge everywhere.

5. Kähler groups in the Cremona group

Here, we recall the construction of the *Picard–Manin space* on which the Cremona group acts faithfully, before proving Theorem 3. For more details on the notions presented here, the reader can consult [BFJ08, Can11, CL10, Fav10, Man86].

5.1 The Picard–Manin space

We consider the set $\mathscr{E}$ of all pairs (X, p) where X is a smooth rational surface and $p : X \to \mathbb{P}^2$ is a birational morphism. Observe that if $(X, p) \in \mathscr{E}$, the map p is a composition of a certain number k of blow-ups and the intersection form $(\cdot, \cdot)_X$ on the group $H^2(X, \mathbf{Z})$ has signature $(1, k)$ (i.e. $H^2(X, \mathbf{Z})$ with its intersection form is isomorphic to $\mathbf{Z}^{k+1}$ with the form $x_1^2 - x_2^2 - \cdots - x_{k+1}^2$). Although one can define the 'action' f^* of a birational map $f : X \dashrightarrow X$ on the group $H^2(X, \mathbf{Z})$ (see [DF01]), the corresponding map $f \mapsto (f^{-1})^*$ is not a group homomorphism. To solve this problem, we look at the action of the Cremona group on a suitably defined limit of all cohomology groups $H^2(X, \mathbf{Z})$ for $(X, p) \in \mathscr{E}$. The construction of this limit is due to Manin [Man86].

Formally, one proceeds as follows. If (X, p) and (Y, q) are elements of $\mathscr{E}$, we will say that a morphism $u : Y \to X$ is admissible if $q = p \circ u$. Note that there is at most one admissible morphism between (X, p) and (Y, q). There is one exactly when the birational map $p^{-1} \circ q$ is holomorphic. If $u : Y \to X$ is admissible, the map

$$u^* : (H^2(X, \mathbf{Z}), (\cdot, \cdot)_X) \to (H^2(Y, \mathbf{Z}), (\cdot, \cdot)_Y)$$

is an isometric embedding. Observe also that given (X, p) and (Y, q) in $\mathscr{E}$, one can always find a third surface $(Z, r) \in \mathscr{E}$ for which there exist admissible morphisms $Z \to X$ and $Z \to Y$. This allows one to define the space

$$\text{Z}(\mathbb{P}^2) = \varinjlim H^2(X, \mathbf{Z}),$$

which is the inductive limit of the groups $H^2(X, \mathbf{Z})$ (for $(X, p) \in \mathscr{E}$) with respect to all admissible morphisms. The abelian group $\text{Z}(\mathbb{P}^2)$ is endowed with an intersection form $(\cdot, \cdot)_{\mathbb{P}^2}$. One defines an action of the group $\text{Bir}(\mathbb{P}^2)$ on $\text{Z}(\mathbb{P}^2)$ as follows. Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map.

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If $\alpha \in H^2(X, \mathbf{Z})$ (for some $(X, p) \in \mathscr{E}$) and $[\alpha]$ denotes the image of α in $\mathrm{Z}(\mathbb{P}^2)$, we want to define $f^*([\alpha])$. The birational map $p^{-1} \circ f : \mathbb{P}^2 \dashrightarrow X$ can be decomposed as a sequence of blow-ups and blow-downs. Hence we can choose a rational surface (Y, q) in $\mathscr{E}$ such that the rational map $g := p^{-1} \circ f \circ q$ is a morphism and define $f^*([\alpha]) := [g^*\alpha]$.

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
\mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^2
\end{array}
$$

This defines a linear isometry $f^* : (\mathrm{Z}(\mathbb{P}^2), (\cdot, \cdot)_{\mathbb{P}^2}) \to (\mathrm{Z}(\mathbb{P}^2), (\cdot, \cdot)_{\mathbb{P}^2})$. We then put

$$f_* := (f^*)^{-1}$$

and observe that $f \mapsto f_*$ is a group homomorphism.

