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A PROBABILISTIC STUDY ON THE VALUE-DISTRIBUTION OF DIRICHLET SERIES ATTACHED TO CERTAIN CUSP FORMS

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§1. Introduction

The existence of the asymptotic probability measure of the Riemann zeta-function was proved in Bohr-Jessen's classical paper [3] [4].

Let $s = \sigma + it$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, and R an arbitrary rectangle with the edges parallel to the axes. Then, for any $\sigma_0 > 1/2$ and $T > 0$, the set

$$\{t \in [0, T] \mid \log \zeta(\sigma_0 + it) \in R\}$$

is Jordan measurable, and we denote the Jordan measure of this set by $V(T, R; \zeta)$. Then, Bohr-Jessen's main result asserts the existence of the limit

$$W(R; \zeta) = \lim_{T\to\infty} V(T, R; \zeta)/T,$$

which we call the asymptotic probability measure of $\log \zeta(s)$ on the line $\sigma = \sigma_0$.

Let N be a positive integer, $\theta_n \in [0, 1)$ $(1 \leqq n \leqq N)$, and we define the mapping S_N from $Q_N = [0, 1)^N$ to the complex plane $\boldsymbol{C}$ by

$$S_N(\theta_1, \cdots, \theta_N; \zeta) = -\sum_{n=1}^{N} \log(1 - p_n^{-\sigma_0} \exp(2\pi i\theta_n)), \tag{1.1}$$

where p_n is the n-th prime number. By $W_N(R; \zeta)$ we mean the N-dimensional Jordan measure of the inverse image $S_N^{-1}(R)$. Then, Bohr-Jessen proved that when N tends to infinity, the limit $\lim W_N(R; \zeta)$ exists, which just coincides our desired $W(R; \zeta)$.

Here we take notice of the property that in the right-hand side of (1.1), each term $\log(1 - p_n^{-\sigma_0} e^{2\pi i\theta_n})$ describes a closed convex curve, as θ_n

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moves from 0 to 1. Hence, $S_N(\theta_1, \cdots, \theta_N)$ is a kind of "sum" of convex curves. Bohr-Jessen's original proof of the existence of $\lim W_N(R; \zeta)$ is based on a rather involved theory on the infinite sums of convex curves [5]. Later, using Fourier transforms of probability measures, an alternative proof was given ([6] [13]), but it also treats the case of convex curves only (see Theorem 13 of [13]).

For more general Euler products, however, the corresponding terms do not always describe convex curves any more. Therefore, if we want to generalize Bohr-Jessen's theory, it is indispensable to develop a method which is independent of convexity. In the present paper, we will study the value-distribution of Dirichlet series attached to cusp forms which are simultaneous eigenfunctions of Hecke operators, as a simple example of non-convex Euler products.

In the following sections, the rectangles we consider are closed and have the edges parallel to the axes. For any $z \in \boldsymbol{C}$ and subset $X \subset \boldsymbol{C}$, the set $\{w - z \mid w \in X\}$ we denote by $X - z$. Also, $\operatorname{dist}(z, X)$ means the lower bound of $\{|z - w| \mid w \in X\}$.

§ 2. Statement of results

As usual, we denote by $\mathrm{SL}(2, \boldsymbol{Z})$ the elliptic modular group. Let k, M be posititive integers, χ a Dirichlet character mod. M, and we define the Hecke congruence subgroup of level M by

$$\Gamma_0(M) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \boldsymbol{Z}) \middle| c \equiv 0 \ (\mathrm{mod.}\ M) \right\}.$$

By $\mathscr{S}_k(M, \chi)$ we mean the space of cusp forms of weight k with respect to $\Gamma_0(M)$ with character χ. If a function $f(w)$ is a non-zero element ʇo $\mathscr{S}_k(M, \chi)$, then $f(w)$ has the Fourier expansion

$$f(w) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n w}$$

at the cusp ∞. Hecke proved that the Dirichlet series

$$L(s) = L(s; f) = \sum_{n=1}^{\infty} c(n) n^{-s}$$

is convergent absolutely in the half-plane $\sigma > (k+1)/2$, and can be continued holomorphically to the whole plane. Furthermore, the functional equation

$$\Lambda(s; f) = i^k \Lambda(k - s; \tilde{f}) \tag{2.1}$$

is valid, where $\Lambda(s; f) = (2\pi/\sqrt{M})^{-s}\Gamma(s)L(s; f)$ and $\tilde{f}(w) = M^{-k/2}w^{-k}f(-1/Mw)$. From (2.1) we see that the "critical strip" of $L(s; f)$ is $\{s \mid (k-1)/2 \leqq \sigma \leqq (k+1)/2\}$, and the "critical line" is $\sigma = k/2$. We consider the value-distribution of $L(s; f)$ in the half-plane $\sigma > k/2$.

