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HALL SUBGROUPS AND STABLE BRAUER CHARACTERS

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Abstract Let H be a Hall π-subgroup of a finite π-separable group G, and let α be an irreducible Brauer character of H. If $\alpha(x) = \alpha(y)$ whenever $x, y \in H$ are p-regular and G-conjugate, then α extends to a Brauer character of G.

Keywords: Brauer characters; Hall subgroups

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1. Introduction

Let G be a finite group and let p be a prime number. Suppose that $U \subseteq G$ and let $\alpha \in \mathrm{IBr}(U)$ be an irreducible Brauer character of U. We say that α is *G-stable* if $\alpha(x) = \alpha(y)$ whenever $x, y \in U$ are p-regular and G-conjugate.

We prove the following theorem.

Theorem A. *Suppose that G is π-separable and let H be a Hall π-subgroup of G. If $\alpha \in \mathrm{IBr}(H)$ is G-stable, then α extends to some Brauer character of G.*

If $H \triangleleft G$, then Theorem A is Gallagher's theorem (for ordinary characters) and Dade's theorem (for Brauer characters; see [**1**] and Theorem (8.13) of [**4**]). If p does not divide $|H|$, Theorem A was obtained by Isaacs in [**2**] (with a different approach). As is well known, extendability of characters and control of fusion are closely related.

Corollary B. *Suppose that G is π-separable and let H be a Hall π-subgroup of G. Then every irreducible Brauer character of H extends to some Brauer character of G if and only if whenever $x, y \in H$ are p-regular and G-conjugate, then x and y are H-conjugate.*

2. Proofs

If $U \subseteq G$, we denote by U^0 the set of p-regular elements of U. Also, $\mathrm{cf}(U^0)$ is the complex space of class functions of U defined on U^0. We say that $\alpha \in \mathrm{cf}(U^0)$ is *G-stable* if $\alpha(x) = \alpha(y)$ whenever $x, y \in U$ are p-regular and G-conjugate.

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Lemma 2.1. *Let $\alpha \in \mathrm{cf}(U^0)$. Then α is G-stable if and only if there exists $\xi \in \mathrm{cf}(G^0)$ such that $\xi_U = \alpha$.*

Proof. Assume that α is G-stable. We define some $\xi \in \mathrm{cf}(G^0)$. We put $\xi(x) = 0$, say, for every p-regular element $x \in G$ lying in no G-conjugate of U. If, on the other hand, $\mathrm{cl}(x) \cap U$ is non-empty, we set $\xi(x) = \alpha(u)$, where u is any element of $\mathrm{cl}(x) \cap U$. Since α is G-stable, it follows that ξ is a well-defined function on G^0 extending α. For the converse, it is clear that if α extends to some $\xi \in \mathrm{cf}(G^0)$, then α is G-stable. □

Lemma 2.2. *Let $\alpha \in \mathrm{cf}(U^0)$. Then α is G-stable if and only if*

$$(\alpha^G)_U = \alpha((1_U)^G)_U.$$

Proof. Assume that α is G-stable. By Lemma 2.1, let $\xi \in \mathrm{cf}(G^0)$ be such that $\xi_U = \alpha$. Now,

$$(\alpha^G)_U = ((\xi_U)^G)_U = (\xi(1_U)^G)_U = \xi_U((1_U)^G)_U = \alpha((1_U)^G)_U,$$

as desired. Assume now that

$$(\alpha^G)_U = \alpha((1_U)^G)_U,$$

and note that $((1_U)^G)_U$ is a G-stable class function of U which is never zero. Then

$$\alpha = \frac{(\alpha^G)_U}{((1_U)^G)_U}$$

is the quotient of two G-stable class functions. Hence α is G-stable. □

From now on, we fix a maximal ideal of the ring of the algebraic integers $\boldsymbol{R}$ containing $p\boldsymbol{R}$, so that for every finite group G we have a uniquely defined set of irreducible Brauer characters $\mathrm{IBr}(G)$ of G. We follow the notation of [**4**].

Let $N \triangleleft G$ and let $\theta \in \mathrm{IBr}(N)$. We denote by $\mathrm{cf}(G^0 \mid \theta)$ the complex linear combinations of $\mathrm{IBr}(G \mid \theta)$.