We now describe a basis of the space $\mathrm{Z}(\mathbb{P}^2)$. On the union

$$\bigcup_{(X,p)\in\mathscr{E}} X$$

of all rational surfaces in $\mathscr{E}$, we define an equivalence relation as follows: we say that a point x in (X, p) is equivalent to a point y in (Y, q) if the map $q^{-1} \circ p$ is well defined at x, maps x onto y and is a local isomorphism near x. We denote by $\mathrm{Ec}(\mathbb{P}^2)$ the quotient of $\bigcup_{(X,p)\in\mathscr{E}} X$ by this equivalence relation. If x is a point in a rational surface X dominating $\mathbb{P}^2$, we let $X_x \to X$ be the blow-up of X at x and $e_x \in H^2(X_x, \mathbf{Z})$ be the class of the exceptional divisor. We still denote by e_x the image of this class in $\mathrm{Z}(\mathbb{P}^2)$ and observe that $e_x \in \mathrm{Z}(\mathbb{P}^2)$ depends only on the equivalence class of x in $\mathrm{Ec}(\mathbb{P}^2)$. Hence for each point $a \in \mathrm{Ec}(\mathbb{P}^2)$ we have a class $e_a \in \mathrm{Z}(\mathbb{P}^2)$ associated to a. We have a natural map from the abelian group

$$A(\mathbb{P}^2) := \mathbf{Z} \oplus \left(\bigoplus_{a \in \mathrm{Ec}(\mathbb{P}^2)} \mathbf{Z}_a \right)$$

to $\mathrm{Z}(\mathbb{P}^2)$: it takes the first factor isomorphically to $H^2(\mathbb{P}^2, \mathbf{Z}) \hookrightarrow \mathrm{Z}(\mathbb{P}^2)$ and the factor $\mathbf{Z}_a$ to $\mathbf{Z}e_a \subset \mathrm{Z}(\mathbb{P}^2)$. The direct sum above can be endowed with the bilinear form $(\cdot, \cdot)$ for which the vector $(t, (\lambda_a)_{a \in \mathrm{Ec}(\mathbb{P}^2)})$ has norm $t^2 - \sum_a \lambda_a^2$. We then have the following result.

PROPOSITION 14 [Man86]. . *The natural map*

$$(A(\mathbb{P}^2), (\cdot, \cdot)) \to (\mathrm{Z}(\mathbb{P}^2), (\cdot, \cdot)_{\mathbb{P}^2})$$

is a surjective isometry.

We will use two consequences of this proposition (see [BFJ08, Can11]).

(i) One can complete $\mathrm{Z}(\mathbb{P}^2)$ to obtain a Hilbert space $\mathscr{Z}(\mathbb{P}^2)$ endowed with a bilinear form (still denoted by $(\cdot, \cdot)_{\mathbb{P}^2}$) of signature $(1, \infty)$. Any isometry of $\mathrm{Z}(\mathbb{P}^2)$ extends to an isometry of $\mathscr{Z}(\mathbb{P}^2)$, and hence we get an action of the group $\mathrm{Bir}(\mathbb{P}^2)$ on $\mathscr{Z}(\mathbb{P}^2)$.

(ii) One can describe the action of a birational map f on $\mathrm{Z}(\mathbb{P}^2)$ in terms of the basis given by the previous proposition. If $x \in \mathbb{P}^2$ and if f is defined at x we have

$$f_*(e_x) = e_{f(x)}$$

(when one identifies x and $f(x)$ to their images in $\mathrm{Ec}(\mathbb{P}^2)$). If x is a point of a rational surface X dominating $\mathbb{P}^2$, and if the image of x in $\mathbb{P}^2$ is not a point of indeterminacy of f, the same

formula remains true, provided that we define $f(x)$ correctly (see [Can11]). In particular, if f is an automorphism of $\mathbb{P}^2$, f_* permutes the vectors of the previous basis of $\mathscr{Z}(\mathbb{P}^2)$.

We will denote by $u_0 \in \mathrm{Z}(\mathbb{P}^2)$ the class of a line in $\mathbb{P}^2$. One can finally define the space $\mathbb{H}_{\mathbb{P}^2}$. This is simply the hyperbolic space associated to the space $(\mathscr{Z}(\mathbb{P}^2), (\cdot, \cdot)_{\mathbb{P}^2})$; that is,

$$\mathbb{H}_{\mathbb{P}^2} := \{u \in \mathscr{Z}(\mathbb{P}^2), (u, u)_{\mathbb{P}^2} = 1, (u, u_0)_{\mathbb{P}^2} > 0\}.$$

Example 1, continued. We denote by x_1, x_2, x_3 the three points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ in $\mathbb{P}^2$. The group $A := \mathbb{C}^* \times \mathbb{C}^*$ can be identified with the diagonal subgroup of $\mathrm{PGL}(3, \mathbb{C})$. It fixes the class u_0 of a line as well as the three points e_{x_1}, e_{x_2} and e_{x_3} in $\mathscr{Z}(\mathbb{P}^2)$. Hence it fixes pointwise the intersection of the four-dimensional subspace spanned by u_0 and the e_{x_i} with $\mathbb{H}_{\mathbb{P}^2}$, which is a copy of the three-dimensional hyperbolic space.