Now we assume $f(w)$ is a primitive form of level M. Then, $f(w)$ is a simultaneous eigenfunction of Hecke operators $T(n)$, defined by

$$(f \mid T(n))(w) = n^{k-1} \sum_{\substack{0<d\mid n \\ ad=n}} \sum_{b=0}^{d-1} \chi(a) d^{-k} f((aw + b)/d),$$

and the corresponding eigenvalue is equal to the n-th Fourier coefficient $c(n)$. The Euler product expansion

$$L(s; f) = \prod_{n=1}^{\infty} (1 - c(p_n)p_n^{-s} + \chi(p_n)p_n^{k-1-2s})^{-1}$$

holds for $\sigma > (k+1)/2$. Hence $L(s) \neq 0$ if $\sigma > (k+1)/2$, so we can define

$$\log L(s) = -\sum_{n=1}^{\infty} \log (1 - c(p_n)p_n^{-s} + \chi(p_n)p_n^{k-1-2s}) \tag{2.2}$$

in this region. Here we comment the rigorous meaning of the right-hand side of the above. If $(p_n, M) = 1$, then it follows from Deligne [8] and Deligne-Serre [9] that we can write

$$1 - c(p_n)p_n^{-s} + \chi(p_n)p_n^{k-1-2s} = (1 - \alpha_n p_n^{-s})(1 - \beta_n p_n^{-s})$$

with $|\alpha_n| \leqq p_n^{(k-1)/2}$ and $|\beta_n| \leqq p_n^{(k-1)/2}$. So the principal value $\operatorname{Log}(1 - \alpha_n p_n^{-s})$, $\operatorname{Log}(1 - \beta_n p_n^{-s})$ is well-defined if $\sigma > (k-1)/2$, and we put

$$\log (1 - c(p_n)p_n^{-s} + \chi(p_n)p_n^{k-1-2s}) = \operatorname{Log}(1 - \alpha_n p_n^{-s}) + \operatorname{Log}(1 - \beta_n p_n^{-s}). \tag{2.3}$$

Next, if $p_n \mid M$, then $\chi(p_n) = 0$, and $|c(p_n)| \leqq p_n^{(k-1)/2}$ since $f(w)$ is primitive. Hence, (2.3) is valid with $\alpha_n = c(p_n)$ and $\beta_n = 0$. Hence, each term in the right-hand side of (2.2) is well-defined for $\sigma > (k-1)/2$, and the sum is convergent absolutely for $\sigma > (k+1)/2$.

Next we define $\log L(s)$ in the strip $k/2 < \sigma \leqq (k+1)/2$. There is a possibility of the existence of zeros of $L(s)$ in this region, so we restrict our consideration to the set

$$G = \{s \mid \sigma > k/2\} - \bigcup_{s_j = \sigma_j + it_j} \{s = \sigma + it_j \mid k/2 < \sigma \leqq \sigma_j\},$$

where s_j's ($j = 1, 2, \cdots$) run through all possible zeros of $L(s)$ in $k/2 < \sigma \leqq (k+1)/2$. For any $s_0 = \sigma_0 + it_0 \in G$, we define $\log L(s_0)$ by the analytic continuation along the path $\{s = \sigma + it_0 \,|\, \sigma \geqq \sigma_0\}$.

We fix a $\sigma_0 > k/2$, and discuss the value-distribution of $\log L(s)$ on the line $\sigma = \sigma_0$. Let R be an arbitrary rectangle, and $T > 0$. The set

$$\{t \in [0, T] \,|\, \sigma_0 + it \in G,\ \log L(\sigma_0 + it) \in R\}$$

consists of several intervals, so it is obviously Jordan measurable, and by $V(T, R) = V(T, R; L)$ we denote the Jordan measure of this set. The principal result of this paper is the following

THEOREM 1. *Let $L(s)$ be the Dirichlet series attached to a primitive form of level M. Then, there exists the limit*

$$W(R) = W(R; L) = \lim_{T \to \infty} V(T, R; L)/T$$

for any $\sigma_0 > k/2$.

The following four sections are devoted mainly to the proof of Theorem 1. In the proof we shall see that W is a probability measure. The evaluation of $W(E)$ for any measurable E is an interesting problem. In this direction, as a generalization of Theorem 19 of Jessen-Wintner [13], we have

THEOREM 2. *Let a, λ be positive numbers. Then, for any W-measurable set E included in $\{z \,||z| > 3a\}$, the inequality*

$$W(E) \leqq Ce^{-\lambda a^2}$$

holds, where C is a positive constant depending only on λ, k and σ_0.

§ 3. Application of the Kronecker-Weyl theorem

Let N be a positive integer, and put

$$L_N(s) = \prod_{n=1}^{N} (1 - c(p_n)p_n^{-s} + \chi(p_n)p_n^{k-1-2s})^{-1}.$$

Then,

$$\log L_N(s) = -\sum_{n=1}^{N} \log(1 - c(p_n)p_n^{-s} + \chi(p_n)p_n^{k-1-2s}),$$

which is well-defined if $\sigma > (k-1)/2$. Let $V_N(T, R) = V_N(T, R; L)$ be the Jordan measure of the set

$$\{t \in [0, T] \mid \log L_N(\sigma_0 + it) \in R\}.$$

Next, let $Q_N = [0, 1)^N$ be the N-dimensional unit torus, and for any $(\theta_1, \cdots, \theta_N) \in [0, 1)^N$, we put

$$S_N(\theta_1, \cdots, \theta_N) = -\sum_{n=1}^{N} \log(1 - c(p_n)p_n^{-\sigma_0} \exp(2\pi i\theta_n) + \chi(p_n)p_n^{k-1-2\sigma_0} \exp(4\pi i\theta_n)).$$

