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ON NEAR-PERFECT NUMBERS WITH TWO DISTINCT PRIME FACTORS

XIAO-ZHI REN and YONG-GAO CHEN✉

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Abstract

Recently, Pollack and Shevelev ['On perfect and near-perfect numbers', *J. Number Theory* **132** (2012), 3037–3046] introduced the concept of near-perfect numbers. A positive integer n is called *near-perfect* if it is the sum of all but one of its proper divisors. In this paper, we determine all near-perfect numbers with two distinct prime factors.

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1. Introduction

A positive integer is called a *perfect number* if it is the sum of all of its proper divisors. It is well known that Euler proved that an even perfect number can be written as $2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are primes. Primes of the form $2^p - 1$ are called *Mersenne primes*. Lenstra, Pomerance, and Wagstaff have conjectured that there are infinitely many Mersenne primes (see the discussion in [1]).

Following Pollack and Shevelev [2], a positive integer n is called a *near-perfect number* if it is the sum of all but one of its proper divisors. The missing divisor d is called *redundant*. That is, n is near-perfect with redundant divisor d if and only if d is a proper divisor of n and $\sigma(n) = 2n + d$. Pollack and Shevelev [2] constructed the following three types of near-perfect numbers.

Type 1. $n = 2^{t-1}(2^t - 2^k - 1)$, where $2^t - 2^k - 1$ is prime, and 2^k is the redundant divisor.

Type 2. $n = 2^{2p-1}(2^p - 1)$, where both p and $2^p - 1$ are prime numbers, and $2^p(2^p - 1)$ is the redundant divisor.

Type 3. $n = 2^{p-1}(2^p - 1)^2$, where both p and $2^p - 1$ are prime numbers, and $2^p - 1$ is the redundant divisor.

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Remark 1.1. If p and $2^p - 1$ are prime numbers, then

$$n = 2^p(2^p - 1) = 2^p(2^{p+1} - 2^p - 1)$$

is a near-perfect number with redundant divisor 2^p. In this case, $n = 2m$, where $m = 2^{p-1}(2^p - 1)$ is an even perfect number. Pollack and Shevelev [2, Proposition 3] proved that if $n = 2^j m$ is a near-perfect number, where m is an even perfect number, then either $n = 2^p(2^p - 1)$ or $n = 2^{2p-1}(2^p - 1)$, where both p and $2^p - 1$ are prime numbers. It is clear that if the Lenstra–Pomerance–Wagstaff conjecture is true, then there are infinitely many near-perfect numbers.

We observe that near-perfect numbers of types 1, 2 and 3 have two distinct prime factors. It is easy to see that 40 is a near-perfect number with redundant divisor 10, and 40 is not of type 1, 2 or 3.

In this paper, we determine all near-perfect numbers with two distinct prime factors.

Theorem 1.2. *All near-perfect numbers with two distinct prime factors are of types* 1, 2 *and* 3, *together with* 40.

Among the first 39 near-perfect numbers which are listed in A181595 in [3], except for the number 40, 21 (12, 20, 56, 88, 104, 368, 464, 992, 1504, 1888, 1952, 16256, 24448, 28544, 30592, 32128, 98048, 122624, 128768, 130304, 507392) are of type 1; three (24, 224, 15872) are of type 2; three (18, 196, 15376) are of type 3, and 11 (234, 650, 3724, 5624, 9112, 11096, 13736, 17816, 77744, 174592, 396896) have three distinct prime factors. D. Johnson has found that $173369889 = 3^4 \times 7^2 \times 11^2 \times 19^2$ the smallest odd near-perfect number and P. Moses has verified that this is the only odd near-perfect number up to 1.4×10^{19} (see A181595 in [3]). (From the definition of near-perfect numbers, one may see that all odd near-perfect numbers are squares.)

We pose the following conjecture.

Conjecture 1.3. For any integer $k \geq 3$, there are only finitely many near-perfect numbers with k distinct prime factors.

2. Preliminary lemmas

To prove Theorem 1.2, we first give the following two lemmas.

Lemma 2.1. *If $n = 2^\alpha q$ is a near-perfect number with redundant divisor $d = 2^s q$, where q is an odd prime, then either $n = 40$ or n is of type* 2.