To prove that the space $(\mathbb{H}_{\mathbb{P}^2})^A$ is actually infinite-dimensional, we observe that when we blow up the three points x_i on $\mathbb{P}^2$, we obtain a rational surface X on which A acts with six fixed points. By blowing up these six fixed points and iterating this procedure, we obtain a sequence of actions of A on some rational surfaces X_n dominating $\mathbb{P}^2$, for which the second Betti numbers $b_2(X_n)$ go to infinity. This proves that the space of fixed points of A in $\mathbb{H}_{\mathbb{P}^2}$ is infinite-dimensional.

We finish this paragraph with a remark, which will be used below. If $q : Y \to \mathbb{P}^2$ is a rational surface dominating $\mathbb{P}^2$, one can repeat everything we have said in this paragraph, replacing $\mathbb{P}^2$ by Y: one can define the spaces $\mathrm{Z}(Y)$, $\mathscr{Z}(Y)$, and the form $(\cdot, \cdot)_Y$. One difference is that when defining the group $A(Y)$, the first $\mathbf{Z}$ factor has to be replaced by the group $H^2(Y, \mathbf{Z})$. The spaces $\mathrm{Z}(Y)$ and $\mathscr{Z}(Y)$ are isomorphic to $\mathrm{Z}(\mathbb{P}^2)$ and $\mathscr{Z}(\mathbb{P}^2)$, and when one identifies $\mathrm{Bir}(\mathbb{P}^2)$ to $\mathrm{Bir}(Y)$ via q, the two actions we get on $\mathscr{Z}(\mathbb{P}^2)$ and $\mathscr{Z}(Y)$ are conjugated. In particular one gets the following conclusion: if a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ lifts to an automorphism of Y which is isotopic to the identity, there is an orthonormal basis of $\mathscr{Z}(\mathbb{P}^2)$ which is permuted by f_*.

5.2 Proof of Theorem 3

We now prove Theorem 3. Let $\rho : \Gamma \to \mathrm{Bir}(\mathbb{P}^2)$ be a non-elementary homomorphism. Let $\mathbb{H}_\rho$ be the unique minimal closed totally geodesic ρ-invariant subspace of $\mathbb{H}_{\mathbb{P}^2}$. We know that the action of Γ on $\mathbb{H}_\rho$ factors through a homomorphism θ onto a subgroup Λ of $\mathrm{PSL}_2(\mathbb{R})$. We need to show that Λ is discrete.

We assume by contradiction that Λ is dense in $\mathrm{PSL}_2(\mathbb{R})$. Let Λ_0 be a torsion-free finite index subgroup of Λ. The group Λ_0 is still dense in $\mathrm{PSL}_2(\mathbb{R})$. Since elliptic elements form an open set of $\mathrm{PSL}_2(\mathbb{R})$, there exists an element $\gamma_0 \in \Lambda_0$ which is elliptic of infinite order. Let g be an element of Γ such that $\theta(g) = \gamma_0$. Then the birational transformation $\rho(g)$ has the following property: there exists a two-dimensional subspace $P \subset \mathscr{Z}(\mathbb{P}^2)$ on which the form $(\cdot, \cdot)_{\mathbb{P}^2}$ is positive definite and on which $\rho(g)_*$ acts as an irrational rotation. Indeed, the harmonic map $f : \Delta \to \mathbb{H}_\rho \subset \mathbb{H}_{\mathbb{P}^2}$ is $\mathrm{PSL}_2(\mathbb{R})$-equivariant in our situation, and hence has rank two at every point of Δ. One can thus take P to be the image of the differential of f at the point $\xi \in \Delta$ fixed by γ_0.

The next proposition proves that such a behavior does not occur inside the Cremona group. Hence Λ cannot be dense and has to be discrete. After eventually applying the Stein factorization theorem, we obtain a fibration of X onto a hyperbolic 2-orbifold Σ, and the kernel H of the homomorphism $\Gamma \to \pi_1^{\mathrm{orb}}(\Sigma)$ has to fix pointwise the subspace $\mathbb{H}_\rho$. Since $\mathbb{H}_\rho$ contains the image of f, its dimension is at least two. This concludes the proof of Theorem 3. $\qquad\square$