Lemma 2.3. *Suppose that $N \subseteq U \subseteq G$ where $N \triangleleft G$. Let $\theta \in \mathrm{IBr}(N)$ and suppose that $\alpha \in \mathrm{IBr}(U \mid \theta)$ is G-stable. Then there exists $\phi \in \mathrm{cf}(G^0 \mid \theta)$ extending α. Also, $TU = G$, where T is the stabilizer of θ in G.*

Proof. By Lemma 2.1, let $\hat{\alpha} \in \mathrm{cf}(G^0)$ be an extension of α. Write

$$\mathrm{cf}(G^0) = \mathrm{cf}(G^0 \mid \theta) \oplus \Delta,$$

where Δ is the $\boldsymbol{C}$-span of those irreducible Brauer characters μ of G that do not lie over θ. We write $\hat{\alpha} = \phi + \psi$, where $\phi \in \mathrm{cf}(G^0 \mid \theta)$ and $\psi \in \Delta$. We claim that $\alpha^G \in \mathrm{cf}(G^0 \mid \theta)$. By Lemma 2.2, we have that

$$(\alpha^G)_U = \alpha((1_U)^G)_U.$$

Then

$$(\alpha^G)_N = \alpha_N((1_U)^G)_N = |G : U|\alpha_N,$$

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and it follows that every irreducible constituent of the Brauer character α^G lies over θ, as claimed. Also, if $\beta \in \mathrm{IBr}(G)$ is an irreducible constituent of α^G, then β_N is contained in $|G : U|\alpha_N$, which is a sum of U-conjugates of θ (by Clifford's Theorem applied to α). Hence, it follows that β_N is a sum of some U-conjugates of θ. Now, if $g \in G$, we have that θ^g is also an irreducible constituent of β_N. Therefore, there is $u \in U$ such that $\theta^g = \theta^u$. Hence, $gu^{-1} \in T$, and we conclude that $TU = G$. Now, since N is contained in the kernel of every irreducible constituent of $(1_U)^G$, it follows that $\phi(1_U)^G \in \mathrm{cf}(G^0 \mid \theta)$ and $\psi(1_U)^G \in \Delta$. Now,

$$\phi(1_U)^G + \psi(1_U)^G = (\phi + \psi)(1_U)^G = \hat{\alpha}(1_U)^G = (\hat{\alpha}_U)^G = \alpha^G,$$

and we conclude that $\psi(1_U)^G = 0$. Then

$$\phi_U((1_U)^G)_U = (\phi_U + \psi_U)((1_U)^G)_U = \alpha((1_U)^G)_U.$$

Since $((1_U)^G)_U$ is never zero, we conclude that $\phi_U = \alpha$, as desired. □

If $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ and $T = I_G(\theta)$, recall that $\xi \mapsto \xi^G$ defines a bijection

$$\mathrm{cf}(T^0 \mid \theta) \to \mathrm{cf}(G^0 \mid \theta)$$

(as easily follows from Theorem (8.9) of [**4**]).

Lemma 2.4. *Suppose that $N \subseteq U \subseteq G$, where $N \lhd G$. Let $\theta \in \mathrm{IBr}(N)$ and suppose that $\alpha \in \mathrm{IBr}(U \mid \theta)$ is G-stable. Write $T = I_G(\theta)$ and let $\xi \in \mathrm{IBr}(T \cap U \mid \theta)$ be the Clifford correspondent of α over θ. Then ξ is T-stable.*

Proof. By Lemma 2.3, we know that $G = TU$. Also, by Lemma 2.3, let $\hat{\alpha} \in \mathrm{cf}(G^0 \mid \theta)$ be such that $\hat{\alpha}_U = \alpha$. Now, let $\hat{\xi} \in \mathrm{cf}(T^0 \mid \theta)$ be such that $\hat{\xi}^G = \hat{\alpha}$. By Mackey, we have that

$$(\hat{\xi}_{T\cap U})^T = (\hat{\xi}^G)_U = \hat{\alpha}_U = \alpha.$$

We have that $\hat{\xi}_{T\cap U} \in \mathrm{cf}((T \cap U)^0 \mid \theta)$, and we conclude that

$$\hat{\xi}_{T\cap U} = \xi$$

by uniqueness. Hence, by Lemma 2.1 we have that ξ is T-stable, as desired. □

The key idea in our proof of Theorem A is to use 'modular character triples'. The reader is referred to Chapter 7 of [**4**] for their definition and main properties.

Theorem 2.5. *Suppose that (G, N, θ) is a modular character triple with N a π-group. Then there exists an isomorphic triple (G^*, N^*, θ^*), where N^* is a π-group contained in $\boldsymbol{Z}(G^*)$.*