For any subset $E \subset C$, we denote the inverse image $S_N^{-1}(E)$ by $\Omega_N(E) = \Omega_N(E; L)$. Then, $\log L_N(\sigma_0 + it) \in R$ if and only if

$$\left(\left\{-\left(\frac{\log p_1}{2\pi}\right)t\right\}, \cdots\cdots, \left\{-\left(\frac{\log p_N}{2\pi}\right)t\right\}\right) \in \Omega_N(R),$$

where the symbol $\{x\}$ denotes the fractional part of x. Hence, if $\Omega_N(R)$ is Jordan measurable, then by using the Kronecker-Weyl theorem (see Titchmarsh [19], § 11.7), we can conclude

$$\lim_{T\to\infty} V_N(T, R; L)/T = W_N(R), \tag{3.1}$$

where $W_N(R) = W_N(R; L)$ is the N-dimensional Jordan measure of $\Omega_N(R)$. Therefore, to establish (3.1), it is sufficient to prove the following

LEMMA 1. *For any rectangle R, the set $\Omega_N(R)$ is Jordan measurable. Furthermore, for any positive ε, there exists a positive η, which is independent of N, and for which $W_N(R) < \varepsilon$ holds for any R with the area $\mu(R) < \eta$.*

This lemma was at first proved by Bohr-Courant [2] for the case of $\zeta(s)$, and then, in § 11 of Bohr-Jessen [5] for general convex curves. Their induction argument can be applied to our present case.

Let

$$z_n = z_n(\theta_n) = -\log(1 - c(p_n)p_n^{-\sigma_0} \exp(2\pi i\theta_n) + \chi(p_n)p_n^{k-1-2\sigma_0} \exp(4\pi i\theta_n)),$$

and $\omega_n = \{z_n(\theta_n) \mid 0 \leqq \theta_n < 1\}$. We prove the lemma by induction.

The set $\Omega_1(R)$ is a union of several intervals, so it is clearly Jordan measurable. To show the second assertion, we first note that if $\mu(R) < \eta$, then the length of at least one edge of R is smaller than $\sqrt{\eta}$, hence it is included in an open strip of width $\sqrt{\eta}$, parallel to the real or imaginary axis. We only treat the former; the argument in the latter case is similar.

For any real x, by an elementary calculation we can show that the number of the roots θ_1 which satisfies $\operatorname{Im} z_1(\theta_1) = x$ is at most four. Let l be an arbitrary line parallel to the real axis, and denote by $z_1(\theta_1^{(\nu)})$ $(1 \leqq \nu \leqq 4)$ the intersection points of l and ω_1. Let $A(l; \eta)$ be the open strip of width η, whose center line is l. For sufficiently small η, the strip $A(l; 2\sqrt{\eta})$ includes only four disjoint pieces $\omega_1^{(\nu)}(l)$ of ω_1 $(1 \leqq \nu \leqq 4)$ on which lies the point $z_1(\theta_1^{(\nu)})$, respectively. Furthermore, we can choose $\eta = \eta(l)$ so small that the length of the set $\{\theta_1 | z_1(\theta_1) \in \omega_1^{(\nu)}(l)\}$ is less than $\varepsilon/4$. Hence we have that the Jordan measure of the set

$$\Omega_1(A(l;\ 2\sqrt{\eta(l)}))$$

is smaller than ε. We define

$$\omega_1(l) = \left(\bigcup_{\nu=1}^{4} \omega_1^{(\nu)}(l)\right) \cap A(l;\ \sqrt{\eta(l)})\,.$$

Since ω_1 is compact, we can choose a finite number of the lines $\{l_j\}$, which gives a finite covering $\{\omega_1(l_j)\}$ of ω_1. And we put

$$\eta = \min_j \{\eta(l_j)\}\,.$$

Then it is obvious that for any l, there exists a line l_j, for which

$$\omega_1 \cap A(l;\ \sqrt{\eta}) \subset A(l_j;\ 2\sqrt{\eta(l_j)})$$

holds. This implies the second assertion for $N = 1$.

The following second step is the same as in the original proof of Bohr-Courant, but we present the argument for the convenience of readers.

We now assume the lemma is valid for N. By the assumption, $W_N(R - z_{N+1})$ is a continuous function of z_{N+1}, so is also a continuous function of θ_{N+1}. Hence the integral

$$I(R) = \int_0^1 W_N(R - z_{N+1})d\theta_{N+1}$$

exists.