Proposition 15. *Let $h : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map such that the isometry h_* fixes a point in $\mathbb{H}_{\mathbb{P}^2}$. Then there is no h_*-invariant two-dimensional subspace $P \subset \mathscr{Z}(\mathbb{P}^2)$ on which h_* acts as an irrational rotation.*

Proof. According to [Can11], and up to replacing h by a power, one can find a birational model Y of $\mathbb{P}^2$ on which h acts by an automorphism isotopic to the identity. The space $\mathscr{Z}(Y)$ is the direct sum of $H^2(Y, \mathbf{R})$ and of $\ell^2(\mathrm{Ec}(Y))$; h_* acts trivially on the first factor and permutes the coordinates in the second. The plane P has to be contained in the second factor. Let $u_i = \sum_{x \in \mathrm{Ec}(Y)} \lambda_{i,x} e_x$ $(i = 1, 2)$ be a basis of P. We know that there exists an angle α (with α/π irrational) such that

$$h_* u_1 = \cos(\alpha) u_1 + \sin(\alpha) u_2,$$
$$h_* u_2 = -\sin(\alpha) u_1 + \cos(\alpha) u_2.$$

We fix a point $x \in \mathrm{Ec}(Y)$ such that $(\lambda_{1,x}, \lambda_{2,x}) \neq (0, 0)$. Looking coordinate by coordinate we obtain, for each integer n, the equalities

$$\lambda_{1, h^n(x)} = \cos(n\alpha) \lambda_{1,x} + \sin(n\alpha) \lambda_{2,x},$$
$$\lambda_{2, h^n(x)} = -\sin(n\alpha) \lambda_{1,x} + \cos(n\alpha) \lambda_{2,x}.$$

Since the sum $\sum_y \lambda_{i,y}^2$ is finite, there must exist an integer N such that $h^N(x) = x$. However, this implies that the vector $(\lambda_{1,x}, \lambda_{2,x}) \in \mathbf{R}^2$ is an eigenvector for the rotation of angle $N\alpha$ in the plane. This is a contradiction. $\square$

Theorem 3 only deals with non-elementary homomorphisms into the Cremona group, but the work of Cantat also gives information about elementary homomorphisms from finitely generated groups into $\mathrm{Bir}(\mathbb{P}^2)$. In particular we have the following consequence of Theorem 4.

If N is a finitely generated subgroup of the Cremona group which preserves a geodesic γ in $\mathbb{H}_{\mathbb{P}^2}$ but is not elliptic, then either N is virtually cyclic or N is conjugated to a subgroup of the group G_{toric}.

Let us explain this fact. Let $u : N \to \mathbf{R}$ be the 'translation homomorphism' induced by the action of N on γ. It is known that its image is cyclic (see [Can11, Fav10]); this uses the fact that N is finitely generated. If N is not virtually cyclic, the group $A := \mathrm{Ker}\, u$ is infinite and elliptic, and we are in the situation of Theorem 4, establishing the above result.

We can now prove Corollary 3.

Proof of Corollary 3. Let $\rho : \Gamma_1 \to \mathrm{Bir}(\mathbb{P}^2)$ be an injective homomorphism where Γ_1 is a cocompact lattice in $\mathrm{SU}(n, 1)$ $(n \geqslant 2)$. Note that if the conclusion of the corollary holds for a normal subgroup of finite index of Γ_1, it holds also for Γ_1. Hence we can assume that Γ_1 is torsion-free. We assume by contradiction that ρ is non-elementary, and apply Corollary 2: Γ_1 has no finite index subgroup isomorphic to a surface group, and hence Γ_1 must be isomorphic to a subgroup of the group G_{toric}. However, Γ_1 has no non-trivial normal abelian subgroup, and hence its intersection with $\mathbb{C}^* \times \mathbb{C}^* \subset G_{\mathrm{toric}}$ must be trivial. This implies that Γ_1 is isomorphic to a subgroup of $\mathrm{GL}(2, \mathbf{Z})$, which is a contradiction. Hence the homomorphism ρ has to be elementary. If $\rho(\Gamma_1)$ preserves a geodesic we apply the remark made above. Since Γ_1 is not virtually cyclic, it has to be isomorphic to a subgroup of the group G_{toric}, yielding a contradiction again.

Finally, if ρ is elementary but does not preserve a geodesic, it has to fix a point in the space $\mathbb{H}_{\mathbb{P}^2}$ or to fix a unique point of the boundary of $\mathbb{H}_{\mathbb{P}^2}$. $\square$

6. Koszul and Malgrange's integrability theorem

In this section, we prove the infinite-dimensional version of Koszul and Malgrange's integrability theorem.