Proof. We argue as in Theorem (5.2) of [**3**]. Let (G^*, N^*, θ^*) be any isomorphic triple where θ^* is linear, and factor $\theta^* = \alpha\beta$, where the order of α is a π-number and the order of β is a π'-number. Notice that both characters are G-invariant, by uniqueness. As in Theorem (5.2) of [**3**] (and using Lemma (8.26) of [**4**] and its previous comments), it suffices to show that β extends to G. By Theorem (8.29) of [**4**], it suffices to prove that θ extends to Q^*, where Q^*/N^* is a Sylow q-subgroup of G^*/N^*. Suppose first that $q \in \pi$. Then the result follows from Theorem (8.23) of [**4**]. So we may assume that $q \in \pi'$. Now, consider the group Q corresponding to Q^* with $N \subseteq Q \subseteq G$. Since N is a π-group, θ extends to some $\psi \in \mathrm{IBr}(Q \mid \theta)$ by Theorem (8.13) of [**4**]. Let $\lambda = \psi^* \in \mathrm{Irr}(Q^* \mid \theta^*)$. Now, λ is linear (because it extends θ^*). Factor λ as $\mu\nu$ with $o(\mu)$ a π-number and $o(\nu)$ a π'-number. Then

$$\mu_{N^*}\nu_{N^*} = \lambda_{N^*} = \theta^* = \alpha\beta,$$

and, by uniqueness, we have that $\mu_{N^*} = \beta$, as desired. □

Now we can prove Theorem A, which we restate here.

Theorem 2.6. *Suppose that G is π-separable and let H be a Hall π-subgroup of G. If $\alpha \in \mathrm{IBr}(H)$ is G-stable, then α extends to some Brauer character of G.*

Proof. We argue by double induction, first on $|G : \boldsymbol{O}_\pi(G)|$ and second on $|G|$. Let $N = \boldsymbol{O}_\pi(G)$ and let $\theta \in \mathrm{IBr}(N)$ be an irreducible constituent of α_N. Let T be the stabilizer of θ in G. We know that $TH = G$ by Lemma 2.3. Let $\beta \in \mathrm{IBr}(T \cap H \mid \theta)$ be the Clifford correspondent of α with respect to θ. By Lemma 2.4, we know that β is T-stable. Assume first that $T < G$. Then $|T : \boldsymbol{O}_\pi(T)| < |G : N|$, and by induction we have that β extends to some $\xi \in \mathrm{IBr}(T)$. Now, by Mackey, we have that

$$(\xi^G)_H = (\xi_{T\cap H})^H = \beta^H = \alpha,$$

as desired. So we may assume that $T = G$. Now, by Theorem 2.5, let (G^*, N^*, θ^*) be an isomorphic modular character triple with N^* a central π-group. Since G/N and G^*/N^* are isomorphic, we have in fact that $N^* = \boldsymbol{O}_\pi(G^*)$. Let $M = \boldsymbol{O}_{\pi'}(G^*)$. If $M = 1$, then $G^* = \boldsymbol{C}_{G^*}(N^*) \subseteq N^*$, and in this case we are clearly done. So we may assume that $M > 1$. Now, we work in G^*/M. Let $N^* \subseteq H^* \subseteq G^*$ correspond to H. Of course, notice that H^* is a Hall subgroup of G^*. Suppose that $\alpha^* \in \mathrm{IBr}(H^*)$ corresponds to α. By Lemmas 2.3 and 2.1, we know that α extends to some $\xi \in \mathrm{cf}(G^0 \mid \theta)$. By the properties of modular character triples, we have that ξ^* extends α^*. Therefore, we have that α^* is G^*-stable. Since $H^* \cap M = 1$, it follows that α^* uniquely extends to some Brauer character $\bar{\alpha} \in \mathrm{IBr}(H^*M)$ that contains M in its kernel. View $\bar{\alpha}$ as an irreducible Brauer character of H^*M/M and notice that $\bar{\alpha}$ is G^*/M-stable. Now, $|G^*/M : \boldsymbol{O}_\pi(G^*/M)| < |G : N|$, and by induction we have that $\bar{\alpha}$ extends to some Brauer character μ of G^*. Hence, α^* extends to G^*. Since μ necessarily lies over θ^*, it follows that α extends to some Brauer character of G. □

Next is Corollary B of § 1.

Corollary 2.7. *Suppose that G is π-separable and let H be a Hall π-subgroup of G. Then every irreducible Brauer character of H extends to some Brauer character of G if and only if whenever $x, y \in H$ are p-regular and G-conjugate, then x and y are H-conjugate.*

Proof. If H controls G-fusion on p-regular elements, it is clear that every irreducible Brauer character of H is G-stable. In this case, by Theorem A, every irreducible Brauer character of H extends to some Brauer character of G. On the other hand, assume that every irreducible Brauer character of H extends to some Brauer character of G. Then every irreducible Brauer character of H is G-stable. Suppose now that $x, y \in H$ are p-regular and G-conjugate. Then $\alpha(x) = \alpha(y)$ for every $\alpha \in \mathrm{IBr}(H)$. In this case, x and y are H-conjugate, since the Brauer character table of H is an invertible matrix (and, therefore, cannot have two identical columns). $\square$

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