Proof. Since $n = 2^\alpha q$ is a near-perfect number with redundant divisor $d = 2^s q$, it follows that $(2^{\alpha+1} - 1)(q + 1) = 2^{\alpha+1} q + 2^s q$. That is, $(2^s + 1)q = 2^{\alpha+1} - 1$. This implies that $s \geq 1$.

Let k and r be two integers with $0 \leq r \leq s - 1$ such that $\alpha + 1 = ks + r$. Then

$$2^{\alpha+1} - 1 \equiv 2^{ks+r} - 1 \equiv (-1)^k 2^r - 1 \pmod{2^s + 1}.$$

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Since $(2^s+1)q = 2^{\alpha+1}-1$, it follows that $2^s+1 \mid (-1)^k 2^r - 1$. Thus, by

$$|(-1)^k 2^r - 1| \le 2^r + 1 < 2^s + 1,$$

we have $(-1)^k 2^r - 1 = 0$. This implies that k is even and $r = 0$. Let $k = 2m$. Thus, by $(2^s+1)q = 2^{\alpha+1}-1 = 2^{2sm}-1$,

$$q = (2^s - 1)\frac{2^{2sm}-1}{2^{2s}-1}.$$

Since q is an odd prime, it follows that either $2^s - 1 = 1$ or $2^{2sm} - 1 = 2^{2s} - 1$. Thus, either $s = 1$ or $m = 1$.

If $s = 1$, then $\alpha + 1 = 2m$. Thus $(2^s+1)q = 2^{\alpha+1}-1$ becomes $3q = (2^m - 1) \times (2^m + 1)$. Since $(2^m - 1, 2^m + 1) = 1$ and $q \ge 3$, it follows that $m = 2$, $q = 5$ and $\alpha = 3$. Thus $n = 40$.

If $m = 1$, then $q = 2^s - 1$ is an odd prime and $\alpha = 2s - 1$. Hence $n = 2^{2s-1}(2^s - 1)$ is of type 2.

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let q be an odd prime, α and β positive integers with $\beta \ge 2$. If $n = 2^\alpha q^\beta$ is a near-perfect number with redundant divisor d, then $q^\beta \nmid d$.*

Proof. Suppose that $d = 2^s q^\beta$ with $0 \le s \le \alpha - 1$. We will derive a contradiction.

Since $\sigma(n) = 2n + d$, it follows that

$$(2^{\alpha+1} - 1)(1 + q + \cdots + q^\beta) = (2^{\alpha+1} + 2^s)q^\beta. \tag{2.1}$$

If β is even, then $1 + q + \cdots + q^\beta$ is odd. Thus $s = 0$. Since

$$(2^{\alpha+1} - 1, 2^{\alpha+1} + 1) = 1, \quad (1 + q + \cdots + q^\beta, q^\beta) = 1,$$

it follows from (2.1) that

$$2^{\alpha+1} - 1 = q^\beta, \quad 1 + q + \cdots + q^\beta = 2^{\alpha+1} + 1.$$

Thus

$$2^{\alpha+1} + 1 = 2^{\alpha+1} - 1 + 2 = q^\beta + 2 < 1 + q + \cdots + q^\beta = 2^{\alpha+1} + 1,$$

a contradiction.

Now we assume that β is odd. Then the left-hand side of (2.1) is even. Thus $s \ge 1$. From (2.1),

$$2^s \mid 1 + q + \cdots + q^\beta \tag{2.2}$$

and

$$(2^{\alpha+1} - 1)(1 + q + \cdots + q^{\beta-1}) = (2^s + 1)q^\beta. \tag{2.3}$$

By (2.3) and $(q^\beta, 1 + q + \cdots + q^{\beta-1}) = 1$,

$$1 + q + \cdots + q^{\beta-1} \mid 2^s + 1. \tag{2.4}$$

We distinguish two cases according to the value of β.

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Case 1: $\beta \equiv 1 \pmod 4$. Since

$$1 + q + \cdots + q^{\beta} = (1 + q)(1 + q^2 + q^4 + \cdots + q^{\beta-1})$$

and

$$1 + q^2 + q^4 + \cdots + q^{\beta-1} \equiv \frac{\beta+1}{2} \not\equiv 0 \pmod 2,$$

it follows from (2.2) that $2^s \mid 1 + q$. Thus, by $\beta \geq 3$, we have $2^s + 1 \leq q + 2 < 1 + q + \cdots + q^{\beta-1}$, a contradiction with (2.4).