Let us first remind the content of the proof in the finite-dimensional case. Assume that $E \to X$ is a smooth complex vector bundle over the complex manifold X and that D is a $(0, 1)$-connection on E, i.e. a differential operator $D : A^0(E) \to A^1(E)$ (where $A^0(E)$ is the space of smooth sections of E and $A^1(E)$ the space of smooth 1-forms on X with values in E) such that

$$D(fs) = \overline{\partial}(f)s + fDs$$

whenever f is a smooth function and s is a section of E. The operator D naturally extends to an operator acting on all differential forms with values in E. The theorem of Koszul and Malgrange asserts that if D satisfies the integrability condition $D^2 = 0$ one can find a holomorphic structure on E for which D is the usual $\overline{\partial}$ operator. This is a simple particular case of the integrability criterion of Newlander and Nirenberg. The proof can be described as follows. Let $s_1, \ldots, s_n$ be any local frame for the bundle $E \to X$, over some open set $U \subset X$. We want to prove that, around any point, one can find a frame made of sections annihilated by D (the transition matrices between any to such frames will then be holomorphic). We can write

$$Ds_j = \sum_{i=1}^n a_{ij} s_i,$$

where $A = (a_{ij})$ is a matrix of $(0, 1)$-forms on U. We look for an invertible matrix-valued map $M = (m_{ij})$ on U for which the sections $u_j = \sum_{i=1}^n m_{ij} s_i$ would satisfy $Du_j = 0$. In other words we want to solve (locally) the equation $\overline{\partial} M = -AM$, knowing the integrability condition

$$\overline{\partial} A + A \wedge A = 0,$$

which is equivalent to the condition $D^2 = 0$. The (matrix-valued) 2-form $A \wedge A$ is defined by

$$A \wedge A = \sum_{j<l} [A_j, A_l] \, d\overline{z}_j \wedge d\overline{z}_l,$$

where $A = \sum_{i=1}^k A_i \, d\overline{z}_i$. Here $(z_1, \ldots, z_k)$ are coordinates on U and the $A_i = A_i(z)$ are $n \times n$ matrices.

In the infinite-dimensional setting, $A = \sum_{i=1}^k A_i \, d\overline{z}_i$ where the A_i are smooth maps with values in $\mathrm{End}(\mathscr{H})$ ($\mathscr{H}$ being a fixed complex Hilbert space), and we look for a smooth map M with values in the open set $\mathrm{GL}(\mathscr{H}) \subset \mathrm{End}(\mathscr{H})$ of invertible endomorphisms of $\mathscr{H}$, solving the equation above. To solve this problem, we use the following proposition.

PROPOSITION 16. *Let C be a smooth map with values in the space $\mathrm{End}(\mathscr{H})$, defined in a neighborhood of 0 in $\mathbb{C}^k$. Assume that C is holomorphic with respect to $z_1, \ldots, z_p$ for some integer $0 \leqslant p < k$. Then there exists a smooth map h defined in a neighborhood of 0 in $\mathbb{C}^k$, with values in the space $\mathrm{GL}(\mathscr{H})$ and holomorphic in $z_1, \ldots, z_p$, such that*

$$h^{-1} \frac{\partial h}{\partial \overline{z}_{p+1}} = C.$$

To deduce the theorem from the proposition, one proceeds exactly as in [KM58]. Note that it is at this point that we use the integrability condition satisfied by A. Since this part of the

proof is exactly the same as in the finite-dimensional case, we refer the reader to [KM58, p. 103]. We now turn to the proof of Proposition 16.

We follow the proof explained in [DK90, p. 50]. We denote by B a ball in $\mathbb{C}$, by E a complex Banach space, and by $C^{l+\alpha}(B, E)$ the usual Hölder space of C^l maps from B to E, with α-Hölder derivatives of order l ($0 < \alpha < 1$). We will use the following result.

PROPOSITION 17. *There exists a continuous linear operator* $P : C^{\alpha}(B, E) \to C^{1+\alpha}(B, E)$ *such that*

$$\frac{\partial}{\partial \bar{z}}(Pg) = g,$$

for all $g \in C^{\alpha}(B, E)$. *Moreover, the operator* P *is continuous from* $C^{l+\alpha}(B, E)$ *to* $C^{l+1+\alpha}(B, E)$ *for every integer* l.