We denote the four vertices of R by $A_u + iB_v$ $(u, v = 1, 2,\ A_1 < A_2,\ B_1 < B_2)$:

$$R = \{z | A_1 \leqq \operatorname{Re}(z) \leqq A_2,\ B_1 \leqq \operatorname{Im}(z) \leqq B_2\}\,.$$

Let $\delta > 0$, and we put

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$$R_i = R_i(\delta) = \{z \mid A_1 + \delta \leqq \mathrm{Re}(z) \leqq A_2 - \delta,\ B_1 + \delta \leqq \mathrm{Im}(z) \leqq B_2 - \delta\}$$

and

$$R_y = R_y(\delta) = \{z \mid A_1 - \delta \leqq \mathrm{Re}(z) \leqq A_2 + \delta,\ B_1 - \delta \leqq \mathrm{Im}(z) \leqq B_2 + \delta\}.$$

For any positive ε, by the assumption there exists a sufficiently small δ, independent of N, for which the inequalities

$$W_N(R - z_{N+1}) - \varepsilon < W_N(R_i - z_{N+1}), \tag{3.2}$$

$$W_N(R_y - z_{N+1}) < W_N(R - z_{N+1}) + \varepsilon \tag{3.3}$$

hold for any $z_{N+1} \in \omega_{N+1}$.

Let us take a sequence $0 = \theta_{N+1}^{(1)} < \theta_{N+1}^{(2)} < \cdots < \theta_{N+1}^{(m)} < \theta_{N+1}^{(m+1)} = 1$, and define

$$I^{(m)}(R) = \sum_{\mu=1}^{m} W_N(R - z_{N+1}^{(\mu)})(\theta_{N+1}^{(\mu+1)} - \theta_{N+1}^{(\mu)}),$$

where $z_{N+1}^{(\mu)} = z_{N+1}(\theta_{N+1}^{(\mu)})$. Under a suitable choice of $\{\theta_{N+1}^{(\mu)}\}$, we have

$$|I(R) - I^{(m)}(R)| < \varepsilon, \tag{3.4}$$

and

$$R_i - z_{N+1}^{(\mu)} \subset R - z_{N+1} \subset R_y - z_{N+1}^{(\mu)}$$

for any μ and any $z_{N+1} = z_{N+1}(\theta_{N+1})$ with $\theta_{N+1}^{(\mu)} \leqq \theta_{N+1} < \theta_{N+1}^{(\mu+1)}$. Hence,

$$\Omega_N(R_i - z_{N+1}^{(\mu)}) \times [\theta_{N+1}^{(\mu)}, \theta_{N+1}^{(\mu+1)}) \subset \Omega_\mu \subset \Omega_N(R_y - z_{N+1}^{(\mu)}) \times [\theta_{N+1}^{(\mu)}, \theta_{N+1}^{(\mu+1)}),$$

where $\Omega_\mu = \{(\theta_1, \cdots, \theta_{N+1}) \in \Omega_{N+1}(R) \mid \theta_{N+1}^{(\mu)} \leqq \theta_{N+1} < \theta_{N+1}^{(\mu+1)}\}$. So it follows that

$$W_N(R_i - z_{N+1}^{(\mu)})(\theta_{N+1}^{(\mu+1)} - \theta_{N+1}^{(\mu)}) \leqq \underline{m}(\Omega_\mu) \leqq \overline{m}(\Omega_\mu) \leqq W_N(R_y - z_{N+1}^{(\mu)})(\theta_{N+1}^{(\mu+1)} - \theta_{N+1}^{(\mu)}),$$

where $\underline{m}(X)$ (resp. $\overline{m}(X)$) denotes the Jordan inner (resp. outer) volume of X, hence the inequality

$$I^{(m)}(R_i) \leqq \underline{m}(\Omega_{N+1}(R)) \leqq \overline{m}(\Omega_{N+1}(R)) \leqq I^{(m)}(R_y)$$

follows. Combining this result with (3.2), (3.3) and (3.4), we have

$$I(R) - 2\varepsilon \leqq \underline{m}(\Omega_{N+1}(R)) \leqq \overline{m}(\Omega_{N+1}(R)) \leqq I(R) + 2\varepsilon,$$

which implies $\Omega_{N+1}(R)$ is Jordan measurable, and

$$W_{N+1}(R) = \int_0^1 W_N(R - z_{N+1})d\theta_{N+1}. \tag{3.5}$$

The second assertion of the lemma is a direct consequence of the expression (3.5).

§ 4. An evaluation of the probability measure W_N

Let E a subset of $\boldsymbol{C}$, for which $\Omega_N(E)$ is Lebesgue measurable. We denote the N-dimensional Lebesgue measure of $\Omega_N(E)$ by $W_N(E)$. Then W_N is clearly a probability measure over $\boldsymbol{C}$, and, due to Lemma 2.4.3 of Itô [12], it is regular. The purpose of this section is to prove the following

LEMMA 2. *Let λ be an arbitrary positive number. Then, there exists a positive constant $a_0 = a_0(\lambda, k, \sigma_0)$, for which the inequality*

$$W_N(E) \leqq Ce^{-\lambda a^2}$$

holds for any $a > a_0$, any Borel set $E \subset \{z \mid |z| > 2a\}$ and any sufficiently large positive integer N, with a positive constant $C = C(\lambda, k, \sigma_0)$.

The basic idea of the following proof is due to Jessen-Wintner [13] (see also Borchsenius-Jessen [6]), though their argument depends on the existence of the density function of W_N.