Case 2: $\beta \equiv 3 \pmod 4$. Since

$$1 + q + \cdots + q^{\beta} = (1 + q^2)(1 + q + q^4 + q^5 + \cdots + q^{\beta-3} + q^{\beta-2})$$

and $4 \nmid 1 + q^2$, it follows from (2.2) that $2^{s-1} \mid 1 + q + q^4 + q^5 + \cdots + q^{\beta-3} + q^{\beta-2}$. Thus, by (2.4) and $\beta \geq 3$,

$$q2^{s-1} \leq q(1 + q + q^4 + q^5 + \cdots + q^{\beta-3} + q^{\beta-2}) \leq 2^s + 1 - 1 = 2^s,$$

a contradiction with q being an odd prime. This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.2

Let $n = p^{\alpha} q^{\beta}$ be a near-perfect number with redundant divisor d, where p and q are two primes with $p < q$, and α, β are two positive integers. Then $\sigma(n) = 2n + d$. If $p \geq 3$, then

$$\sigma(n) = \frac{p^{\alpha+1} - 1}{p - 1} \frac{q^{\beta+1} - 1}{q - 1} < \frac{p^{\alpha+1}}{p - 1} \frac{q^{\beta+1}}{q - 1} = n \frac{p}{p - 1} \frac{q}{q - 1} \leq \frac{3 \times 5}{2 \times 4} n < 2n.$$

Hence $p = 2$. Let $d = 2^s q^t$ with $0 \leq s \leq \alpha$ and $0 \leq t \leq \beta$. Thus, $\sigma(n) = 2n + d$ becomes

$$(2^{\alpha+1} - 1)(1 + q + \cdots + q^{\beta}) = 2^{\alpha+1} q^{\beta} + 2^s q^t. \tag{3.1}$$

We distinguish three cases according to the value of β.

Case 1: $\beta = 1$. Then $t \in \{0, 1\}$. If $t = 0$, then, by (3.1), $q = 2^{\alpha+1} - 2^s - 1$ is an odd prime and $d = 2^s$. Thus $n = 2^{\alpha}(2^{\alpha+1} - 2^s - 1)$ is of type 1. If $t = 1$, then $n = 2^{\alpha} q$ is a near-perfect number with redundant divisor $d = 2^s q$. By Lemma 2.1, either $n = 40$ or n is of type 2.

Case 2: $\beta = 2$. By Lemma 2.2, $q^2 \nmid d$. Thus $t \in \{0, 1\}$. By (3.1), $s = 0$. If $t = 0$, then, by (3.1), $(2^{\alpha+1} - q)(1 + q) = 2$, a contradiction. Hence $t = 1$. By (3.1), $q = 2^{\alpha+1} - 1$ is an odd prime. Therefore, $n = 2^{\alpha}(2^{\alpha+1} - 1)^2$ is of type 3.

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Case 3: $\beta \geq 3$. By Lemma 2.2, $0 \leq t \leq \beta - 1$. The equality (3.1) can be rewritten as

$$(2^{\alpha+1} - q)(1 + q + \cdots + q^{\beta-1}) = 1 + 2^s q^t. \tag{3.2}$$

If $q < 2^\alpha$, then the left-hand side of (3.2) is more than $2^\alpha(1 + q + \cdots + q^{\beta-1})$. Noting that $\beta \geq 3$, the right-hand side of (3.2) does not exceed $1 + 2^\alpha q^{\beta-1} < 2^\alpha(1 + q + \cdots + q^{\beta-1})$, a contradiction. Hence $q \geq 2^\alpha$. Since the left-hand side of (3.2) is at least

$$1 + q + \cdots + q^{\beta-1} > 1 + q^2 \geq 1 + 2^\alpha q \geq 1 + 2^s q,$$

we have $t \geq 2$. By (3.1), $q \mid 2^{\alpha+1} - 1$, a contradiction with $q \geq 2^\alpha$.

This completes the proof of Theorem 1.2.

Acknowledgements

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XIAO-ZHI REN, School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, PR China

YONG-GAO CHEN, School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, PR China
e-mail: ygchen@njnu.edu.cn