Proof. The construction of the operator P is classical in the case of complex-valued functions; see [Vek62, ch. I, §§ 6, 8, 9] for its construction and continuity between various Hölder and Sobolev spaces. Here we are dealing with functions with values in a complex Banach space. Observe that all the integrals of functions with values in a Banach space that we need to consider to define P are integrals of *continuous functions*. They can thus be defined by limits of Riemann sums, and the separability (or not) of the space E plays no role here. We shall indeed apply this proposition to some non-separable Banach space, namely, the space of endomorphisms of a Hilbert space.

Therefore one sees readily that the proofs given in [Vek62] extend to this context. We only give the formula defining P and refer to [Vek62] for its properties:

$$P(g)(z) = \frac{1}{2i\pi} \int_B \frac{g(\xi)}{\xi - z} \, d\xi \wedge d\bar{\xi}. \qquad \square$$

We now prove Proposition 16. Write $z = z_{p+1}$ and $z' = (z_1, \ldots, z_p, z_{p+2}, \ldots, z_k)$. We first treat z' as a parameter. We thus want to solve the equation

$$\frac{\partial h}{\partial \bar{z}} = hC,$$

in a neighborhood of $0 \in \mathbb{C}$. A simple renormalization argument (see [DK90, p. 51] for instance) implies that it is enough to solve this equation when C is defined on the ball B of radius 1 around 0 in $\mathbb{C}$ and the C^1 norm of C is as small as we wish. By looking for h under the form $h = \mathrm{Id} + u$ and by using the operator P given by the previous proposition (with the Banach space E being the space of endomorphisms of $\mathscr{H}$), we see that it is enough to find a solution of the equation

$$u = P(C + uC),$$

where $u \in C^{\alpha}(B, E)$. If C is small enough, we can apply the contraction mapping principle to the operator $u \mapsto P(C + uC)$ in the space $C^{\alpha}(B, E)$. Its fixed point will be close to 0, and hence the map $h = \mathrm{Id} + u$ will take its values in $\mathrm{GL}(\mathscr{H})$.

The fact that the solution we obtain depends smoothly on z' and holomorphically on the z_i ($i \leqslant p$) is a consequence of the implicit function theorem. This proves Proposition 16.

7. About the regularity of harmonic maps

The purpose of this section is to briefly explain the proof of Proposition 6. We first state a version of Proposition 6 for the flat Laplacian Δ, i.e. the Laplacian associated to the Euclidean

metric on $\mathbb{C}^n$. Here again, B is a closed ball in $\widetilde{X}$ that we can identify with a ball in $\mathbb{C}^n$ via some choice of coordinates.

PROPOSITION 18. *Let* $p > 1$. *There exists a constant* C *such that if* $u \in W^{1,p}(B, \mathscr{H})$ *and* $\Delta u \in L^p(B, \mathscr{H})$ *then* $u \in W^{2,p}(B, \mathscr{H})$ *and we have* $|D^2 u|_p \leqslant C|\Delta u|_p$. *Here* $D^2 u$ *is the matrix of partial derivatives of order two of* u *and* $|D^2 u|_p^p = \sum_{i,j} |\partial_i \partial_j u|_p^p$.

Proof. This proof is classical. It follows from the fact that for each index α, there exists a linear operator R_α (the Riesz transform) mapping continuously $L^p(B, \mathscr{H})$ into itself for each $p \in (1, +\infty)$ and such that $\partial^2 u / \partial x_\alpha \partial x_\beta = -R_\alpha R_\beta(\Delta u)$. See [Ste70, ch. 2] for the discussion of the continuity properties of various operators defined by singular integrals. The case of vector-valued functions is discussed in § 5 of Chapter 2. The particular case of the Riesz transforms is discussed in Chapter 3. □

With Proposition 18 at hand, one proves Proposition 6 exactly as in the finite-dimensional case. One first proves the following *a priori* estimate.

PROPOSITION 19. *Let* $B' \subset\subset B''$ *be closed balls contained in the interior of* B. *There exists a constant* C *such that if* $u \in W_{\mathrm{loc}}^{2,p}(B)$ *then*

$$|D^2 u|_{p,B'} \leqslant C|\Delta_g u|_{p,B''}.$$

From this statement, one easily deduces the result of Proposition 6. We refer the reader to [GT01, ch. 9] for more details.

ACKNOWLEDGEMENTS

Appendix. Elliptic subgroups of the Cremona group with a large normalizer

Serge Cantat

In this appendix, we prove the following theorem (Theorem 4 in the introduction). Its statement uses the action of $\mathrm{Bir}(\mathbb{P}^2)$ on $\mathbb{H}_{\mathbb{P}^2}$. Recall that the distance $d(u,v)$ on $\mathbb{H}_{\mathbb{P}^2}$ is given by the intersection form $(u,v)_{\mathbb{P}^2} : \cosh(d(u,v)) = (u,v)_{\mathbb{P}^2}$. Birational transformations determine three kind of isometries of this space: elliptic, parabolic, or hyperbolic.