Let r be a positive integer, $N > r$, and put

$$S_{r,N}(\theta_{r+1}, \cdots, \theta_N) = -\sum_{n=r+1}^{N} \log(1 - c(p_n)p_n^{-\sigma_0}\exp(2\pi i\theta_n) + \chi(p_n)p_n^{k-1-2\sigma_0}\exp(4\pi i\theta_n)) .$$

For any Borel set E, the inverse image $\Omega_{r,N}(E) = S_{r,N}^{-1}(E)$ is Lebesbue measurable, so we can define a probability measure $W_{r,N}(E)$, which is equal to the $(N-r)$-dimensional Lebesgue measure of $\Omega_{r,N}(E)$. By Fubini's theorem we have

$$\text{(4.1)} \qquad W_N(E) = \int_{Q_{N-r}} W_r(E - S_{r,N}(\theta_{r+1}, \cdots, \theta_N))dm(\theta_{r+1}, \cdots, \theta_N) = \int_{\boldsymbol{C}} W_r(E - z)dW_{r,N}(z) ,$$

where m is the $(N-r)$-dimensional Lebesgue measure.

The set

$$\sum_r = \{S_r(\theta_1, \cdots, \theta_r) \mid \theta_n \in [0, 1) \ (1 \leqq n \leqq r)\}$$

is bounded; there exists a positive number $a_0 = a_0(r, k, \sigma_0)$ for which $\sum_r \subset \{z \mid |z| \leqq a_0\}$ holds. Let $a > a_0$ and E an arbitrary Borel set included in

$\{z||z|>2a\}$. If $|z|\leqq a$, then $(E-z)\cap\sum_r=\emptyset$, which yields $W_r(E-z)=0$. Therefore, from (4.1), we have

$$W_N(E)=\int_{|z|>a}W_r(E-z)dW_{r,N}(z) \tag{4.2}$$
$$\leqq\int_{|z|>a}dW_{r,N}(z)=W_{r,N}(\{z||z|>a\}).$$

To evaluate the right-hand side of the above, we prepare the following

LEMMA 3. *Let* $\lambda>0$, $b>0$, *and* B *a bounded set which satisfies* $B\subset\{z||z|\leqq b\}$. *Then, under a suitable choice of* $r=r(\lambda,k,\sigma_0)$, *there exists a positive constant* $C_1=C_1(\lambda,k,\sigma_0)$, *for which*

$$W_{r,N}(z_0-B)\leqq C_1\exp(-4\lambda|z_0|^2)$$

holds for any $z_0\in\{z||z|>2b\}$.

Proof. At first we note that if $\theta=(\theta_{r+1},\cdots,\theta_N)\in\Omega_{r,N}(z_0-B)$, then $|S_{r,N}(\theta)|>|z_0|/2$. Hence,

$$\exp(4\lambda|z_0|^2)W_{r,N}(z_0-B)=\int_{\Omega_{r,N}(z_0-B)}\exp(4\lambda|z_0|^2)dm(\theta) \tag{4.3}$$
$$\leqq\int_{\Omega_{r,N}(z_0-B)}\exp(16\lambda|S_{r,N}(\theta)|^2)dm(\theta)$$
$$\leqq\int_{Q_{N-r}}\exp(16\lambda|S_{r,N}(\theta)|^2)dm(\theta).$$

Next, since $\sigma_0>k/2$, we have

$$|\alpha_n p_n^{-\sigma_0}\exp(2\pi i\theta_n)|\leqq p_n^{(k-1)/2-\sigma_0}\leqq 2^{(k-1)/2-\sigma_0}<1/\sqrt{2}<1,$$

and the same estimate holds for $\beta_n p_n^{-\sigma_0}\exp(2\pi i\theta_n)$. There is an absolute constant C_2, for which

$$|-\log(1-z)-z|\leqq C_2|z|^2$$

holds for any $z\in\{|z|\leqq 1/\sqrt{2}\}$. Hence, if we put

$$S^*_{r,N}(\theta)=\sum_{n=r+1}^{N}(\alpha_n+\beta_n)p_n^{-\sigma_0}\exp(2\pi i\theta_n),$$

then

$$|S_{r,N}(\theta)-S^*_{r,N}(\theta)|\leqq C_2\sum_{n=r+1}^{N}(|\alpha_n|^2+|\beta_n|^2)p_n^{-2\sigma_0}\leqq 2C_2C_3,$$

where

$$C_3 = C_3(k, \sigma_0) = \sum_{n=1}^{\infty} p_n^{k-1-2\sigma_0}.$$

In general, if $|u - v| \leqq w$, then $|u|^2 \leqq 2(|v|^2 + w^2)$. Therefore,

$$\begin{aligned}\int_{Q_{N-r}} &\exp(16\lambda|S_{r,N}(\theta)|^2)dm(\theta) \\ &\leqq \int_{Q_{N-r}} \exp(32\lambda(|S_{r,N}^*(\theta)|^2 + 4C_2^2C_3^2))dm(\theta) \\ &= \exp(128C_2^2C_3^2\lambda)\sum_{j=0}^{\infty}\frac{(32\lambda)^j}{j!}\int_{Q_{N-r}}|S_{r,N}^*(\theta)|^{2j}dm(\theta).\end{aligned}$$