THEOREM A.1. *Let* N *be a subgroup of the Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$. *Assume that there exists a short exact sequence*

$$1 \to A \to N \to B \to 1$$

where N *contains at least one hyperbolic element, and* A *is infinite elliptic. Then* N *is conjugate to a subgroup of the group* G_{toric} *of automorphisms of* $(\mathbb{C}^*)^2$.

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Remark A.1. The same conclusion holds if we assume that A is infinite, finitely generated, and all its elements are elliptic, because this new assumption implies that A is elliptic (see [Can11, Theorem 6.4, Proposition 6.12]).

Let Y be a rational surface. The group $\mathrm{Aut}(Y)$ of automorphisms of Y is a complex Lie group. It may have an infinite number of connected components, but its connected component of the identity $\mathrm{Aut}(Y)^0$ is easily described: $\mathrm{Aut}(Y)^0$ is isomorphic to an algebraic subgroup of $\mathrm{PGL}_{n+1}(\mathbb{C})$ for some positive integer n; this isomorphism is given by an equivariant embedding of Y in $\mathbb{P}^n(\mathbb{C})$. All pairs $(Y, \mathrm{Aut}(Y)^0)$ with $\dim(\mathrm{Aut}(Y)^0) \geqslant 1$ are described in [Akh95, Bla09].

Let $u_0 \in \mathscr{Z}(\mathbb{P}^2)$ be the class of a line in $\mathbb{P}^2$. Since A is elliptic, it fixes a point u in $\mathbb{H}_{\mathbb{P}^2}$ and we have

$$(a_*(u_0), u_0)_{\mathbb{P}^2} \leqslant \cosh(2d(u_0, u)),$$

for all a in A. Since $(a_*(u_0), u_0)_{\mathbb{P}^2}$ is the degree of a, A is contained in the algebraic variety $\mathrm{Bir}_D(\mathbb{P}^2)$ of birational transformations of the plane of degree $\leqslant D$, for $D = [\cosh(2\mathrm{dist}(u_0, u))] + 1$. Let $\overline{A}$ be the Zariski closure of A in $\mathrm{Bir}_D(\mathbb{P}^2)$; since A is infinite, $\dim(\overline{A}) \geqslant 1$; since A is normalized by N, so is $\overline{A}$. From Weil and Rosenlicht theorem (see [Bla09, Ros56, Wei55]), there exist a surface Y and a birational map $\pi : Y \dashrightarrow \mathbb{P}^2(\mathbb{C})$ such that $\pi^{-1} \circ \overline{A} \circ \pi$ is contained in the group $\mathrm{Aut}(Y)$ and intersects a finite number of its components. After conjugation by π, we can and do assume that N is contained in $\mathrm{Bir}(Y)$ and $\overline{A}$ is a Zariski closed complex Lie subgroup of $\mathrm{Aut}(Y)$. Changing A into $\overline{A}$ and changing N accordingly, we have now a short exact sequence

$$1 \to \overline{A} \to \overline{N} \to \overline{B} \to 1$$

in which $\overline{N}$ contains hyperbolic elements and $\overline{A}$ is a Lie subgroup of $\mathrm{Aut}(Y)$ with finitely many components. Let $\overline{A}_0$ be the connected component of the identity in $\overline{A}$.

LEMMA A.1. *The group $\overline{N}$ normalizes $\overline{A}_0$, and the group $\overline{A}_0$ has a Zariski open orbit in Y.*

Proof. Since $\overline{N}$ normalizes $\overline{A}$, it normalizes the connected component $\overline{A}_0$. Since $\overline{A}_0$ is Zariski-closed in the algebraic group $\mathrm{Aut}(Y)^0$, its orbits in Y are Zariski open subsets of Zariski closed subsets in Y. Thus, two cases can occur. Either $\overline{A}_0$ has an open orbit, or the orbits of $\overline{A}_0$ determine a pencil of curves on Y. In the second case, the group $\overline{N}$ permutes the members of the pencil; in particular, all elements of N preserve a meromorphic fibration on Y, and N does not contain any hyperbolic element (see [CF03]). From this contradiction, we deduce that $\overline{A}_0$ has an open orbit in Y; in particular, $\dim(\overline{A}_0) \geqslant 2$.