By using Parseval's equation, we have

$$\begin{aligned}\int_{Q_{N-r}}|S_{r,N}^*(\theta)|^{2j}dm(\theta) &= \sum_{j_{r+1}+\cdots+j_N=j}\left|\frac{j!}{j_{r+1}!\cdots j_N!}\prod_{n=r+1}^{N}((\alpha_n+\beta_n)p_n^{-\sigma_0})^{j_n}\right|^2 \\ &\leqq j!\left(\sum_{n=r+1}^{N}|(\alpha_n+\beta_n)p_n^{-\sigma_0}|^2\right)^j.\end{aligned}$$

Now we choose $r = r(\lambda, k, \sigma_0)$ so large that

$$d = 1 - 32\lambda\sum_{n=r+1}^{\infty}|(\alpha_n+\beta_n)p_n^{-\sigma_0}|^2 \geqq 1/2$$

holds. Then we have

$$\begin{aligned}\int_{Q_{N-r}} &\exp(16\lambda|S_{r,N}(\theta)|^2)dm(\theta) \\ &\leqq \exp(128C_2^2C_3^2\lambda)\sum_{j=0}^{\infty}(1-d)^j \leqq 2\cdot\exp(128C_2^2C_3^2\lambda).\end{aligned}$$

This inequality with (4.3) leads to the assertion of Lemma 3.

Now we complete the proof of Lemma 2. Let

$$\begin{aligned}\Delta = \Delta(\mu,\nu) = \{z\,|\,\mu(a_0/2\sqrt{2}) &\leqq \mathrm{Re}(z) \leqq (\mu+1)(a_0/2\sqrt{2}), \\ \nu(a_0/2\sqrt{2}) &\leqq \mathrm{Im}(z) \leqq (\nu+1)(a_0/2\sqrt{2})\}\end{aligned}$$

for any integers μ and ν. Then it is obvious that

$$W_{r,N}(\{z\,|\,|z| > a\}) \leqq \sum_{\Delta} W_{r,N}(\Delta), \tag{4.4}$$

where the sum runs through all Δ which satisfies the condition $\Delta \cap \{z\,|\,|z| > a\} \neq \varnothing$. Let z_Δ be the vertex of Δ which is the most distant from the origin. Then we can write $\Delta = z_\Delta - \Delta_0$, where Δ_0 is one of the squares $\Delta(-1,-1)$, $\Delta(-1,0)$, $\Delta(0,-1)$ and $\Delta(0,0)$. Since $|z_\Delta| > a$ and

$$\Delta_0 \subset \{z \,|\, |z| \leqq \sqrt{2}\,(a_0/2\sqrt{2}) = a_0/2\}\,,$$

we can apply Lemma 3 with $z_0 = z_\Delta$, $B = \Delta_0$ and $b = a_0/2$. The result is that

$$W_{r,N}(\Delta) \leqq C_1 \exp(-4\lambda |z_\Delta|^2)\,.$$

The inequality $|z| \leqq |z_\Delta|$ holds for any $z \in \Delta$, so we have

$$\exp(-4\lambda |z_\Delta|^2) \leqq (a_0/2\sqrt{2})^{-2} \int_\Delta \exp(-4\lambda |z|^2) dz\,.$$

Substituting these results in (4.4), we have

$$\begin{aligned} W_{r,N}(\{z \,|\, |z| > a\}) &\leqq 8C_1 a_0^{-2} \int_{|z| \geqq a/2} \exp(-4\lambda |z|^2) dz \\ &= (2C_1\pi/\lambda a_0^2) \exp(-\lambda a^2)\,. \end{aligned}$$

The result of the lemma follows from this inequality and (4.2).

§ 5. The existence of the asymptotic probability measure

Borchsenius-Jessen's proof [6] of the existence of $\lim W_N(R;\zeta)$ is based on Lévy's convergence theorem, and their argument can be generalized to our present case. However, by using the result of Lemma 2, we can give a very simple proof of this fact.

Let P_1, P_2 be two regular probability measures over $\boldsymbol{C}$, and ε_{12} be the lower bound of those ε, for which

$$P_1(F) < P_2(\{z \,|\, \mathrm{dist}(z, F) < \varepsilon\}) + \varepsilon$$

holds for any closed subset F. Similarly we define the number ε_{21}, and put

$$\rho(P_1, P_2) = \max\{\varepsilon_{12}, \varepsilon_{21}\}\,.$$

It can be shown that ρ is a distance function, which we call Prokhorov's distance. Prokhorov [16] proved that with this metric, the space $\mathscr{D}$ of all regular probability measures over $\boldsymbol{C}$ is a complete separable metric space. The convergence with respect to this metric is equivalent to the weak convergence.