Now let f be an element of $\overline{N}$. Since f normalizes $\overline{A}_0$ and $\overline{A}_0$ is contained in $\mathrm{Aut}(Y)$, the group $\overline{A}_0$ permutes the indeterminacy set $\mathrm{Ind}(f)$ and the set of exceptional curves $\mathrm{Exc}(f)$ ($\mathrm{Exc}(f)$ is the zero locus of the Jacobian determinant of f). In particular, both $\mathrm{Ind}(f)$ and $\mathrm{Exc}(f)$ are contained in the complement of the open orbit of $\overline{A}_0$; this proves the following lemma.

LEMMA A.2. *The union of all indeterminacy sets $\mathrm{Ind}(f)$ and exceptional sets $\mathrm{Exc}(f)$ of elements of $\overline{N}$ is a proper Zariski closed set $Z \subset Y$. The group $\overline{N}$ acts by automorphisms on the Zariski open set $U = Y \setminus Z$.*

We can now restrict our study to the Zariski open set $U \subset Y$. Each element f of $\overline{N}$ determines an automorphism φ_f of the algebraic group $\overline{A}_0$ such that

$$f \circ a \circ f^{-1} = \varphi_f(a)$$

for all a in $\overline{A}_0$. The group $\overline{A}_0$ splits into a φ_f-invariant exact sequence

$$1 \to R \to \overline{A}_0 \to S \to 1$$

where R is the solvable radical of $\overline{A}_0$ and S is semi-simple.

Assume, first, that R is trivial. Then $\overline{A}_0 = S$ is semi-simple and the group of exterior automorphisms of $\overline{A}_0$ is finite. Let f be a hyperbolic element of $\overline{N}$. Changing f into one of its positive iterates, φ_f is an interior automorphism: there exists b in S such that $f \circ a \circ f^{-1} = b \circ a \circ b^{-1}$ for all a in S. In particular, f commutes to b; since b is in $\text{Aut}^0(Y)$ and f is hyperbolic, we deduce from [Can11, Theorem B], that b has finite order. Changing f into a new positive iterate, we can assume that b is the identity and thus f commutes to the action of S. Theorem B of [Can11] provides a contradiction because f is hyperbolic. This shows that *the solvable radical of $\overline{A}_0$ is non-trivial.*

The radical R and all members of its central series $R_i = [R_{i-1}, R_{i-1}]$ are φ_f-invariant. Thus, changing $\overline{A}_0$ into the last non-trivial derived subgroup $\overline{A}_1$ of R, we have a new exact sequence $1 \to \overline{A}_1 \to \overline{N} \to Q \to 1$ where $\overline{A}_1$ is a connected abelian group with an open orbit $V \subset U$.

There are three possibilities for this algebraic group $\overline{A}_1$: $\mathbb{C}^2$, $\mathbb{C} \times \mathbb{C}^*$, and $\mathbb{C}^* \times \mathbb{C}^*$. To finish, we need to exclude the first two cases. In the first case, $\overline{A}_1 = \mathbb{C}^2$, the unique open orbit of $\overline{A}_1$ in U can be identified to $\mathbb{C}^2$ and all elements f of $\overline{N}$ act by affine automorphisms on $V = \mathbb{C}^2$ (with linear part given by $\varphi_f \in \text{GL}(2, \mathbb{C})$). In particular, the degree of f^n is bounded and $\overline{G}$ does not contain any hyperbolic element, contradicting our assumption. In the second case, $\overline{A}_1 = \mathbb{C} \times \mathbb{C}^*$. The group of algebraic automorphisms of $\overline{A}_1$ is made of diagonal transformations $(x, y) \mapsto (ax, \epsilon y)$ with $a \in \mathbb{C}^*$ and $\epsilon = \pm 1$; thus, the same argument applies to exclude this case.

As a consequence, $\overline{A}_1 = \mathbb{C}^* \times \mathbb{C}^*$ and there is a $\overline{N}$-invariant Zariski-open subset $V \subset Y$ such that V is isomorphic to $\overline{A}_1$ and $\overline{N}$ acts by regular transformations on V. Since the group of regular automorphisms of $\mathbb{C}^* \times \mathbb{C}^*$ is the semi-direct product of $\mathbb{C}^* \times \mathbb{C}^*$ (acting by translations) and $\text{GL}(2, \mathbf{Z})$ (acting by group automorphisms, i.e. by monomial transformations), Theorem A.1 is proved.

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