Let $\{P_\alpha\}_{\alpha \in \Lambda}$ be a subset of $\mathscr{D}$. We call $\{P_\alpha\}$ is tight if for any positive ε, there exists a compact set $K = K(\varepsilon) \subset \boldsymbol{C}$, for which the inequality

$$P_\alpha(\boldsymbol{C} - K) < \varepsilon$$

holds for any $\alpha \in \Lambda$. Now we quote the following

LEMMA 4 (Prokhorov [16]). *In order for* $\{P_\alpha\}$ *to be tight it is necessary and sufficient that* $\{P_\alpha\}$ *is totally bounded with respect to the Prokhorov metric.*

If $\sigma_0 > (k+1)/2$, then $S_N(\theta_1, \cdots, \theta_N)$ is uniformly bounded for any N, so it is obvious that $\{W_N\}$ is a tight subset. Lemma 2 implies that the tightness is valid for any $\sigma_0 > k/2$. Hence, from Lemma 4, there exists a subsequence $\{W_{N(j)}\}_{j=1}^{\infty}$, which is convergent weakly to a measure $W \in \mathscr{D}$. In the next section we will prove that this W is just the desired limit in Theorem 1.

Here we note that Theorem 2 is now a immediate consequence of Lemma 2 and the above claim. In fact, let $a > a_0$, and E be an W-measurable set included in $\{z \mid |z| > 3a\}$. We can assume E is compact, because W is K-regular. Let G_E be an open set which satisfies

$$E \subset G_E \subset \{z \mid |z| > 2a\}.$$

Then, there exists a continuous function g_E which is equal to 1 on E, equal to 0 on G_E^c, and satisfies $0 \leqq g_E(z) \leqq 1$ if $z \in G_E - E$. Then it follows that

$$W(E) \leqq \int_C g_E(z) dW(z) = \lim_{j \to \infty} \int_C g_E(z) dW_{N(j)}(z) \leqq \liminf_{j \to \infty} W_{N(j)}(G_E).$$

Lemma 2 shows $W_{N(j)}(G_E) \leqq Ce^{-\lambda a^2}$, hence $W(E) \leqq Ce^{-\lambda a^2}$. To verify Theorem 2 in case $a \leqq a_0$, it is enough to change the value of C, if necessary.

§ 6. Completion of the proof of Theorem 1

Let ε be an arbitrary positive number. The second assertion of Lemma 1 (and its proof) implies that there exists a $\delta > 0$, for which

$$W_N(R_y(2\delta) - R_i(2\delta)) < \varepsilon/2$$

holds for any rectangle R and any N. We define a continuous function g_R by

$$g_R(z) = \begin{cases} 1 & \text{if } z \text{ is included in the closure of } R_y(\delta) - R_i(\delta), \\ 0 & \text{if } z \text{ is not included in the open kernel of } \\ & \qquad R_y(2\delta) - R_i(2\delta), \end{cases}$$

and $0 \leqq g_R(z) \leqq 1$ if $z \in (R_i(\delta) - R_i(2\delta)) \cup (R_y(2\delta) - R_y(\delta))$. Then,

$$W(R_y(\delta)-R_i(\delta)) \leqq \int_C g_R(z)dW(z) = \lim_{j\to\infty}\int_C g_R(z)dW_{N(j)}(z)$$
$$\leqq \liminf_{j\to\infty} W_{N(j)}(R_y(2\delta)-R_i(2\delta)),$$

which yields

$$(6.1)\qquad |W(R)-W(R_i(\delta))|<\varepsilon/2, \qquad |W(R)-W(R_y(\delta))|<\varepsilon/2.$$

In particular, any rectangle is a continuity set with respect to W. Hence, there exists a sufficiently large positive J_1, for which

$$(6.2)\qquad |W_{N(j)}(R_i)-W(R_i)|<\varepsilon/2, \qquad |W_{N(j)}(R_y)-W(R_y)|<\varepsilon/2$$

holds for any $j \geqq J_1$.

Now we assume $\sigma_0 > (k+1)/2$. Then we have

$$(6.3)\quad |\log L(\sigma_0+it)-\log L_{N(j)}(\sigma_0+it)| \leqq C_4 \sum_{n=N(j)+1}^{\infty}(|\alpha_n p_n^{-s}|+|\beta_n p_n^{-s}|)<\delta$$

for any real t and any $j \geqq J_2$, with a sufficiently large $J_2 = J_2(\delta, k, \sigma_0)$ and an absolute constant C_4. Hence,

$$V_{N(j)}(T, R_i(\delta)) \leqq V(T, R) \leqq V_{N(j)}(T, R_y(\delta)),$$

and so, from (3.1), we have

$$W_{N(j)}(R_i) \leqq \liminf_{T\to\infty} V(T,R)/T \leqq \limsup_{T\to\infty} V(T,R)/T \leqq W_{N(j)}(R_y).$$

Hence, with (6.1) and (6.2),

$$W(R)-\varepsilon \leqq \liminf_{T\to\infty} V(T,R)/T \leqq \limsup_{T\to\infty} V(T,R)/T \leqq W(R)+\varepsilon,$$

which leads to the assertion of Theorem 1 in the domain of absolute convergence.

Next we proceed to the case $k/2 < \sigma_0 \leqq (k+1)/2$. By $k_N^\delta(T)$ we denote the measure of the set

$$K_N^\delta(T) = \{t \in [0, T] \mid \sigma_0+it \in G, |\log L(\sigma_0+it)-\log L_N(\sigma_0+it)| \geqq \delta\}.$$

Then it follows that

$$(6.4)\qquad V_{N(j)}(T, R_i(\delta)) - k_{N(j)}^\delta(T) \leqq V(T,R) \leqq V_{N(j)}(T, R_y(\delta)) + k_{N(j)}^\delta(T)$$

for any j. Let t_0 be a real number, $k/2 < \alpha_0 < \sigma_0$,

$$H(t_0) = \{s = \sigma+it \mid \sigma > \alpha_0,\ t_0-\tfrac{1}{2} < t < t_0+\tfrac{1}{2}\}$$

and

$$\varphi_N^\delta(t_0) = \begin{cases} 0 & \text{if } H(t_0) \subset G, \text{ and if } |\log L(s) - \log L_N(s)| < \delta \\ & \qquad\qquad\qquad\qquad \text{for any } s \in H(t_0), \\ 1 & \text{otherwise}. \end{cases}$$

Then it is obvious that

$$k_N^\delta(T) \leqq \int_0^T \varphi_N^\delta(t_0) dt_0 .$$

Hence we have

$$\begin{aligned} W_{N(j)}(R_i) - \Phi_{N(j)} &\leqq \liminf_{T\to\infty} V(T, R)/T \\ &\leqq \limsup_{T\to\infty} V(T, R)/T \leqq W_{N(j)}(R_y) + \Phi_{N(j)} \end{aligned}$$

from (6.4), where

$$\Phi_N = \limsup_{T\to\infty} T^{-1} \int_0^T \varphi_N^\delta(t_0) dt_0 .$$

Therefore, if we can show

$$\lim_{N\to\infty} \Phi_N = 0 , \tag{6.5}$$

then, by a way similar to the case of $\sigma_0 > (k+1)/2$, we can complete the proof of Theorem 1 in the critical strip.

In the case of the Riemann zeta-function, the result corresponding to (6.5) is Hilfssatz 5 of Bohr [1]. Bohr's proof of Hilfssatz 5 is based on Hilfssatz 2 in the same paper. The analogue of Hilfssatz 2 in our case can be stated as follows:

LEMMA 5. *Let $k/2 < \sigma_1 < \sigma_2$, and ε be an arbitrary positive number. Then there exists a positive $N_0 = N_0(\sigma_1, \sigma_2, \varepsilon)$, for which the inequality*

$$\iint_{\substack{\sigma_1 \leqq \sigma \leqq \sigma_2 \\ 0 \leqq t \leqq T}} |L(s)/L_N(s) - 1|^2 d\sigma dt < \varepsilon T$$

holds for any $N \geqq N_0$ and any $T \geqq T_0$, with a positive $T_0 = T_0(N)$.

As we have already mentioned in [14], we can skip Bohr's technical argument in the proof of Hilfssatz 2, by using a general mean-value theorem of Carlson.

By virtue of Hecke's estimate (Satz 7 of [11]), we can apply Potter's general result (Theorem 3 of [15]) to our case, and the result is the asymptotic formula

$$\int_0^T |L(\sigma_0 + it)|^2 dt = T \sum_{n=1}^{\infty} |c(n)|^2 n^{-2\sigma_0} + o(T) \tag{6.6}$$

which is valid for $\sigma_0 > k/2$. It can be easily shown that

$$|L_N(\sigma_0 + it)|^{-1} \leqq \exp(C_4 N^{(k+1)/2-\sigma_0}),$$

so from (6.6) we see

$$T^{-1} \int_0^T |L(\sigma_0 + it)/L_N(\sigma_0 + it) - 1|^2 dt$$

is also bounded. Hence, by using Carlson's theorem [7] (see also § 9.51 of Titchmarsh [18]), we have

$$\lim_{T\to\infty} T^{-1} \int_0^T |L(\sigma_0 + it)/L_N(\sigma_0 + it) - 1|^2 dt = \sum_{\substack{(m, p_1 p_2 \cdots p_N)=1 \\ m \neq 1}} |c(m)|^2 m^{-2\sigma_0} \tag{6.7}$$

for any $\sigma_0 > k/2$, because the Dirichlet series expansion

$$L(s)/L_N(s) = \sum_{(m, p_1 p_2 \cdots p_N)=1} c(m) m^{-s}$$

holds. From the well-known result

$$\sum_{m \leqq x} |c(m)|^2 = C_5 x^k + O(x^{k-2/5})$$

(C_5 being a constant depending on k, M and f) in Rankin's classical work [17], it follows immediately that the right-hand side of (6.7) can be estimated by $O(N^{k-2\sigma_0})$ (cf. Lemma 5 of Good [10]). This completes the proof of the lemma.

The method of the deduction of (6.5) from Lemma 5 is quite the same as the original proof of Bohr [1], so we omit the details. Consequently, our Theorem 1 is now proved.

Note added in proof.

The results in the present paper are now generalized to the case of more general Euler products. A generalization of Theorem 1, with a simplified proof, is written in the author's paper entitled "Value-distribution of zeta-functions", which will be published in "The Proceedings for the Japanese-French Symposium on Analytic Number Theory", ed. by E. Fouvry and K. Nagasaka, a volume in Lecture Notes in Math. Ser., Springer-Verlag.

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